THE HOMOTOPY GROUPS OF THE INVERSE LIMIT OF A TOWER OF FIBRATIONS

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Abstract. We carefully present an elementary proof of the well known theorem that each homotopy group (or, in degree zero, pointed set) of the inverse limit of a tower of fibrations maps naturally onto the inverse limit of the homotopy groups (or, in degree zero, pointed sets) of the spaces in the tower, with kernel naturally isomorphic to $\lim^1$ of the tower of homotopy groups of one dimension higher.

The main theorem here is Theorem 2.1. This is due to Gray [G], Quillen [Q, Proposition 3.8], Vogt [V], Cohen [C1, C2], and Bousfield and Kan [BK, Theorem 3.1 in Chapter IX, section 3], some of which only consider the case of simply connected spaces or spaces with abelian fundamental groups. The proof given here is at least morally the one in [C2].

1. $\lim^1$ of a tower of not necessarily abelian groups

This definition is as in [V] and [BK, Chapter IX, section 2].

Definition 1.1. If

$$\cdots \longrightarrow G_{n+1} \xrightarrow{p_{n+1}} G_n \xrightarrow{p_n} G_{n-1} \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_1} G_0$$

is a tower of groups, then we define a left action of the product group $\prod_{n\geq 0} G_i$ on the product set $\prod_{n\geq 0} G_i$ by letting the action of an element $(g_n)_{n\geq 0} = (g_0, g_1, g_2, \ldots)$ on an element $(h_n)_{n\geq 0} = (h_0, h_1, h_2, \ldots)$ be

$$\bigl(g_n h_n\bigl(p_{n+1}(g_{n+1}^{-1})\bigr)\bigr)_{n\geq 0} = \bigl(g_0 h_0\bigl(p_1(g_1^{-1})\bigr), g_1 h_1\bigl(p_2(g_2^{-1})\bigr), g_2 h_2\bigl(p_3(g_3^{-1})\bigr), \ldots\bigr) \ .$$

We define $\lim^1$ of that tower of groups to be the pointed set that is the set of orbits of this action. If all of the groups G_n are abelian, then $\lim^1$ of the tower is isomorphic to the underlying pointed set of the cokernel of the homomorphism $f\colon \prod_{n\geq 0} G_n \to \prod_{n\geq 0} G_n$ defined by

$$f(g_0, g_1, g_2, \ldots) = \bigl(g_0\bigl(p_1(g_1^{-1})\bigr), g_1\bigl(p_2(g_2^{-1})\bigr), g_2\bigl(p_3(g_3^{-1})\bigr), \ldots\bigr) \ ,$$

and thus has a natural abelian group structure.

2. The main theorems

Theorem 2.1. *Let*

$$\cdots \longrightarrow X_{n+1} \xrightarrow{p_{n+1}} X_n \xrightarrow{p_n} X_{n-1} \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_1} X_0$$

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be a tower of fibrations of pointed spaces. For every choice of basepoint in $\lim_n X_n$ *(whose image is then taken as the basepoint in each* X_n *for* $n \geq 0$*) and every* $k \geq 0$ *there is a natural short exact sequence*

$$1 \longrightarrow \lim_n^1 \pi_{k+1} X_n \longrightarrow \pi_k \lim_n X_n \xrightarrow{P} \lim_n \pi_k X_n \longrightarrow 1$$

of abelian groups if $k \geq 2$*, of groups if* $k = 1$*, and of pointed sets if* $k = 0$*. If* $k = 0$*, all of the spaces* X_n *are H-spaces, and all of the maps* p_n *are H-maps, then this is a short exact sequence of groups.*

The proof of Theorem 2.1 is in Section 4.

Theorem 2.2. *If*

$$\begin{CD}
\cdots @>>> X_{n+1} @>{p_{n+1}}>> X_n @>{p_n}>> X_{n-1} @>{p_{n-1}}>> \cdots @>{p_1}>> X_0 \\
@. @V{f_{n+1}}VV @V{f_n}VV @V{f_{n-1}}VV @. @V{f_0}VV \\
\cdots @>>> Y_{n+1} @>>{q_{n+1}}> Y_n @>>{q_n}> Y_{n-1} @>>{q_{n-1}}> \cdots @>>{q_1}> Y_0
\end{CD}$$

is a map of towers of fibrations of topological spaces such that f_n *is a weak equivalence for every* $n \geq 0$*, then the induced map of limits* $f\colon \lim_n X_n \to \lim_n Y_n$ *is a weak equivalence.*

Proof. This theorem mostly follows from Theorem 2.1 and the five lemma, but care must be taken to show that the set of path components is mapped isomorphically. We must first show that $\lim_n X_n$ is nonempty if and only if $\lim_n Y_n$ is nonempty. If $\lim_n X_n$ is nonempty, then the existence of the natural map $\lim_n X_n \to \lim_n Y_n$ implies that $\lim_n Y_n$ is nonempty.

Conversely, if $\lim_n Y_n$ is nonempty, then we can choose a point $(y_n)_{n\geq 0}$ in $\lim_n Y_n$, and for each $n \geq 0$ we can choose a point $x_n \in X_n$ such that $f_n(x_n)$ is in the same path component as y_n. We will now inductively define points $x'_n \in X_n$ such that x'_n is in the same path component as x_n and $p_{n+1}(x'_{n+1}) = x'_n$ for all $n \geq 0$. We begin the induction by letting $x'_0 = x_0$. If $n \geq 0$ and we've defined x'_n, then $q_{n+1}f_{n+1}(x_{n+1}) = f_n p_{n+1}(x_{n+1})$ is in the same path component as $q_{n+1}(y_{n+1}) = y_n$, and so (since f_n is a weak equivalence) $p_{n+1}(x_{n+1})$ is in the same path component as x'_n. Thus, we can choose a path $\alpha\colon I \to X_n$ from $p_{n+1}(x_{n+1})$ to x'_n and then lift α to a path $\widetilde{\alpha}\colon I \to X_{n+1}$ such that $\widetilde{\alpha}(0) = x_{n+1}$; we let $x'_{n+1} = \widetilde{\alpha}(1)$. This completes the induction, and so $\lim_n X_n$ contains the point $(x'_n)_{n\geq 0}$.

In the case in which $\lim_n X_n$ and $\lim_n Y_n$ are nonempty, for every choice of basepoint in $\lim_n X_n$ (whose image is then taken as the basepoint in each of $\lim_n Y_n$, X_n, and Y_n for $n \geq 0$) and every $k \geq 0$ Theorem 2.1 implies that we have a map of short exact sequences

$$\begin{CD}
1 @>>> \lim_n^1 \pi_{k+1} X_n @>>> \pi_k \lim_n X_n @>{P}>> \lim_n \pi_k X_n @>>> 1 \\
@. @V{\phi}VV @V{f_*}VV @V{\psi}VV @. \\
1 @>>> \lim_n^1 \pi_{k+1} Y_n @>>> \pi_k \lim_n Y_n @>{Q}>> \lim_n \pi_k Y_n @>>> 1
\end{CD} \tag{2.3}$$

of groups if $k > 0$ and of pointed sets if $k = 0$, and both ϕ and ψ are isomorphisms. For $k > 0$ the five lemma (non-abelian, if $k = 1$) applied to Diagram 2.3 shows that for every choice of basepoint in $\lim_n X_n$ the map $\pi_k \lim_n X_n \to \pi_k \lim_n Y_n$ is an isomorphism, and so it remains only to show that the set of path components of $\lim_n X_n$ maps isomorphically to the set of path components of $\lim_n Y_n$.

To see that the map of path components is surjective, choose some basepoint for $\lim_n X_n$ and consider Diagram 2.3. Let $a \in \pi_0 \lim_n Y_n$; then we can choose $b \in \pi_0 \lim_n X_n$ such that $P(b) = \psi^{-1}Q(a)$, and we will have that $Q\big(f_*(b)\big) = Q(a)$. Now choose a new basepoint for $\lim_n X_n$ that is in the path component b of $\lim_n X_n$, and consider this version of Diagram 2.3. The path component $f_*(b)$ of $\lim_n Y_n$ is now the path component of the basepoint, and so $a \in \lim^1_n \pi_{k+1} Y_n$. Thus, there is an element $a' \in \lim^1_n \pi_{k+1} X_n$ such that $\phi(a') = a$, and a' is an element of $\pi_0 \lim_n X_n$ that goes to a under f_*.

To see that the map of path components is injective, let a and b be path components of $\lim_n X_n$ that go to the same path component of $\lim_n Y_n$. Choose a basepoint in the path component a, and consider Diagram 2.3. We have $f_*(a) = f_*(b)$, and so $P(a) = P(b)$, and since a is the path component of the basepoint both a and b are elements of $\lim^1_n \pi_{k+1} X_n$. If $a \neq b$, then $\phi(a) \neq \phi(b)$, and so $\phi(a)$ and $\phi(b)$ are distinct elements of $\pi_0 \lim_n Y_n$. Since those are the same element of $\pi_0 \lim_n Y_n$, it must be that $a = b$ in $\lim_n X_n$. □

3. Homotopy groups

Notation 3.1. We let I denote the interval $[0, 1]$, and we let I^0 denote a single point. If $k \geq 1$ we will often denote a point of I^k as (p, t), where $p \in I^{k-1}$ and $t \in I$.

Definition 3.2. If X is a space and $k \geq 0$, we will represent elements of $\pi_k X$ by maps $\alpha\colon I^k \to X$ that take the boundary of I^k to the basepoint, and we will denote the element of $\pi_k X$ represented by α as $[\alpha]$.

We will almost always multiply and take inverses of elements of $\pi_k X$ using the last coordinate of I^k.

(1) If $k \geq 1$ and $\alpha\colon I^k \to X$ is a map, then the *inverse* $\alpha^{-1}\colon I^k \to X$ of α is the map defined by $\alpha^{-1}(p, t) = \alpha(p, 1 - t)$. If $k = 1$ this is the usual definition of the inverse path.

(2) If $k \geq 1$ and $\alpha, \beta\colon I^k \to X$ are maps such that $\alpha(p, 1) = \beta(p, 0)$ for all $p \in I^{k-1}$, then the *composition* of α and β is the map $\alpha * \beta\colon I^k \to X$ defined by

$$(\alpha * \beta)(p, t) = \begin{cases} \alpha(p, 2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \beta(p, 2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}.$$

If $k = 1$, this is the usual definition of the composition of paths.

Notation 3.3. If $k \geq 0$ and $\alpha, \beta\colon I^k \to X$ are maps such that $\alpha(p) = \beta(p)$ for all $p \in \partial(I^k)$, then $\alpha \simeq \beta$ will mean that α is homotopic to β relative to the boundary of I^k. If $k = 1$, this is the usual notion of path homotopy.

Definition 3.4. If $k \geq 0$ and $\alpha\colon I^k \to X$ is a map taking the boundary of I^k to the basepoint of X, then by a *nullhomotopy* of α we will mean a map $\beta\colon I^{k+1} \to X$

such that

$$\begin{aligned} \beta(p,0) &= * && \text{for all } p \in I^k \\ \beta(p,1) &= \alpha(p) && \text{for all } p \in I^k\text{, and} \\ \beta(p,t) &= * && \text{for all } p \in \partial I^k \ . \end{aligned}$$

There exists such a nullhomotopy for α if and only if $[\alpha] = 0$ in $\pi_k X$.

We will occasionally need to compose maps of cubes using the *next to last coordinate*.

Definition 3.5. If $k \geq 1$ and we have maps $\beta, \bar{\beta} \colon I^{k+1} \to X$ such that

$$\beta(t_1, t_2, t_{k-1}, 1, t_{k+1}) = \bar{\beta}(t_1, t_2, t_{k-1}, 0, t_{k+1})$$

then we define $\beta_n \hat{*} \bar{\beta}_n \colon I^{k+1} \to X_n$ by

$$(\beta_n \hat{*} \bar{\beta}_n)(t_1, t_2, \ldots, t_{k+1}) = \begin{cases} \beta_n(t_1, t_2, \ldots, t_{k-1}, 2t_k, t_{k+1}) & \text{if } 0 \leq t_k \leq \frac{1}{2} \\ \bar{\beta}_n(t_1, t_2, \ldots, t_{k-1}, 2t_k - 1, t_{k+1}) & \text{if } \frac{1}{2} \leq t_k \leq 1 \end{cases}$$

If both β and $\bar{\beta}$ take the boundary of I^{k+1} to the basepoint of X, and thus represent elements of $\pi_{k+1} X$, then we have $[\beta][\bar{\beta}] = [\beta * \bar{\beta}] = [\beta \hat{*} \bar{\beta}]$ in $\pi_{k+1} X$.

4. Proof of Theorem 2.1

We prove that the natural map $P \colon \pi_k \lim_n X_n \to \lim_n \pi_k X$ is surjective in Proposition 4.2, and in the remainder of this section we discuss the kernel of P and the case in which the tower consists of H-maps between H-spaces.

Notation 4.1. Given a tower of spaces as in Theorem 2.1, for each $i \geq 0$ we let $P_i \colon \lim_n X_n \to X_i$ denote the natural projection from the limit.

Proposition 4.2. *If $k \geq 0$, the natural map $P \colon \pi_k \lim_n X_n \to \lim_n \pi_k X_n$ of Theorem 2.1 is surjective.*

Proof. Let $([\alpha_n])_{n \geq 0}$ define an element of $\lim_n \pi_k X_n$, so that the $\alpha_n \colon I^k \to X_n$ for all $n \geq 0$ are maps taking the boundary of I^k to the basepoint of X_n and such that $p_{n+1} \circ \alpha_{n+1} \simeq \alpha_n$. We will inductively define maps $\bar{\alpha}_n \colon I^k \to X_n$ such that $\bar{\alpha}_n \simeq \alpha_n$ and $p_{n+1} \circ \bar{\alpha}_{n+1} = \bar{\alpha}_n$; the collection $(\bar{\alpha}_n)_{n \geq 0}$ will then define a map $\bar{\alpha} \colon I^k \to \lim_n X_n$ such that $P([\bar{\alpha}]) = ([\alpha_n])_{n \geq 0}$.

We begin the induction by letting $\bar{\alpha}_0 = \alpha_0$. If $n \geq 0$ and we've defined $\bar{\alpha}_n$, then $p_{n+1} \circ \alpha_{n+1} \simeq \bar{\alpha}_n$, and we can lift a homotopy to obtain a homotopy in X_{n+1} that begins at α_{n+1} and ends at a map we'll call $\bar{\alpha}_{n+1}$ such that $p_{n+1} \circ \bar{\alpha}_{n+1} = \bar{\alpha}_n$. □

We turn now to the kernel of P. Let $k \geq 0$, and let $\alpha \colon I^k \to \lim_n X_n$ represent an element of $\pi_k \lim_n X_n$ in the kernel of P. For each $n \geq 0$ we let $\alpha_n = P_n \circ \alpha \colon I^k \to X_n$ and we can choose a nullhomotopy $\beta_n \colon I^{k+1} \to X_n$ of α_n (see Definition 3.4). The map $\beta_n * (p_{n+1} \circ \beta_{n+1}^{-1}) \colon I^{k+1} \to X$ takes the boundary of I^{k+1} to the basepoint of X_n and thus defines an element $[\beta_n * (p_{n+1} \circ \beta_{n+1}^{-1})]$ of $\pi_{k+1} X_n$, and these elements for all $n \geq 0$ define an element $\phi(\alpha)$ of $\lim^1_n \pi_{k+1} X_n$.

Proposition 4.3. *The element $\phi(\alpha) \in \lim^1_n \pi_{k+1} X_n$ is independent of the choice of nullhomotopies $(\beta_n)_{n \geq 0}$.*

Proof. If for each $n \geq 0$ we choose a different nullhomotopy $\bar{\beta}_n$, then the map $\bar{\beta}_n * \beta_n^{-1} \colon I^{k+1} \to X$ takes the boundary of I^{k+1} to the basepoint of X and thus defines an element $g_n = [\bar{\beta}_n * \beta_n^{-1}]$ of $\pi_{k+1} X_n$, and the element

$$g_n[\beta_n * (p_{n+1} \circ \beta_{n+1}^{-1})]\big((p_{n+1})_* g_{n+1}^{-1}\big)$$

of $\pi_{k+1} X_n$ is represented by the map

$$\begin{aligned}
&(\bar{\beta}_n * \beta_n^{-1}) * \big(\beta_n * (p_{n+1} \circ \beta_{n+1}^{-1})\big) * \big(p_{n+1} \circ (\bar{\beta}_{n+1} * \beta_{n+1}^{-1})\big)^{-1} \\
\simeq\ &(\bar{\beta}_n * \beta_n^{-1}) * \big(\beta_n * (p_{n+1} \circ \beta_{n+1}^{-1})\big) * \big((p_{n+1} \circ \bar{\beta}_{n+1}) * (p_{n+1} \circ \beta_{n+1}^{-1})\big)^{-1} \\
\simeq\ &\bar{\beta}_n * \beta_n^{-1} * \beta_n * (p_{n+1} \circ \beta_{n+1}^{-1}) * (p_{n+1} \circ \beta_{n+1}) * (p_{n+1} \circ \bar{\beta}_{n+1}^{-1}) \\
\simeq\ &\bar{\beta}_n * (p_{n+1} \circ \bar{\beta}_{n+1}^{-1})
\end{aligned}$$

and so the element of $\lim^1_n \pi_{k+1} X_n$ represented by $([\beta_n * (p_{n+1} \circ \beta_{n+1}^{-1})])_{n \geq 0}$ equals the element represented by $([\bar{\beta}_n * (p_{n+1} \circ \bar{\beta}_{n+1}^{-1})])_{n \geq 0}$. □

Proposition 4.4. *If $\alpha, \bar{\alpha} \colon I^k \to \lim_n X_n$ represent elements of the kernel of P and $[\alpha] = [\bar{\alpha}]$, then $\phi(\alpha) = \phi(\bar{\alpha})$, and so ϕ defines a function $\widetilde{\phi} \colon \operatorname{Ker} P \to \lim^1_n \pi_{k+1} X_n$.*

Proof. Since α and $\bar{\alpha}$ represent the same element of $\pi_k \lim_n X_n$, there is a homotopy $\gamma \colon I^{k+1} \to \lim_n X_n$ such that

$$\gamma(p,t) = \begin{cases} \alpha(p) & \text{if } t = 0 \\ \bar{\alpha}(p) & \text{if } t = 1, \text{ and} \\ * & \text{if } p \in \partial I^k \ . \end{cases}$$

For every $n \geq 0$ we let $\alpha_n = P_n \circ \alpha$, $\bar{\alpha}_n = P_n \circ \bar{\alpha}$, and $\gamma_n = P_n \circ \gamma$, and we choose a nullhomotopy β_n of α_n. For every $n \geq 0$ we can then let $\bar{\beta}_n = \beta_n * \gamma_n$, and $\bar{\beta}_n$ is then a nullhomotopy of $\bar{\alpha}_n$, and we have

$$\begin{aligned}
\bar{\beta}_n * (p_{n+1} \circ \bar{\beta}_{n+1}^{-1}) &= (\beta_n * \gamma_n) * \big(p_{n+1} \circ (\gamma_{n+1}^{-1} * \beta_{n+1}^{-1})\big) \\
&= (\beta_n * \gamma_n) * \big((p_{n+1} \circ \gamma_{n+1}^{-1}) * (p_{n+1} \circ \beta_{n+1}^{-1})\big) \\
&= (\beta_n * \gamma_n) * \big(\gamma_n^{-1} * (p_{n+1} \circ \beta_{n+1}^{-1})\big) \\
&\simeq \beta_n * (p_{n+1} \circ \beta_{n+1}^{-1})
\end{aligned}$$

and so $\phi(\alpha) = \phi(\bar{\alpha})$. □

Proposition 4.5. *If $k \geq 1$, then $\widetilde{\phi} \colon \operatorname{Ker} P \to \lim^1_n \pi_{k+1} X_n$ is a homomorphism.*

Proof. Let $\alpha, \bar{\alpha} \colon I^k \to \lim_n X_n$ represent elements of $\pi_k \lim_n X_n$ in the kernel of P. For every $n \geq 0$, let $\alpha_n = P_n \circ \alpha$, $\bar{\alpha}_n = P_n \circ \bar{\alpha}$, and choose nullhomotopies β_n of α_n and $\bar{\beta}_n$ of $\bar{\alpha}_n$. If $\beta_n \mathbin{\hat{*}} \bar{\beta}_n \colon I^{k+1} \to X_n$ is defined as in Definition 3.5, then $\beta_n \mathbin{\hat{*}} \bar{\beta}_n$ is a nullhomotopy of $\alpha_n * \bar{\alpha}_n$. Thus, $\widetilde{\phi}([\alpha][\bar{\alpha}])$ in level n is represented by

$$(\beta_n \mathbin{\hat{*}} \bar{\beta}_n) * \big(p_{n+1} \circ (\beta_{n+1} \mathbin{\hat{*}} \bar{\beta}_{n+1})^{-1}\big) = (\beta_n \mathbin{\hat{*}} \bar{\beta}_n) * \big((p_{n+1} \circ \beta_{n+1}) \mathbin{\hat{*}} (p_{n+1} \circ \bar{\beta}_{n+1})\big)^{-1}$$

which on the point $(t_1, t_2, \ldots, t_{k+1})$ takes the value

$$\begin{aligned}
\beta(t_1, t_2, \ldots, t_{k-1}, 2t_k, 2t_{k+1}) &\quad \text{if } 0 \leq t_k \leq \tfrac{1}{2} \text{ and } 0 \leq t_{k+1} \leq \tfrac{1}{2} \\
\bar{\beta}(t_1, t_2, \ldots, t_{k-1}, 2t_k - 1, 2t_{k+1}) &\quad \text{if } \tfrac{1}{2} \leq t_k \leq 1 \text{ and } 0 \leq t_{k+1} \leq \tfrac{1}{2} \\
(p_{n+1} \circ \beta)(t_1, t_2, \ldots, t_{k-1}, 2t_k, 1 - (2t_{k+1})) &\quad \text{if } 0 \leq t_k \leq \tfrac{1}{2} \text{ and } \tfrac{1}{2} \leq t_{k+1} \leq 1 \\
(p_{n+1} \circ \bar{\beta})(t_1, t_2, \ldots, t_{k-1}, 2t_k - 1, 1 - (2t_{k+1})) &\quad \text{if } \tfrac{1}{2} \leq t_k \leq 1 \text{ and } \tfrac{1}{2} \leq t_{k+1} \leq 1 \ .
\end{aligned}$$

This is also the definition of

$$\big(\beta_n * (p_{n+1} \circ \beta_{n+1}^{-1})\big) \mathbin{\hat{*}} \big(\bar{\beta}_n * (p_{n+1} \circ \bar{\beta}_{n+1}^{-1})\big)$$

which is the level n representative of $\phi(\alpha) \mathbin{\hat{*}} \phi(\bar{\alpha})$. Since multiplication of elements of a homotopy group can be defined using any of the coordinates of the cube, we have $\widetilde{\phi}\big([\alpha][\bar{\alpha}]\big) = \widetilde{\phi}\big([\alpha]\big)\widetilde{\phi}\big([\bar{\alpha}]\big)$. □

Proposition 4.6. *The function* $\widetilde{\phi}\colon \operatorname{Ker} P \to \lim^1_n \pi_{k+1} X_n$ *is surjective for all* $k \geq 0$.

Proof. Fix a value of $k \geq 0$; every element of $\lim^1_n \pi_{k+1} X_n$ can be represented by $\big([\gamma_n]\big)_{n\geq 0} \in \prod_{n\geq 0} \pi_{k+1} X_n$ where $\gamma_n\colon I^{k+1} \to X_n$ for every $n \geq 0$ is a map taking the boundary of I^{k+1} to the basepoint. We will inductively define $\alpha_n\colon I^k \to X_n$ taking the boundary of I^k to the basepoint and a nullhomotopy $\beta_n\colon I^{k+1} \to X_n$ of α_n. We will arrange it so that $p_{n+1} \circ \alpha_{n+1} = \alpha_n$ for all $n \geq 0$, so that the $\big(\alpha_n\big)_{n\geq 0}$ will define a map $\alpha\colon I^k \to \lim_n X_n$ taking the boundary of I^k to the basepoint, and such that $\widetilde{\phi}\big([\alpha]\big)$ is the element of $\lim^1_n \pi_{k+1} X_n$ represented by $\big([\gamma_n]\big)_{n\geq 0}$. We begin by letting $\alpha_0\colon I^k \to X_0$ and $\beta_0\colon I^{k+1} \to X_0$ both be constant maps to the basepoint.

For the inductive step, let $n \geq 0$ and assume that we've defined α_n and β_n. The map $\gamma_n^{-1} * \beta_n\colon I^{k+1} \to X_n$ is a nullhomotopy of α_n, and its restriction to $(I^k \times \{0\}) \cup (\partial I^k \times I)$ is the constant map to the basepoint. Thus, we can lift that restriction to the constant map to the basepoint of X_{n+1}. Since there is a homeomorphism of I^{k+1} to itself that takes $(I^k \times \{0\}) \cup (\partial I^k \times I)$ onto $I^k \times \{0\}$, we can extend that lift to a map $\beta_{n+1}\colon I^{k+1} \to X_{n+1}$. We define $\alpha_{n+1}(t_1, t_2, \ldots, t_k) = \beta_{n+1}(t_1, t_2, \ldots, t_k, 1)$, and β_{n+1} is a nullhomotopy of α_{n+1}.

We now have $p_{n+1} \circ \alpha_{n+1} = \alpha_n$ for all $n \geq 0$, and so the maps $(\alpha_n)_{n\geq 0}$ define $\alpha\colon I^k \to \lim_n X_n$, which represents an element of $\pi_k \lim_n X_n$ in the kernel of P. For every $n \geq 0$ we have $p_{n+1} \circ \beta_{n+1} = \gamma_n^{-1} * \beta_n$, and so

$$\begin{aligned} \beta_n * (p_{n+1} \circ \beta_{n+1}^{-1}) &= \beta_n * (\beta_n^{-1} * \gamma_n) \\ &\simeq \gamma_n \quad , \end{aligned}$$

and so $\widetilde{\phi}\big([\alpha]\big)$ is represented by $\big([\gamma_n]\big)_{n\geq 0}$. Thus, $\widetilde{\phi}$ is surjective. □

Proposition 4.7. *The function* $\widetilde{\phi}\colon \operatorname{Ker} P \to \lim^1_n \pi_{k+1} X_n$ *is injective for all* $k \geq 0$.

Proof. Let $\alpha, \bar{\alpha}\colon I^k \to \lim_n X_n$ represent elements of the kernel of P such that $\phi(\alpha) = \phi(\bar{\alpha})$ in $\lim^1_n \pi_{k+1} X_n$. For each $n \geq 0$ let $\alpha_n = P_n \circ \alpha$, let $\bar{\alpha}_n = P_n \circ \bar{\alpha}$, and choose nullhomotopies β_n of α_n and $\bar{\beta}_n$ of $\bar{\alpha}_n$. Since $\phi(\alpha) = \phi(\bar{\alpha})$ in $\lim^1_n \pi_{k+1} X_n$, there exists an element $([g_n])_{n\geq 0}$ of $\prod_{n\geq 0} \pi_{k+1} X_n$ (where each map $g_n\colon I^{k+1} \to X_n$ takes the boundary of I^{k+1} to the basepoint) so that

$$g_n * \big(\beta_n * (p_{n+1} \circ \beta_{n+1}^{-1})\big) * (p_{n+1} \circ g_{n+1}^{-1}) \simeq \bar{\beta}_n * (p_{n+1} \circ \bar{\beta}_{n+1}^{-1})$$

and so

$$g_n * \beta_n * \big(p_{n+1} \circ (\beta_{n+1}^{-1} * g_{n+1}^{-1})\big) \simeq \bar{\beta}_n * (p_{n+1} \circ \bar{\beta}_{n+1}^{-1})$$

for all $n \geq 0$. For every $n \geq 0$ let $\gamma_n = g_n * \beta_n$; we then have

$$\gamma_n * (p_{n+1} \circ \gamma_{n+1}^{-1}) \simeq \bar{\beta}_n * (p_{n+1} \circ \bar{\beta}_{n+1}^{-1})$$

and so

$$\bar{\beta}_n^{-1} * \gamma_n \simeq (p_{n+1} \circ \bar{\beta}_{n+1}^{-1}) * (p_{n+1} \circ \gamma_{n+1}) = p_{n+1} \circ (\bar{\beta}_{n+1}^{-1} * \gamma_{n+1}) \quad .$$

Thus, for every $n \geq 0$, we have

- $\bar{\beta}_n^{-1} * \gamma_n$ is a homotopy in X_n from $\bar{\alpha}_n$ to α_n, and
- $p_{n+1} \circ (\bar{\beta}_{n+1}^{-1} * \gamma_{n+1}) \simeq \bar{\beta}_n^{-1} * \gamma_n$.

We will now inductively replace each homotopy $\bar{\beta}_n^{-1} * \gamma_n$ with a homotopic homotopy δ_n so that $p_{n+1} \circ \delta_{n+1} = \delta_n$; the $(\delta_n)_{n \geq 0}$ will then define a homotopy in $\lim_n X_n$ from $\bar{\alpha}$ to α. We begin by letting $\delta_0 = \bar{\beta}_0^{-1} * \gamma_0$.

If $n \geq 0$ and we've defined δ_n, then $p_{n+1} \circ (\bar{\beta}_{n+1}^{-1} * \gamma_{n+1})$ is homotopic to δ_n, i.e., there is a homotopy $H\colon I^{k+2} \to X_n$ such that

$$\begin{aligned} H(p,t) &= \big(p_{n+1} \circ (\bar{\beta}_{n+1}^{-1} * \gamma_{n+1})\big)(p) && \text{for } (p,t) \in (I^{k+1} \times \{0\}) \cup (\partial I^{k+1} \times I) \\ H(p,1) &= \delta_n(p) && \text{for } p \in I^{k+1} \end{aligned}$$

If we define $H'\colon (I^{k+1} \times \{0\}) \cup (\partial I^{k+1} \times I) \to X_{n+1}$ by $H'(p,t) = (\bar{\beta}_{n+1}^{-1} * \gamma_{n+1})(p)$, then H' is a lift of the restriction of H to $(I^{k+1} \times \{0\}) \cup (\partial I^{k+1} \times I)$. There is a homeomorphism of I^{k+2} to itself that takes $(I^{k+1} \times \{0\}) \cup (\partial I^{k+1} \times I)$ onto $I^{k+1} \times \{0\}$, and so we can extend H' to a lift $H'\colon I^{k+2} \to X_{n+1}$ of H. We define $\delta_{n+1}\colon I^{k+1} \to X_{n+1}$ by letting $\delta_{n+1}(p) = H'(p,1)$, and δ_{n+1} is a homotopy from $\bar{\alpha}_{n+1}$ to α_{n+1}, homotopic to $\bar{\beta}_{n+1}^{-1} * \gamma_{n+1}$, such that $p_{n+1} \circ \delta_{n+1} = \delta_n$. This completes the induction, and so the $(\delta_n)_{n \geq 0}$ define a homotopy δ in $\lim_n X_n$ from $\bar{\alpha}$ to α, and so $\widetilde{\phi}$ is injective. □

Proposition 4.8. *For every $k \geq 0$ the natural map $P\colon \pi_k \lim_n X_n \to \lim_n \pi_k X_n$ of Theorem 2.1 is surjective and its kernel is naturally isomorphic to $\lim^1_n \pi_{k+1} X_n$.*

Proof. Proposition 4.2 shows that P is surjective, Proposition 4.3 and Proposition 4.4 define a natural map from the kernel of P to $\lim^1_n \pi_{k+1} X_n$, Proposition 4.5 shows that if $k \geq 1$ then that natural map is a homomorphism, and Proposition 4.6 and Proposition 4.7 show that it is an isomorphism. □

Proposition 4.9. *If $k = 0$, all of the spaces X_n are H-spaces, and all of the maps p_n are H-maps, then $\widetilde{\phi}\colon \operatorname{Ker} P \to \lim^1_n \pi_1 X_n$ is a homomorphism.*

Proof. If W is a space, X is an H-space, and $f, g\colon W \to X$ are maps, then we denote the product of f and g defined using the H-space structure of X as $f \,\tilde{*}\, g$.

Let $\alpha, \bar{\alpha}\colon I^0 \to \lim_n X_n$ be maps such that the elements $[\alpha]$ and $[\bar{\alpha}]$ of $\pi_0 \lim_n X_n$ are in the kernel of P. For every $n \geq 0$ we let $\alpha_n = P_n \circ \alpha$, let $\bar{\alpha}_n = P_n \circ \bar{\alpha}$, and choose nullhomotopies $\beta_n\colon I \to X_n$ of α_n and $\bar{\beta}_n\colon I \to X_n$ of $\bar{\alpha}$.

The multiplication in both π_0 and π_1 of an H-space can be defined using the H-space product, and so the product of $[\alpha]$ and $[\bar{\alpha}]$ in $\pi_0 \lim_n X_n$ is $[\alpha \,\tilde{*}\, \bar{\alpha}]$. If for all $n \geq 0$ we let $(\alpha \,\tilde{*}\, \bar{\alpha})_n = P_n \circ (\alpha \,\tilde{*}\, \bar{\alpha})$, then $\beta_n \,\tilde{*}\, \bar{\beta}_n$ is a nullhomotopy of $(\alpha \,\tilde{*}\, \bar{\alpha})_n = \alpha_n \,\tilde{*}\, \bar{\alpha}_n$. Thus, $\phi(\alpha \,\tilde{*}\, \bar{\alpha})$ is represented by

$$\big((\beta_n \,\tilde{*}\, \bar{\beta}_n) * \big(p_{n+1} \circ (\beta_{n+1} \,\tilde{*}\, \bar{\beta}_{n+1})^{-1}\big)\big)_{n \geq 0}$$

while $\phi(\alpha) \,\tilde{*}\, \phi(\bar{\alpha})$ is represented by

$$\big(\big(\beta_n * (p_{n+1} \circ \beta_{n+1}^{-1})\big) \,\tilde{*}\, \big(\bar{\beta}_n * (p_{n+1} \circ \bar{\beta}_{n+1}^{-1})\big)\big)_{n \geq 0} \;.$$

Since those two maps are equal, $\phi(\alpha \,\tilde{*}\, \bar{\alpha}) = \phi(\alpha) \,\tilde{*}\, \phi(\bar{\alpha})$, and so $\widetilde{\phi}\big([\alpha][\bar{\alpha}]\big) = \widetilde{\phi}\big([\alpha]\big)\widetilde{\phi}\big([\bar{\alpha}]\big)$. □

Proof of Theorem 2.1. This follows from Proposition 4.2, Proposition 4.3, Proposition 4.4, Proposition 4.5, Proposition 4.6, Proposition 4.7, and Proposition 4.9. □

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