

THE DIAGONAL OF A MULTICOSIMPLICIAL OBJECT

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CONTENTS

1. Introduction	1
2. Definitions and notation	2
2.1. The diagonal	2
2.2. Matching objects	2
3. The diagonal is a right Quillen functor	3
3.1. The multicosimplicial product of standard simplices	4
3.2. Quillen functors, but not Quillen equivalences	6
4. Proof of Proposition 3.2	7
5. Proof of Proposition 4.3	9
6. Proof of Proposition 4.4	10
7. Frames, homotopy cotensors, and homotopy function complexes	13
7.1. Frames	13
7.2. Homotopy cotensors	14
7.3. Homotopy function complexes	15
8. Total objects	16
8.1. The total object of a cosimplicial object	17
8.2. The total object of a multicosimplicial object	17
8.3. The total object of the diagonal	18
9. Homotopy limits and total objects	19
9.1. Cosimplicial objects	19
9.2. Multicosimplicial objects	19
9.3. The homotopy limit and total object of the diagonal	20
References	22

1. INTRODUCTION

After constructing a multicosimplicial object, it is common to pass to its diagonal cosimplicial object. In order for the total object of the diagonal to have homotopy meaning, though, you need to know that that diagonal is fibrant.

We prove in Theorem 3.5 that the functor that takes a multicosimplicial object in a model category to its diagonal cosimplicial object is a right Quillen functor. This implies that the diagonal of a fibrant multicosimplicial object is a fibrant cosimplicial object, which has applications to the calculus of functors (see [1]). We

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also show in Theorem 3.10 and Corollary 3.11 that, although the diagonal functor is a Quillen functor, it is not a Quillen equivalence for multicosimplicial spaces.

In Section 8 we discuss total objects, and show that the total object of a multicosimplicial object is isomorphic to the total object of the diagonal.

In Section 9 we discuss homotopy limits, and show that the diagonal embedding of the cosimplicial indexing category into the multicosimplicial indexing category is homotopy left cofinal, which implies that the homotopy limits are weakly equivalent if the multicosimplicial object is at least objectwise fibrant.

2. DEFINITIONS AND NOTATION

Notation 2.1. If n is a nonnegative integer, we let $[n]$ denote the ordered set $(0, 1, 2, \dots, n)$. We will use Δ to denote the *cosimplicial indexing category*, which is the category with objects the $[n]$ for $n \geq 0$ and with morphisms $\Delta([n], [k])$ the weakly monotone functions $[n] \rightarrow [k]$.

Definition 2.2. If \mathcal{M} is a category, a *cosimplicial object in \mathcal{M}* is a functor $\Delta \rightarrow \mathcal{M}$, and the *category of cosimplicial objects in \mathcal{M}* is the functor category \mathcal{M}^{Δ} . If \mathbf{X} is a cosimplicial object in \mathcal{M} , then we will generally denote the value of \mathbf{X} on $[k]$ as \mathbf{X}^k .

Notation 2.3. If n is a positive integer, then we will let Δ^n denote the product category $\underbrace{\Delta \times \Delta \times \dots \times \Delta}_{n \text{ times}}$.

Definition 2.4. If \mathcal{M} is a category and n is a positive integer, then an *n -cosimplicial object in \mathcal{M}* is a functor $\Delta^n \rightarrow \mathcal{M}$. If \mathbf{X} is an n -cosimplicial object in \mathcal{M} , then we will generally denote the value of \mathbf{X} on $([k_1], [k_2], \dots, [k_n])$ by $\mathbf{X}^{(k_1, k_2, \dots, k_n)}$.

2.1. The diagonal. An n -cosimplicial object in a category \mathcal{M} is a functor $\mathbf{X}: \Delta^n \rightarrow \mathcal{M}$, and we can restrict that functor to the “diagonal subcategory” of Δ^n to obtain a cosimplicial object $\text{diag } \mathbf{X}$ in \mathcal{M} .

Definition 2.5. Let n be a positive integer.

- (1) The *diagonal embedding* of the category Δ into Δ^n is the functor $D: \Delta \rightarrow \Delta^n$ that takes the object $[k]$ of Δ to the object $\underbrace{([k], [k], \dots, [k])}_{n \text{ times}}$ of Δ^n and the morphism $\phi: [p] \rightarrow [q]$ of Δ to the morphism (ϕ^n) of Δ^n .
- (2) If \mathcal{M} is a category and \mathbf{X} is an n -cosimplicial object in \mathcal{M} , then the *diagonal* $\text{diag } \mathbf{X}$ of \mathbf{X} is the cosimplicial object in \mathcal{M} that is the composition $\Delta \xrightarrow{D} \Delta^n \xrightarrow{\mathbf{X}} \mathcal{M}$, so that $(\text{diag } \mathbf{X})^k = \mathbf{X}^{(k, k, \dots, k)}$.

2.2. Matching objects. If \mathcal{C} is a Reedy category (see [6, Def. 15.1.2]), \mathcal{M} is a model category, \mathbf{X} is a \mathcal{C} -diagram in \mathcal{M} , and α is an object of \mathcal{C} , then we will use the notation of [6, Def. 15.2.5] and denote the *matching object* of \mathbf{X} at α by $M_{\alpha} \mathbf{X}$, or by $M_{\alpha}^{\mathcal{C}} \mathbf{X}$ if the indexing category isn’t obvious.

Note that in the case of cosimplicial objects, our notation for matching objects (see [6, Def. 15.2.5]) differs from that of [4, Ch. X, §4], in that we index a matching object by the degree at which it is the matching object, whereas [4, Ch. X, §4] indexes it by one less than that. Thus, our notation for the matching map of a cosimplicial object \mathbf{X} at $[k]$ is $\mathbf{X}^k \rightarrow M_k \mathbf{X}$, while the notation of [4, Ch. X, §4] is $\mathbf{X}^k \rightarrow M^{k-1} \mathbf{X}$.

Definition 2.6. Let \mathcal{C} be a Reedy category, let \mathcal{M} be a model category, let \mathbf{X} be a \mathcal{C} -diagram in \mathcal{M} , and let α be an object of \mathcal{C} .

- (1) The *matching category* $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ of \mathcal{C} at α is the full subcategory of $(\alpha \downarrow \overleftarrow{\mathcal{C}})$ containing all of the objects except the identity map of α .
- (2) The *matching object* of \mathbf{X} at α is $M_\alpha \mathbf{X} = \lim_{\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})} \mathbf{X}$ and the *matching map* of \mathbf{X} at α is the natural map $\mathbf{X}_\alpha \rightarrow M_\alpha \mathbf{X}$. We will use $M_\alpha^{\mathcal{C}} \mathbf{X}$ to denote the matching object if the indexing category isn't obvious.

Definition 2.7 ([6, Def. 15.3.3]). Let \mathcal{C} be a Reedy category and let \mathcal{M} be a model category. A map $f: \mathbf{X} \rightarrow \mathbf{Y}$ of \mathcal{C} -diagrams in \mathcal{M} is a *Reedy fibration* if for every object α of \mathcal{C} the relative matching map $\mathbf{X}_\alpha \rightarrow \mathbf{Y}_\alpha \times_{M_\alpha \mathbf{Y}} M_\alpha \mathbf{X}$ is a fibration in \mathcal{M} .

3. THE DIAGONAL IS A RIGHT QUILLEN FUNCTOR

We prove in Theorem 3.5 that the functor that takes a multicosimplicial object to its diagonal cosimplicial object is a right Quillen functor.

Definition 3.1. Let \mathcal{M} be a model category, let n be a positive integer, and let $\mathbf{X} \rightarrow \mathbf{Y}$ be a Reedy fibration of n -cosimplicial objects in \mathcal{M} . For every nonnegative integer k , the matching objects of \mathbf{X} in \mathcal{M}^{Δ^n} at $([k], [k], \dots, [k])$ and of $\text{diag } \mathbf{X}$ in \mathcal{M}^Δ at $[k]$ are

$$M_{([k],[k],\dots,[k])}^{\Delta^n} \mathbf{X} = \lim_{\partial(([k],[k],\dots,[k]] \downarrow \overleftarrow{\Delta^n})} \mathbf{X} \quad \text{and} \quad M_{[k]}^\Delta \text{diag } \mathbf{X} = \lim_{\partial([k] \downarrow \overleftarrow{\Delta})} \text{diag } \mathbf{X}$$

(with similar formulas for \mathbf{Y}), and we define $P_k^{\Delta^n}$ and P_k^Δ by letting the following diagrams be pullbacks:

$$\begin{array}{ccc} P_k^{\Delta^n} \cdots \cdots \cdots \rightarrow \mathbf{Y}^{(k,k,\dots,k)} & & P_k^\Delta \cdots \cdots \cdots \rightarrow \mathbf{Y}^{(k,k,\dots,k)} \\ \downarrow \text{dotted} & \downarrow & \downarrow \text{dotted} \\ M_{([k],[k],\dots,[k])}^{\Delta^n} \mathbf{X} \longrightarrow M_{([k],[k],\dots,[k])}^{\Delta^n} \mathbf{Y} & & M_{[k]}^\Delta \text{diag } \mathbf{X} \longrightarrow M_{[k]}^\Delta \text{diag } \mathbf{Y} \end{array}$$

Thus,

- if the map $\mathbf{X} \rightarrow \mathbf{Y}$ is a Reedy fibration of n -cosimplicial objects then the natural map $\mathbf{X}^{(k,k,\dots,k)} \rightarrow P_k^{\Delta^n}$ is a fibration for all $k \geq 0$, and
- the map $\text{diag } \mathbf{X} \rightarrow \text{diag } \mathbf{Y}$ is a Reedy fibration of cosimplicial objects if and only if the natural map $\mathbf{X}^{(k,k,\dots,k)} \rightarrow P_k^\Delta$ is a fibration for all $k \geq 0$

(see Definition 2.7).

Proposition 3.2. Let \mathcal{M} be a model category, let n be a positive integer, and let $\mathbf{X} \rightarrow \mathbf{Y}$ be a Reedy fibration of n -cosimplicial objects in \mathcal{M} . Since we are viewing Δ as a subcategory of Δ^n , for every nonnegative integer k there are natural maps

$$\begin{array}{ccc} \lim_{\partial(([k],[k],\dots,[k]] \downarrow \overleftarrow{\Delta^n})} \mathbf{X} \longrightarrow \lim_{\partial([k] \downarrow \overleftarrow{\Delta})} \text{diag } \mathbf{X} & \text{and} & \\ \lim_{\partial(([k],[k],\dots,[k]] \downarrow \overleftarrow{\Delta^n})} \mathbf{Y} \longrightarrow \lim_{\partial([k] \downarrow \overleftarrow{\Delta})} \text{diag } \mathbf{Y} & & \end{array}$$

and those induce a natural map

$$P_k^{\Delta^n} \longrightarrow P_k^\Delta$$

(see Definition 3.1). That natural map is a fibration.

The proof of Proposition 3.2 is in Section 4.

Theorem 3.3. *If \mathcal{M} is a model category, n is a positive integer, and $\mathbf{X} \rightarrow \mathbf{Y}$ is a Reedy fibration of n -cosimplicial objects in \mathcal{M} , then the induced map of diagonals*

$$\text{diag } \mathbf{X} \longrightarrow \text{diag } \mathbf{Y}$$

is a Reedy fibration of cosimplicial objects.

Proof. We must show that for every nonnegative integer k the map

$$(\text{diag } \mathbf{X})^k = \mathbf{X}^{(k,k,\dots,k)} \longrightarrow P_k^\Delta$$

(see Definition 3.1) is a fibration. That map is the composition

$$\mathbf{X}^{(k,k,\dots,k)} \longrightarrow P_k^{\Delta^n} \longrightarrow P_k^\Delta .$$

The first of those is a fibration because the map $\mathbf{X} \rightarrow \mathbf{Y}$ is a Reedy fibration of n -cosimplicial objects, and Proposition 3.2 is the statement that the second is also a fibration. \square

Special cases of the following corollary are already known (see [9, Lem. 7.1], in view of [5, Prop. 5.8]).

Corollary 3.4. *If \mathcal{M} is a model category, n is a positive integer, and \mathbf{X} is a Reedy fibrant n -cosimplicial object in \mathcal{M} , then the diagonal cosimplicial object $\text{diag } \mathbf{X}$ is Reedy fibrant.*

Proof. This follows from Theorem 3.3 by letting \mathbf{Y} be the constant n -cosimplicial object at the terminal object of \mathcal{M} . \square

Theorem 3.5. *If \mathcal{M} is a model category and n is a positive integer, then the diagonal functor $\text{diag}: \mathcal{M}^{\Delta^n} \rightarrow \mathcal{M}^\Delta$, which takes an n -cosimplicial object in \mathcal{M} to its diagonal cosimplicial object, is a right Quillen functor (see [6, Def. 8.5.2]).*

Proof. Since a model category is cocomplete, the left Kan extension of a cosimplicial object along the diagonal inclusion $\Delta \rightarrow \Delta^n$ always exists (see [2, Thm. 3.7.2] or [8, Thm. 17.1.6] or, for the dual statement, [7, Cor. X.3.2]), and so the diagonal functor has a left adjoint. Thus, we need only show that the diagonal functor preserves both fibrations and trivial fibrations (see [6, Prop. 8.5.3]).

Theorem 3.3 implies that the diagonal preserves fibrations. Since the weak equivalences of cosimplicial objects and of n -cosimplicial objects are defined degree-wise, the restriction of an n -cosimplicial object to its diagonal preserves *all* weak equivalences, and so it also preserves trivial fibrations. \square

3.1. The multicosimplicial product of standard simplices. The main result of this section is Theorem 3.9, which we will use in the following section to show that the right Quillen functor of Theorem 3.5 is not a Quillen equivalence. It will also be used in Theorem 8.10 to show that the total object of a multicosimplicial object is isomorphic to the total object of its diagonal cosimplicial object.

Definition 3.6. If $F: \mathcal{A} \rightarrow \mathcal{B}$ is a functor between small categories and X is an object of \mathcal{B} , then $\mathcal{B}(F-, X)$ is the \mathcal{A}^{op} -diagram of sets that on an object W of \mathcal{A} is the set $\mathcal{B}(FW, X)$. This is natural in X , and thus defines a \mathcal{B} -diagram of \mathcal{A}^{op} -diagrams of sets.

If n is a positive integer and $F: \mathcal{A} \rightarrow \mathcal{B}$ is the diagonal embedding $D: \mathbf{\Delta} \rightarrow \mathbf{\Delta}^n$ (see Definition 2.5), then an \mathcal{A}^{op} -diagram of sets is a $\mathbf{\Delta}^{\text{op}}$ -diagram of sets, i.e., a simplicial set, and so this defines a $\mathbf{\Delta}^n$ -diagram of simplicial sets, i.e., an n -cosimplicial simplicial set, which we will denote by $\Delta^{(n)}$. If $([p_1], [p_2], \dots, [p_n])$ is an object of $\mathbf{\Delta}^n$, then the simplicial set $\mathbf{\Delta}^n(D-, ([p_1], [p_2], \dots, [p_n]))$ has as its k -simplices the n -tuples of maps

$$(\alpha_1, \alpha_2, \dots, \alpha_n): ([k], [k], \dots, [k]) \longrightarrow ([p_1], [p_2], \dots, [p_n])$$

where each $\alpha_i: [k] \rightarrow [p_i]$ is a weakly monotone map. Thus, each k -simplex is the product for $1 \leq i \leq n$ of a k -simplex of $\Delta[p_i]$, i.e., a k -simplex of $\Delta[p_1] \times \Delta[p_2] \times \dots \times \Delta[p_n]$, and so $\Delta^{(n)}([p_1], [p_2], \dots, [p_n])$ is the product of standard simplices $\Delta[p_1] \times \Delta[p_2] \times \dots \times \Delta[p_n]$. That is, $\Delta^{(n)}: \mathbf{\Delta}^n \rightarrow \text{SS}$ is an n -cosimplicial simplicial set whose value on the object $([p_1], [p_2], \dots, [p_n])$ is $\Delta[p_1] \times \Delta[p_2] \times \dots \times \Delta[p_n]$. We will call it the *n -cosimplicial product of standard simplices*.

If $n = 1$, so that $F: \mathcal{A} \rightarrow \mathcal{B}$ is the identity functor of $\mathbf{\Delta}$, then this defines a cosimplicial object in the category of simplicial sets, i.e., a cosimplicial simplicial set, which we will denote by Δ . If k is a nonnegative integer, then for a nonnegative integer i the i -simplices of $\mathbf{\Delta}(-, [k])$ are the maps $\mathbf{\Delta}([i], [k])$, i.e., the weakly monotone functions $[i] \rightarrow [k]$, and so the simplicial set $\mathbf{\Delta}(-, [k])$ is the *standard k -simplex*, which we will denote by $\Delta[k]$. That is, $\Delta: \mathbf{\Delta} \rightarrow \text{SS}$ is a cosimplicial simplicial set, and its value on the object $[k]$ is $\Delta[k]$. We will call it the *cosimplicial standard simplex*.

Lemma 3.7. *If n is a positive integer, then the n -cosimplicial product of standard simplices $\Delta^{(n)}$ (see Definition 3.6) is a Reedy cofibrant (see [6, Def. 15.3.3]) n -cosimplicial simplicial set.*

Proof. The latching map at the object $([p_1], [p_2], \dots, [p_n])$ of $\mathbf{\Delta}^n$ is the inclusion of the boundary of $\Delta[p_1] \times \Delta[p_2] \times \dots \times \Delta[p_n]$, and is thus a cofibration. \square

Lemma 3.8. *If K is a simplicial set and ΔK is the category of simplices of K (which has as objects the simplices of K and as morphisms from σ to τ the simplicial operators that take τ to σ ; see [6, Def. 15.1.16]), then K is naturally isomorphic to the colimit of the ΔK -diagram of simplicial sets that*

- takes a k -simplex of K to the standard k -simplex $\Delta[k]$,
- when $\partial_i(\tau) = \sigma$, takes ∂_i to the inclusion of the image of σ as the i 'th face of the image of τ , and
- when $s^i(\tau) = \sigma$, takes s^i to the collapse of the image of σ to the image of τ that identifies vertices i and $i + 1$

under an isomorphism that for a k -simplex σ takes the nondegenerate k -simplex of $\Delta[k]$ to σ .

Proof. See [6, Prop. 15.1.20]. \square

The following theorem is the degree 0 part of [5, Remark on p. 172] and [9, Prop. 8.1]. The full statement, for a general model category, is Theorem 8.10.

Theorem 3.9. *If n is a positive integer, then the left Kan extension of the cosimplicial standard simplex Δ (see Definition 3.6) along the diagonal embedding $\mathbf{\Delta} \rightarrow \mathbf{\Delta}^n$ (see Definition 2.5) is the n -cosimplicial product of standard simplices $\Delta^{(n)}$ (see Definition 3.6) with the natural transformation $\alpha: \Delta \rightarrow \text{diag } \Delta^{(n)}$ that on the object*

$[k]$ of $\mathbf{\Delta}$ is the diagonal map $\Delta[k] \rightarrow \Delta[k] \times \Delta[k] \times \cdots \times \Delta[k]$. Thus, for every n -cosimplicial simplicial set \mathbf{X} there is a natural isomorphism of sets

$$\mathrm{SS}^{\mathbf{\Delta}^n}(\Delta^{(n)}, \mathbf{X}) \approx \mathrm{SS}^{\mathbf{\Delta}}(\Delta, \mathrm{diag} \mathbf{X})$$

between the set of maps of n -cosimplicial simplicial sets $\Delta^{(n)} \rightarrow \mathbf{X}$ and the set of maps of cosimplicial simplicial sets $\Delta \rightarrow \mathrm{diag} \mathbf{X}$ (see Definition 2.5). That isomorphism takes a map $f: \Delta^{(n)} \rightarrow \mathbf{X}$ in $\mathrm{SS}^{\mathbf{\Delta}^n}$ to the composition $\Delta \xrightarrow{\alpha} \mathrm{diag} \Delta^{(n)} \xrightarrow{\mathrm{diag} f} \mathrm{diag} \mathbf{X}$ in $\mathrm{SS}^{\mathbf{\Delta}}$ (see [2, Def. 3.7.1]).

Proof. Since the category of simplicial sets is cocomplete, the left Kan extension $\mathbf{L}\Delta$ of Δ exists and can be constructed pointwise (see [2, Thm. 3.7.2]). We view $\mathbf{\Delta}$ as the diagonal subcategory of $\mathbf{\Delta}^n$, and so for each object $([p_1], [p_2], \dots, [p_n])$ of $\mathbf{\Delta}^n$, the simplicial set $\mathbf{L}\Delta([p_1], [p_2], \dots, [p_n])$ is the colimit of the $(\mathbf{\Delta} \downarrow ([p_1], [p_2], \dots, [p_n]))$ -diagram of simplicial sets that takes the object

$$(\alpha_1, \alpha_2, \dots, \alpha_n): ([k], [k], \dots, [k]) \longrightarrow ([p_1], [p_2], \dots, [p_n])$$

of $(\mathbf{\Delta} \downarrow ([p_1], [p_2], \dots, [p_n]))$ to the standard k -simplex $\Delta[k]$. That object is the product for $1 \leq i \leq n$ of morphisms $\alpha_i: [k] \rightarrow [p_i]$ in $\mathbf{\Delta}$, i.e., the product for $1 \leq i \leq n$ of a k -simplex of $\Delta[p_i]$, i.e., a k -simplex of $\Delta[p_1] \times \Delta[p_2] \times \cdots \times \Delta[p_n]$.

Thus, $\mathbf{L}\Delta([p_1], [p_2], \dots, [p_n])$ is the colimit of the diagram indexed by the category of simplices of $\Delta[p_1] \times \Delta[p_2] \times \cdots \times \Delta[p_n]$ (see Lemma 3.8) that takes each k -simplex of $\Delta[p_1] \times \Delta[p_2] \times \cdots \times \Delta[p_n]$ to the standard k -simplex $\Delta[k]$, and so Lemma 3.8 implies that $\mathbf{L}\Delta([p_1], [p_2], \dots, [p_n]) \approx \Delta[p_1] \times \Delta[p_2] \times \cdots \times \Delta[p_n]$.

For the natural transformation α , note that a map of simplicial sets with domain $\Delta[k]$ is entirely determined by what it does to the nondegenerate k -simplex of $\Delta[k]$, which is the identity map of $[k]$. The natural transformation $\alpha: \Delta \rightarrow \mathrm{diag} \Delta^{(n)}$ on the object $[k]$ of $\mathbf{\Delta}$ takes that nondegenerate k -simplex of $\Delta[k]$ to the k -simplex of $\Delta[k] \times \Delta[k] \times \cdots \times \Delta[k]$ that is the image under the diagonal embedding of the identity map of $[k]$, which is the map $[k] \rightarrow ([k], [k], \dots, [k])$ whose projection onto each factor is the identity map of $[k]$, i.e., the product of the nondegenerate k -simplices of each factor. \square

3.2. Quillen functors, but not Quillen equivalences. Let \mathcal{M} be a model category. We show in Theorem 3.10 and Corollary 3.11 that the right Quillen functor $\mathrm{diag}: \mathcal{M}^{\mathbf{\Delta}^n} \rightarrow \mathcal{M}^{\mathbf{\Delta}}$ (see Theorem 3.5) and its left adjoint $\mathbf{L}: \mathcal{M}^{\mathbf{\Delta}} \rightarrow \mathcal{M}^{\mathbf{\Delta}^n}$ are not Quillen equivalences when \mathcal{M} is either the model category of simplicial sets or the model category of topological spaces (see [6, Def. 8.5.20]).

Theorem 3.10. *If $\mathcal{M} = \mathrm{SS}$, the model category of simplicial sets, and $n \geq 2$, then the Quillen functors $\mathrm{diag}: \mathcal{M}^{\mathbf{\Delta}^n} \rightarrow \mathcal{M}^{\mathbf{\Delta}}$ and its left adjoint $\mathbf{L}: \mathcal{M}^{\mathbf{\Delta}} \rightarrow \mathcal{M}^{\mathbf{\Delta}^n}$ are not Quillen equivalences.*

Proof. We will discuss the case $n = 2$; the other cases are similar. We will construct a cofibrant cosimplicial simplicial set \mathbf{X} , a fibrant bicosimplicial simplicial set \mathbf{Y} , and a weak equivalence $f: \mathbf{X} \rightarrow \mathrm{diag} \mathbf{Y}$ such that the corresponding map $\mathbf{L}\mathbf{X} \rightarrow \mathbf{Y}$ is not a weak equivalence.

For each $k \geq 0$, let $(\Delta[k])^0$ denote the 0-skeleton of $\Delta[k]$, and let \mathbf{X} be the cosimplicial simplicial set that is the degreewise 0-skeleton of the cosimplicial standard simplex Δ , so that $\mathbf{X}^k = (\Delta[k])^0$. Since the maximal augmentation of \mathbf{X} is empty, \mathbf{X} is cofibrant (see [6, Cor. 15.9.10]).

Let \mathbf{W} be the bicosimplicial simplicial set obtained from \mathbf{X} by making it constant in the second index, i.e., $\mathbf{W}^{(p,q)} = \mathbf{X}^p = (\Delta[p])^0$, and let $\mathbf{W} \rightarrow \mathbf{Y}$ be a fibrant approximation to \mathbf{W} , so that $\mathbf{W} \rightarrow \mathbf{Y}$ is a weak equivalence of bicosimplicial simplicial sets and \mathbf{Y} is fibrant. There is an obvious isomorphism of cosimplicial simplicial sets $\mathbf{X} \rightarrow \text{diag } \mathbf{W}$, and our map $f: \mathbf{X} \rightarrow \text{diag } \mathbf{Y}$ is the composition $\mathbf{X} \rightarrow \text{diag } \mathbf{W} \rightarrow \text{diag } \mathbf{Y}$; it is the composition of an isomorphism and an objectwise weak equivalence, and so it is an objectwise weak equivalence, i.e., a weak equivalence of cosimplicial simplicial sets.

The functor $\mathbf{L}: \text{SS}^\Delta \rightarrow \text{SS}^{\Delta^2}$ takes the cosimplicial standard simplex Δ to the bicosimplicial product of standard simplices $\Delta^{(2)}$ (see Theorem 3.9). Since the colimit of a diagram of simplicial sets is constructed degreewise, and every simplicial set in the diagram whose colimit is $(\mathbf{L}\mathbf{X})^{(p,q)}$ is discrete (i.e., has all face and degeneracy operators isomorphisms), each $(\mathbf{L}\mathbf{X})^{(p,q)}$ is also discrete, and so $\mathbf{L}\mathbf{X} = (\Delta^{(2)})^0$, the degreewise 0-skeleton of $\Delta^{(2)}$. Thus, $(\mathbf{L} \text{diag } \mathbf{X})^{(1,1)}$ is the 0-skeleton of $\Delta[1] \times \Delta[1]$, and has four path components, while $\mathbf{Y}^{(1,1)}$ is weakly equivalent to $(\Delta[1])^0$, and has two path components. Thus, the map $\mathbf{L}\mathbf{X} \rightarrow \mathbf{Y}$ is not a weak equivalence. \square

Corollary 3.11. *If $\mathcal{M} = \text{Top}$, the model category of topological spaces, and $n \geq 2$, then the Quillen functors $\text{diag}: \mathcal{M}^{\Delta^n} \rightarrow \mathcal{M}^\Delta$ and its left adjoint $\mathbf{L}: \mathcal{M}^\Delta \rightarrow \mathcal{M}^{\Delta^n}$ are not Quillen equivalences.*

Proof. The geometric realization of the example in Theorem 3.10 is a cofibrant cosimplicial space \mathbf{X} , a fibrant bisimplicial space \mathbf{Y} , and a weak equivalence $f: \mathbf{X} \rightarrow \mathbf{Y}$ such that the corresponding map $\mathbf{L}\mathbf{X} \rightarrow \mathbf{Y}$ is not a weak equivalence. \square

4. PROOF OF PROPOSITION 3.2

Since $\overleftarrow{\Delta}^n = \overleftarrow{\Delta} \times \overleftarrow{\Delta} \times \cdots \times \overleftarrow{\Delta}$ (see [6, Prop. 15.1.6]), the matching category $\partial(([k], [k], \dots, [k]) \downarrow \overleftarrow{\Delta}^n)$ has as objects the maps

$$([k] \rightarrow [p_1], [k] \rightarrow [p_2], \dots, [k] \rightarrow [p_n])$$

in Δ^n such that each $[k] \rightarrow [p_i]$ is a surjection and such that at least one of them is not the identity map. The diagonal embedding of Δ into Δ^n (see Definition 2.5) takes the matching category $\partial([k] \downarrow \overleftarrow{\Delta})$ to the full subcategory of $\partial(([k], [k], \dots, [k]) \downarrow \overleftarrow{\Delta}^n)$ with objects the maps

$$\{\phi^n \mid \phi: [k] \rightarrow [p] \text{ is a surjection and is not the identity map}\}$$

and we will identify $\partial([k] \downarrow \overleftarrow{\Delta})$ with its image in $\partial(([k], [k], \dots, [k]) \downarrow \overleftarrow{\Delta}^n)$. Our map

$$\begin{aligned} P_k^{\Delta^n} &= \left(\lim_{\partial((([k], [k], \dots, [k]) \downarrow \overleftarrow{\Delta}^n)} \mathbf{X} \right) \times_{\left(\lim_{\partial((([k], [k], \dots, [k]) \downarrow \overleftarrow{\Delta}^n)} \mathbf{Y} \right)} \mathbf{Y}^{(k,k,\dots,k)} \\ &\longrightarrow P_k^\Delta = \left(\lim_{\partial([k] \downarrow \overleftarrow{\Delta})} \mathbf{X} \right) \times_{\left(\lim_{\partial([k] \downarrow \overleftarrow{\Delta})} \mathbf{Y} \right)} \mathbf{Y}^{(k,k,\dots,k)} \end{aligned}$$

(see Definition 3.1) is induced by restricting the functors \mathbf{X} and \mathbf{Y} to this subcategory. We will define a nested sequence of subcategories of $\partial(([k], [k], \dots, [k]) \downarrow \overleftarrow{\Delta}^n)$

$$\partial([k] \downarrow \overleftarrow{\Delta}) = \mathcal{C}_{-1} \subset \mathcal{C}_0 \subset \cdots \subset \mathcal{C}_{n-1} = \partial((([k], [k], \dots, [k]) \downarrow \overleftarrow{\Delta}^n)$$

and for $-1 \leq i \leq nk - 1$ we will let P_i be the pullback

$$\begin{array}{ccc} P_i & \xrightarrow{\quad \quad \quad} & \mathbf{Y}^{(k,k,\dots,k)} \\ \vdots & & \downarrow \\ \lim_{\mathcal{C}_i} \mathbf{X} & \longrightarrow & \lim_{\mathcal{C}_i} \mathbf{Y} . \end{array}$$

Thus, we will have a factorization of our map $P_k^{\Delta^n} \rightarrow P_k^{\Delta}$ as

$$P_k^{\Delta^n} = P_{nk-1} \longrightarrow P_{nk-2} \longrightarrow \cdots \longrightarrow P_{-1} = P_k^{\Delta}$$

and we will show that the map $P_{i+1} \rightarrow P_i$ is a fibration for $-1 \leq i \leq nk - 2$.

Definition 4.1. If n is a positive integer, k is a nonnegative integer, and $-1 \leq i \leq nk - 1$, we let \mathcal{C}_i be the full subcategory of $\partial([k], [k], \dots, [k] \downarrow \overleftarrow{\Delta^n})$ with objects the union of

- the objects of $\partial([k], [k], \dots, [k] \downarrow \overleftarrow{\Delta^n})$ whose target is of degree at most i , and
- the objects of $\partial([k], [k], \dots, [k] \downarrow \overleftarrow{\Delta^n})$ in the image of the embedding of $\partial([k] \downarrow \overleftarrow{\Delta})$.

That is, we let \mathcal{C}_i be the full subcategory of $\partial([k], [k], \dots, [k] \downarrow \overleftarrow{\Delta^n})$ with objects the maps

$$(\phi_1, \phi_2, \dots, \phi_n): ([k], [k], \dots, [k]) \longrightarrow ([p_1], [p_2], \dots, [p_n])$$

such that either $p_1 + p_2 + \cdots + p_n \leq i$ or $\phi_1 = \phi_2 = \cdots = \phi_n$. Note that, since every map in $\overleftarrow{\Delta}$ is an epimorphism, every map in $\partial([k], [k], \dots, [k] \downarrow \overleftarrow{\Delta^n})$ between objects in the image of the embedding of $\partial([k] \downarrow \overleftarrow{\Delta})$ must also be in the image of that embedding; thus, $\mathcal{C}_{-1} = \partial([k] \downarrow \overleftarrow{\Delta})$.

Proposition 4.2. *If $-1 \leq i \leq nk - 2$, then the map $P_{i+1} \rightarrow P_i$ is a fibration.*

Proof. The objects of \mathcal{C}_{i+1} that aren't in \mathcal{C}_i are maps $([k] \rightarrow [p_1], [k] \rightarrow [p_2], \dots, [k] \rightarrow [p_n])$ such that $p_1 + p_2 + \cdots + p_n = i + 1$ (though not necessarily all such maps), and this set of maps can be divided into two subsets:

- the set S_{i+1} of maps for which there exists an epimorphism $\psi: [k] \rightarrow [j]$ with $j < k$ and a factorization through $\psi^n: ([k], [k], \dots, [k]) \rightarrow ([j], [j], \dots, [j])$,
- the set T_{i+1} of maps for which there is no such factorization.

We let \mathcal{C}'_{i+1} be the full subcategory of $\partial([k], [k], \dots, [k] \downarrow \overleftarrow{\Delta^n})$ with objects the union of S_{i+1} with the objects of \mathcal{C}_i , and define P'_{i+1} as the pullback

$$\begin{array}{ccc} P'_{i+1} & \xrightarrow{\quad \quad \quad} & \mathbf{Y}^{(k,k,\dots,k)} \\ \vdots & & \downarrow \\ \lim_{\mathcal{C}'_{i+1}} \mathbf{X} & \longrightarrow & \lim_{\mathcal{C}'_{i+1}} \mathbf{Y} . \end{array}$$

We have inclusions of categories $\mathcal{C}_i \subset \mathcal{C}'_{i+1} \subset \mathcal{C}_{i+1}$, and the maps

$$\lim_{\mathcal{C}_{i+1}} \mathbf{X} \longrightarrow \lim_{\mathcal{C}_i} \mathbf{X} \quad \text{and} \quad \lim_{\mathcal{C}_{i+1}} \mathbf{Y} \longrightarrow \lim_{\mathcal{C}_i} \mathbf{Y}$$

factor as

$$\lim_{\mathcal{C}_{i+1}} \mathbf{X} \longrightarrow \lim_{\mathcal{C}'_{i+1}} \mathbf{X} \longrightarrow \lim_{\mathcal{C}_i} \mathbf{X} \quad \text{and} \quad \lim_{\mathcal{C}_{i+1}} \mathbf{Y} \longrightarrow \lim_{\mathcal{C}'_{i+1}} \mathbf{Y} \longrightarrow \lim_{\mathcal{C}_i} \mathbf{Y} .$$

These factorizations induce a factorization

$$P_{i+1} \longrightarrow P'_{i+1} \longrightarrow P_i \quad \text{of the map} \quad P_{i+1} \longrightarrow P_i .$$

Proposition 4.3 asserts that the map $P'_{i+1} \rightarrow P_i$ is an isomorphism and Proposition 4.4 asserts that the map $P_{i+1} \rightarrow P'_{i+1}$ is a fibration. \square

Proposition 4.3. *For $-1 \leq i \leq nk - 2$, the map $P'_{i+1} \rightarrow P_i$ is an isomorphism.*

The proof of Proposition 4.3 is in Section 5.

Proposition 4.4. *For $-1 \leq i \leq nk - 1$, the map $P_{i+1} \rightarrow P'_{i+1}$ is a fibration.*

The proof of Proposition 4.4 is in Section 6.

5. PROOF OF PROPOSITION 4.3

Lemma 5.1. *Every morphism*

$$(\alpha_1, \alpha_2, \dots, \alpha_n): ([k], [k], \dots, [k]) \longrightarrow ([p_1], [p_2], \dots, [p_n])$$

in $\overleftarrow{\Delta}^n$ with domain a diagonal object has a terminal factorization through a diagonal morphism, i.e., an epimorphism $\beta: [k] \rightarrow [q]$ and a factorization

$$([k], [k], \dots, [k]) \xrightarrow{\beta^n} ([q], [q], \dots, [q]) \longrightarrow ([p_1], [p_2], \dots, [p_n])$$

of $(\alpha_1, \alpha_2, \dots, \alpha_n)$ such that every epimorphism $\gamma: [k] \rightarrow [r]$ and factorization

$$([k], [k], \dots, [k]) \xrightarrow{\gamma^n} ([r], [r], \dots, [r]) \longrightarrow ([p_1], [p_2], \dots, [p_n])$$

of $(\alpha_1, \alpha_2, \dots, \alpha_n)$ through a diagonal morphism of Δ^n factors uniquely as

$$([k], [k], \dots, [k]) \xrightarrow{\gamma^n} ([r], [r], \dots, [r]) \xrightarrow{\delta^n} ([q], [q], \dots, [q]) \longrightarrow ([p_1], [p_2], \dots, [p_n])$$

with $\delta\gamma = \beta$.

Proof. Each of the epimorphisms $\alpha_j: [k] \rightarrow [p_j]$ is determined by the set U_j of integers i such that $\alpha_j(i) = \alpha_j(i+1)$. We let $U = \bigcap_{1 \leq j \leq n} U_j$. The set U now determines an epimorphism $\beta: [k] \rightarrow [q]$ for some $q \leq k$, and the terminal factorization of α is the factorization through $\beta^n: ([k], [k], \dots, [k]) \rightarrow ([q], [q], \dots, [q])$. \square

Proposition 5.2. *For $-1 \leq i \leq nk - 2$, the inclusion of categories $\mathcal{C}_i \subset \mathcal{C}'_{i+1}$ is left cofinal (see [6, Def. 14.2.1]).*

Proof. Let $\alpha = ([k] \rightarrow [p_1], [k] \rightarrow [p_2], \dots, [k] \rightarrow [p_n])$ be an object of \mathcal{C}'_{i+1} that isn't in \mathcal{C}_i . Since every morphism in $\overleftarrow{\Delta}^n$ lowers degree, the only objects of $(\mathcal{C}_i \downarrow \alpha)$ are factorizations of α through $\phi^n: ([k], [k], \dots, [k]) \rightarrow ([j], [j], \dots, [j])$ for some epimorphism $\phi: [k] \rightarrow [j]$ with $j < k$, and Lemma 5.1 implies that there is a terminal such factorization, i.e., one through which all other factorizations factor uniquely. That terminal factorization is a terminal object of the overcategory $(\mathcal{C}_i \downarrow \alpha)$, and so that overcategory is nonempty and connected, and so the inclusion $\mathcal{C}_i \subset \mathcal{C}'_{i+1}$ is left cofinal. \square

Proof of Proposition 4.3. For $-1 \leq i \leq nk - 2$, Proposition 5.2 implies that the inclusion of categories $\mathcal{C}_i \subset \mathcal{C}'_{i+1}$ is left cofinal, and so the maps $\lim_{\mathcal{C}'_{i+1}} \mathbf{X} \rightarrow \lim_{\mathcal{C}_i} \mathbf{X}$ and $\lim_{\mathcal{C}'_{i+1}} \mathbf{Y} \rightarrow \lim_{\mathcal{C}_i} \mathbf{Y}$ are isomorphisms (see [6, Thm. 14.2.5]), and so the induced map $P'_{i+1} \rightarrow P_i$ is an isomorphism. \square

6. PROOF OF PROPOSITION 4.4

Notation 6.1. To save space, we will use the following compact notation:

- We will use $[\vec{p}]$ to denote an object $([p_1], [p_2], \dots, [p_n])$ of Δ^n .
- If $k \geq 0$, we will use $[k]^n \rightarrow [\vec{p}]$ to denote an object $([k], [k], \dots, [k]) \rightarrow ([p_1], [p_2], \dots, [p_n])$ of the matching category $\partial(([k], [k], \dots, [k]) \downarrow \overleftarrow{\Delta^n})$.

Lemma 6.2. *For each n -cosimplicial object \mathbf{X} in \mathcal{M} there is a pullback square*

$$(6.3) \quad \begin{array}{ccc} \lim_{\mathcal{C}_{i+1}} \mathbf{X} & \longrightarrow & \lim_{\mathcal{C}'_{i+1}} \mathbf{X} \\ \downarrow & & \downarrow \\ \prod_{([k]^n \rightarrow [\vec{p}]) \in T_{i+1}} \mathbf{X}_{[\vec{p}]} & \longrightarrow & \prod_{([k]^n \rightarrow [\vec{p}]) \in T_{i+1}} \lim_{\partial([\vec{p}] \downarrow \overleftarrow{\Delta^n})} \mathbf{X} . \end{array}$$

Proof. For every element $[k]^n \rightarrow [\vec{p}]$ of T_{i+1} , every object of the matching category $\partial([\vec{p}] \downarrow \overleftarrow{\Delta^n})$ is a map to an object of degree at most i , and so there is a functor $\partial([\vec{p}] \downarrow \overleftarrow{\Delta^n}) \rightarrow \mathcal{C}'_{i+1}$ that takes the object $([p_1], [p_2], \dots, [p_n]) \rightarrow ([q_1], [q_2], \dots, [q_n])$ to the composition $([k], [k], \dots, [k]) \rightarrow ([p_1], [p_2], \dots, [p_n]) \rightarrow ([q_1], [q_2], \dots, [q_n])$; this induces a map $\lim_{\mathcal{C}'_{i+1}} \mathbf{X} \rightarrow \lim_{\partial([\vec{p}] \downarrow \overleftarrow{\Delta^n})} \mathbf{X}$ that is the projection of the right hand vertical map onto the factor indexed by $[k]^n \rightarrow [\vec{p}]$. We thus have a commutative square as in Diagram 6.3.

The objects of \mathcal{C}_{i+1} are the objects of \mathcal{C}'_{i+1} together with the elements of T_{i+1} , and so a map to $\lim_{\mathcal{C}_{i+1}} \mathbf{X}$ is determined by a map to $\lim_{\mathcal{C}'_{i+1}} \mathbf{X}$ and a map to $\prod_{([k]^n \rightarrow [\vec{p}]) \in T_{i+1}} \mathbf{X}_{[\vec{p}]}$. Since there are no non-identity morphisms in \mathcal{C}_{i+1} with codomain an element of T_{i+1} , and the only non-identity morphisms with domain an element $[k]^n \rightarrow [\vec{p}]$ of T_{i+1} are the objects of the matching category $\partial([\vec{p}] \downarrow \overleftarrow{\Delta^n})$, maps to $\lim_{\mathcal{C}'_{i+1}} \mathbf{X}$ and to $\prod_{([k]^n \rightarrow [\vec{p}]) \in T_{i+1}} \mathbf{X}_{[\vec{p}]}$ determine a map to $\lim_{\mathcal{C}_{i+1}} \mathbf{X}$ if and only if their compositions to $\prod_{([k]^n \rightarrow [\vec{p}]) \in T_{i+1}} \lim_{\partial([\vec{p}] \downarrow \overleftarrow{\Delta^n})} \mathbf{X}$ agree. Thus, the diagram is a pullback square. \square

Define Q and R by letting the squares

$$(6.4) \quad \begin{array}{ccc} Q & \cdots \longrightarrow & \lim_{\mathcal{C}'_{i+1}} \mathbf{X} \\ \downarrow & & \downarrow \\ \lim_{\mathcal{C}_{i+1}} \mathbf{Y} & \longrightarrow & \lim_{\mathcal{C}'_{i+1}} \mathbf{Y} \end{array} \quad \text{and} \quad \begin{array}{ccc} R & \cdots \longrightarrow & \prod_{([k]^n \rightarrow [\vec{p}]) \in T_{i+1}} \lim_{\partial([\vec{p}] \downarrow \overleftarrow{\Delta^n})} \mathbf{X} \\ \downarrow & & \downarrow \\ \prod_{([k]^n \rightarrow [\vec{p}]) \in T_{i+1}} \mathbf{Y}_{[\vec{p}]} & \longrightarrow & \prod_{([k]^n \rightarrow [\vec{p}]) \in T_{i+1}} \lim_{\partial([\vec{p}] \downarrow \overleftarrow{\Delta^n})} \mathbf{Y} \end{array}$$

be pullbacks, and consider the commutative diagram

$$(6.5) \quad \begin{array}{ccccc} \lim_{\mathcal{C}_{i+1}} \mathbf{X} & \xrightarrow{s} & \lim_{\mathcal{C}'_{i+1}} \mathbf{X} & & \\ \downarrow u & \searrow a & \nearrow c & \searrow \beta & \\ & & Q & & \\ & \searrow \delta & \downarrow d & & \\ & & \lim_{\mathcal{C}_{i+1}} \mathbf{Y} & \xrightarrow{s'} & \lim_{\mathcal{C}'_{i+1}} \mathbf{Y} \\ & & \downarrow g & \downarrow v & \\ \prod_{([k]^n \rightarrow [\bar{p}]) \in T_{i+1}} \mathbf{X}_{[\bar{p}]} & \xrightarrow{t} & \prod_{([k]^n \rightarrow [\bar{p}]) \in T_{i+1}} \lim_{\partial([\bar{p}] \downarrow \Delta^n)} \mathbf{X} & & \\ & \searrow b & \nearrow e & & \\ & & R & & \\ & \searrow \gamma & \downarrow f & & \\ & & \prod_{([k]^n \rightarrow [\bar{p}]) \in T_{i+1}} \mathbf{Y}_{[\bar{p}]} & \xrightarrow{t'} & \prod_{([k]^n \rightarrow [\bar{p}]) \in T_{i+1}} \lim_{\partial([\bar{p}] \downarrow \Delta^n)} \mathbf{Y} \\ & & \downarrow u' & & \downarrow v' \end{array}$$

Lemma 6.2 implies that the front and back rectangles are pullbacks.

Lemma 6.6. *The square*

$$(6.7) \quad \begin{array}{ccc} \lim_{\mathcal{C}_{i+1}} \mathbf{X} & \xrightarrow{a} & Q \\ u \downarrow & & \downarrow g \\ \prod_{([k]^n \rightarrow [\bar{p}]) \in T_{i+1}} \mathbf{X}_{[\bar{p}]} & \xrightarrow{b} & R \end{array}$$

is a pullback.

Proof. Let W be an object of \mathcal{M} and let $h: W \rightarrow \prod_{([k]^n \rightarrow [\bar{p}]) \in T_{i+1}} \mathbf{X}_{[\bar{p}]}$ and $k: W \rightarrow Q$ be maps such that $gk = bh$; we will show that there is a unique map $\phi: W \rightarrow \lim_{\mathcal{C}_{i+1}} \mathbf{X}$ such that $a\phi = k$ and $u\phi = h$.

$$\begin{array}{ccc} W & \xrightarrow{k} & Q \\ \searrow \phi & & \downarrow g \\ \lim_{\mathcal{C}_{i+1}} \mathbf{X} & \xrightarrow{a} & Q \\ \downarrow u & & \downarrow g \\ \prod_{([k]^n \rightarrow [\bar{p}]) \in T_{i+1}} \mathbf{X}_{[\bar{p}]} & \xrightarrow{b} & R \\ \downarrow h & & \end{array}$$

The map $ck: W \rightarrow \lim_{\mathcal{C}'_{i+1}} \mathbf{X}$ has the property that $v(ck) = egk = ebh = th$, and since the back rectangle of Diagram 6.5 is a pullback, the maps ck and h induce a map $\phi: W \rightarrow \lim_{\mathcal{C}_{i+1}} \mathbf{X}$ such that $u\phi = h$ and $s\phi = ck$. We must show that $a\phi = k$, and since Q is a pullback as in Diagram 6.4, this is equivalent to showing that $ca\phi = ck$ and $da\phi = dk$.

Since $ck = s\phi = ca\phi$, we need only show that $da\phi = dk$. Since the front rectangle of Diagram 6.5 is a pullback, it is sufficient to show that $s'da\phi = s'dk$ and $u'da\phi = u'dk$. For the first of those, we have

$$s'da\phi = s'\delta\phi = \beta s\phi = \beta ck = s'dk$$

and for the second, we have

$$u'da\phi = u'\delta\phi = \gamma u\phi = fbu\phi = fbh = fgk = u'dk$$

and so the map ϕ satisfies $a\phi = k$ and $u\phi = h$.

To see that ϕ is the unique such map, let $\psi: W \rightarrow \lim_{\mathcal{C}_{i+1}} \mathbf{X}$ be another map such that $a\psi = k$ and $u\psi = h$. We will show that $s\psi = s\phi$ and $u\psi = u\phi$; since the back rectangle of Diagram 6.5 is a pullback, this will imply that $\psi = \phi$.

Since $u\psi = h = u\phi$, we need only show that $s\psi = s\phi$, which follows because $s\psi = ca\psi = ck = s\phi$. \square

Lemma 6.8. *If $\mathbf{X} \rightarrow \mathbf{Y}$ is a fibration of n -cosimplicial objects, then the natural map*

$$\lim_{\mathcal{C}_{i+1}} \mathbf{X} \longrightarrow Q = \lim_{\mathcal{C}'_{i+1}} \mathbf{X} \times_{(\lim_{\mathcal{C}'_{i+1}} \mathbf{Y})} \lim_{\mathcal{C}_{i+1}} \mathbf{Y}$$

is a fibration.

Proof. Lemma 6.6 gives us the pullback square in Diagram 6.7, where Q and R are defined by the pullbacks in Diagram 6.4. Since $\mathbf{X} \rightarrow \mathbf{Y}$ is a fibration of n -cosimplicial objects, the map $\prod_{([k]^n \rightarrow [p]) \in \mathcal{T}_{i+1}} \mathbf{X}_{[p]} \rightarrow R$ is a product of fibrations and is thus a fibration, and so the map $\lim_{\mathcal{C}_{i+1}} \mathbf{X} \rightarrow Q = \lim_{\mathcal{C}'_{i+1}} \mathbf{X} \times_{(\lim_{\mathcal{C}'_{i+1}} \mathbf{Y})} \lim_{\mathcal{C}_{i+1}} \mathbf{Y}$ is a pullback of a fibration and is thus a fibration. \square

Lemma 6.9 (Reedy). *If both the front and back squares in the diagram*

$$\begin{array}{ccccc} A & \longrightarrow & B & & \\ \downarrow & \searrow f_A & \downarrow & \searrow f_B & \\ & & A' & \longrightarrow & B' \\ \downarrow & & \downarrow & & \downarrow \\ C & \longrightarrow & D & & \\ \downarrow & \searrow f_C & \downarrow & \searrow f_D & \\ & & C' & \longrightarrow & D' \end{array}$$

are pullbacks and both $f_B: B \rightarrow B'$ and $C \rightarrow C' \times_{D'} D$ are fibrations, then $f_A: A \rightarrow A'$ is a fibration.

Proof. This is the dual of a lemma of Reedy (see [6, Lem. 7.2.15 and Rem. 7.1.10]). \square

Proof of Proposition 4.4. We have a commutative diagram

$$\begin{array}{ccccc}
 P_{i+1} & \longrightarrow & Y^{(k,k,\dots,k)} & & \\
 \searrow & & \downarrow & \searrow & \\
 & P'_{i+1} & \longrightarrow & Y^{(k,k,\dots,k)} & \\
 \downarrow & \downarrow & \downarrow & \downarrow & \\
 \lim_{\mathcal{C}_{i+1}} \mathbf{X} & \longrightarrow & \lim_{\mathcal{C}_{i+1}} \mathbf{Y} & & \\
 \searrow & \downarrow & \searrow & \downarrow & \\
 & \lim_{\mathcal{C}'_{i+1}} \mathbf{X} & \longrightarrow & \lim_{\mathcal{C}'_{i+1}} \mathbf{Y} &
 \end{array}$$

in which the front and back squares are pullbacks, and so Lemma 6.9 implies that it is sufficient to show that the map

$$\lim_{\mathcal{C}_{i+1}} \mathbf{X} \longrightarrow \lim_{\mathcal{C}'_{i+1}} \mathbf{X} \times_{(\lim_{\mathcal{C}'_{i+1}} \mathbf{Y})} \lim_{\mathcal{C}_{i+1}} \mathbf{Y}$$

is a fibration; that is the statement of Lemma 6.8. \square

7. FRAMES, HOMOTOPY COTENSORS, AND HOMOTOPY FUNCTION COMPLEXES

If \mathbf{X} is a cosimplicial simplicial set, then both the total space $\text{Tot } \mathbf{X}$ and the homotopy limit $\text{holim } \mathbf{X}$ are built from simplicial sets of the form $(\mathbf{X}^n)^K$ for $n \geq 0$ and K a simplicial set, where $(\mathbf{X}^n)^K$ is the simplicial set of maps $K \rightarrow \mathbf{X}^n$. If \mathcal{M} is a simplicial model category and \mathbf{X} is a cosimplicial object in \mathcal{M} , then the simplicial model category structure includes objects $(\mathbf{X}^n)^K$ of \mathcal{M} , called the *cotensor* of \mathbf{X}^n and K , and $\text{Tot } \mathbf{X}$ and $\text{holim } \mathbf{X}$ are built from those. When \mathcal{M} is a (possibly non-simplicial) model category and \mathbf{X} is a cosimplicial object in \mathcal{M} , we need objects of \mathcal{M} that can play the role of the $(\mathbf{X}^n)^K$, and to define those we choose a *simplicial frame* on the objects of \mathcal{M} . A simplicial frame on an object X of \mathcal{M} is a simplicial object $\widehat{\mathbf{X}}$ of \mathcal{M} such that

- $\widehat{\mathbf{X}}_0$ is isomorphic to X ,
- all the face and degeneracy operators of $\widehat{\mathbf{X}}$ are weak equivalences, and
- if X is a fibrant object of \mathcal{M} then $\widehat{\mathbf{X}}$ is a Reedy fibrant simplicial object.

Given such a simplicial frame on X , we use $\widehat{\mathbf{X}}_n$ to play the role of $X^{\Delta[n]}$, and if K is a simplicial set we construct the *homotopy cotensor* $\widehat{\mathbf{X}}^K$ of X and K as the limit of a diagram of the $\widehat{\mathbf{X}}_n$ indexed by the opposite of the category of simplices of K (see Definition 7.3).

7.1. Frames.

Definition 7.1 (Frames and Reedy frames).

- (1) If \mathcal{M} is a model category and X is an object of \mathcal{M} , then a *simplicial frame* on X is a simplicial object $\widehat{\mathbf{X}}$ together with an isomorphism $X \approx \widehat{\mathbf{X}}_0$ such that
 - all the face and degeneracy operators of $\widehat{\mathbf{X}}$ are weak equivalences and
 - if X is a fibrant object of \mathcal{M} then $\widehat{\mathbf{X}}$ is a Reedy fibrant simplicial object in \mathcal{M} .

Equivalently, a simplicial frame on X is a simplicial object $\widehat{\mathbf{X}}$ in \mathcal{M} together with a weak equivalence $\text{cs}_* X \rightarrow \widehat{\mathbf{X}}$ (where $\text{cs}_* X$ is the constant simplicial object on X) in the Reedy model category structure on $\mathcal{M}^{\Delta^{\text{op}}}$ such that

- the induced map $X \rightarrow \widehat{\mathbf{X}}_0$ is an isomorphism, and

- if X is a fibrant object of \mathcal{M} , then $\widehat{\mathbf{X}}$ is a Reedy fibrant simplicial object.

We often refer to $\widehat{\mathbf{X}}$ as a simplicial frame on X , without explicitly mentioning the map $\text{cs}_*X \rightarrow \widehat{\mathbf{X}}$.

If \mathcal{M} is a simplicial model category and X is an object of \mathcal{M} , the *standard simplicial frame on X* is the simplicial object $\widehat{\mathbf{X}}$ in which $\widehat{\mathbf{X}}_n = X^{\Delta[n]}$ (see [6, Prop. 16.6.4]).

- (2) If \mathcal{M} is a model category, \mathcal{C} is a small category, and \mathbf{X} is a \mathcal{C} -diagram in \mathcal{M} , then a *simplicial frame on \mathbf{X}* is a \mathcal{C} -diagram $\widehat{\mathbf{X}}$ of simplicial objects in \mathcal{M} together with a map of diagrams $j: \text{cs}_*\mathbf{X} \rightarrow \widehat{\mathbf{X}}$ from the diagram of constant simplicial objects such that, for every object α of \mathcal{C} , the map $j_\alpha: \text{cs}_*\mathbf{X}_\alpha \rightarrow \widehat{\mathbf{X}}_\alpha$ is a simplicial frame on \mathbf{X}_α .

If \mathcal{M} is a simplicial model category, then the *standard simplicial frame on \mathbf{X}* is the frame $\widehat{\mathbf{X}}$ on \mathbf{X} such that $\widehat{\mathbf{X}}_\alpha$ is the standard simplicial frame on \mathbf{X}_α for every object α of \mathcal{C} .

- (3) If \mathcal{M} is a model category, \mathcal{C} is a Reedy category, and \mathbf{X} is a \mathcal{C} -diagram in \mathcal{M} , then a *Reedy simplicial frame on \mathbf{X}* is a simplicial frame $\widehat{\mathbf{X}}$ on \mathbf{X} such that if \mathbf{X} is a Reedy fibrant \mathcal{C} -diagram in \mathcal{M} , then $\widehat{\mathbf{X}}$ is a Reedy fibrant \mathcal{C} -diagram in $\mathcal{M}^{\Delta^{\text{op}}}$.

If \mathcal{M} is a simplicial model category and \mathcal{C} is a Reedy category, then for any \mathcal{C} -diagram \mathbf{X} in \mathcal{M} the standard simplicial frame $\widehat{\mathbf{X}}$ on \mathbf{X} (see Definition 7.1) is a Reedy simplicial frame (see [6, Prop. 16.7.9]).

Proposition 7.2 (Existence and essential uniqueness of frames).

- (1) If \mathcal{M} is a model category and X is an object of \mathcal{M} , then there exists a simplicial frame on X and any two simplicial frames on X are connected by an essentially unique zig-zag (see [6, Def. 14.4.2]) of weak equivalences of simplicial frames on X .
- (2) If \mathcal{M} is a model category, \mathcal{C} is a small category, and \mathbf{X} is a \mathcal{C} -diagram in \mathcal{M} , then there exists a simplicial frame on \mathbf{X} and any two simplicial frames on \mathbf{X} are connected by an essentially unique zig-zag (see [6, Def. 14.4.2]) of weak equivalences of simplicial frames on \mathbf{X} .
- (3) If \mathcal{M} is a model category, \mathcal{C} is a Reedy category, and \mathbf{X} is a \mathcal{C} -diagram in \mathcal{M} , then there exists a Reedy simplicial frame on \mathbf{X} and any two Reedy simplicial frames on \mathbf{X} are connected by an essentially unique zig-zag (see [6, Def. 14.4.2]) of weak equivalences of Reedy simplicial frames on \mathbf{X} .
- (4) If \mathcal{M} is a model category and \mathcal{C} is a Reedy category, then there exists a functorial Reedy simplicial frame on every \mathcal{C} -diagram on \mathcal{M} .

Proof. See [6, Thm. 16.6.18, Thm. 16.7.6, Prop. 16.7.11, and Thm. 16.7.14]. \square

7.2. Homotopy cotensors.

Definition 7.3. If \mathcal{M} is a model category, X is an object of \mathcal{M} , $\widehat{\mathbf{X}}$ is a simplicial frame on X , and K is a simplicial set, then the *homotopy cotensor* $\widehat{\mathbf{X}}^K$ is defined to be the object of \mathcal{M} that is the limit of the $(\Delta K)^{\text{op}}$ -diagram in \mathcal{M} (see Lemma 3.8) that takes the object $\Delta[n] \rightarrow K$ of $(\Delta K)^{\text{op}} = (\Delta \downarrow K)^{\text{op}}$ to $\widehat{\mathbf{X}}_n$ and takes the

commutative triangle

$$\begin{array}{ccc} \Delta[n] & \xrightarrow{\alpha} & \Delta[k] \\ & \searrow & \swarrow \\ & K & \end{array}$$

to the map $\alpha^*: \widehat{\mathbf{X}}_k \rightarrow \widehat{\mathbf{X}}_n$ (see [6, Def. 16.3.1]).

Proposition 7.4. *If \mathcal{M} is a simplicial model category, X is an object of \mathcal{M} , $\widehat{\mathbf{X}}$ is the standard simplicial frame on X (see Definition 7.1), and K is a simplicial set, then $\widehat{\mathbf{X}}^K$ is naturally isomorphic to X^K .*

Proof. See [6, Prop. 16.6.6]. □

7.3. Homotopy function complexes. Although homotopy function complexes have many important properties (see [6, Chap. 17]), our only use for them here is the adjointness result Theorem 7.8, which will be used in the proofs of Theorem 8.10 and Theorem 9.8.

Definition 7.5. Let \mathcal{M} be a model category and let W be an object of \mathcal{M} .

- (1) If X is an object of \mathcal{M} and $\widehat{\mathbf{X}}$ is a simplicial frame on X , then $\text{map}_{\widehat{\mathbf{X}}}(W, X)$ will denote the simplicial set, natural in both W and $\widehat{\mathbf{X}}$, defined by

$$\text{map}_{\widehat{\mathbf{X}}}(W, X)_n = \mathcal{M}(W, \widehat{\mathbf{X}}_n)$$

with face and degeneracy maps induced by those in $\widehat{\mathbf{X}}$. If W is cofibrant and X is fibrant, then $\text{map}_{\widehat{\mathbf{X}}}(W, X)$ is a right homotopy function complex from W to X (see [6, Def. 17.2.1]).

- (2) If \mathcal{C} is a small category, \mathbf{X} is a \mathcal{C} -diagram in \mathcal{M} , and $\widehat{\mathbf{X}}$ is a simplicial frame on \mathbf{X} , then $\text{map}_{\widehat{\mathbf{X}}}(W, \mathbf{X})$ will denote the \mathcal{C} -diagram of simplicial sets that on an object α of \mathcal{C} is the simplicial set $\text{map}_{\widehat{\mathbf{X}}}(W, \mathbf{X}_\alpha)$.

Proposition 7.6. *If \mathcal{M} is a simplicial model category, W and X are objects of \mathcal{M} , and $\widehat{\mathbf{X}}$ is the standard simplicial frame on X , then $\text{map}_{\widehat{\mathbf{X}}}(W, X)$ is naturally isomorphic to $\text{Map}(W, X)$, the simplicial set of maps that is part of the structure of the simplicial model category \mathcal{M} .*

Proof. We have natural isomorphisms

$$\begin{aligned} \text{map}_{\widehat{\mathbf{X}}}(W, X)_n &= \mathcal{M}(W, \widehat{\mathbf{X}}_n) \\ &= \mathcal{M}(W, X^{\Delta[n]}) \\ &\approx \text{SS}(\Delta[n], \text{Map}(W, X)) \\ &\approx \text{Map}(W, X)_n . \end{aligned} \quad \square$$

Proposition 7.7. *Let \mathcal{M} be a model category and let W be an object of \mathcal{M} . If X is an object of \mathcal{M} , $\widehat{\mathbf{X}}$ is a simplicial resolution of X , and K is a simplicial set, then there is a natural isomorphism of sets*

$$\text{SS}(K, \text{map}_{\widehat{\mathbf{X}}}(W, X)) \approx \mathcal{M}(W, \widehat{\mathbf{X}}^K) .$$

Proof. Since $\widehat{\mathbf{X}}^K$ is defined as a limit (see Definition 7.3), an element of $\mathcal{M}(W, \widehat{\mathbf{X}}^K)$ is a collection of maps $W \rightarrow \widehat{\mathbf{X}}_n$, one for each n -simplex of K , that commute with

the face and degeneracy operators. This is also a description of an element of $\text{SS}(K, \text{map}_{\widehat{\mathbf{X}}}(W, X))$ (see also [6, Thm. 16.4.2]). \square

Theorem 7.8. *Let \mathcal{M} be a model category and let \mathcal{C} be a small category. If \mathbf{X} is a \mathcal{C} -diagram in \mathcal{M} , $\widehat{\mathbf{X}}$ is a simplicial frame on \mathbf{X} , \mathbf{K} is a \mathcal{C} -diagram of simplicial sets, and W is an object of \mathcal{M} , then there is a natural isomorphism of sets*

$$\mathcal{M}(W, \text{hom}_{\widehat{\mathbf{X}}}^{\mathcal{C}}(\mathbf{K}, \mathbf{X})) \approx \text{SS}^{\mathcal{C}}(\mathbf{K}, \text{map}_{\widehat{\mathbf{X}}}(W, \mathbf{X}))$$

(where $\text{map}_{\widehat{\mathbf{X}}}(W, \mathbf{X})$ is as in Definition 7.5 and $\text{hom}_{\widehat{\mathbf{X}}}^{\mathcal{C}}(\mathbf{K}, \mathbf{X})$ is the end of the functor $\widehat{\mathbf{X}}^{\mathbf{K}}: \mathcal{C} \times \mathcal{C}^{\text{op}} \rightarrow \mathcal{M}$; see [6, Def. 19.2.2]).

Proof. The object $\text{hom}_{\widehat{\mathbf{X}}}^{\mathcal{C}}(\mathbf{K}, \mathbf{X})$ is defined (see [6, Def. 19.2.2]) as the limit of the diagram

$$\prod_{\alpha \in \text{Ob}(\mathcal{C})} (\widehat{\mathbf{X}}_{\alpha})^{\mathbf{K}_{\alpha}} \begin{array}{c} \xrightarrow{\phi} \\ \xrightarrow{\psi} \end{array} \prod_{(\sigma: \alpha \rightarrow \alpha') \in \mathcal{C}} (\widehat{\mathbf{X}}_{\alpha'})^{\mathbf{K}_{\alpha}}$$

where the projection of the map ϕ on the factor indexed by $\sigma: \alpha \rightarrow \alpha'$ is the composition of a natural projection from the product with the map

$$\sigma_*^{1_{\mathbf{K}_{\alpha}}}: (\widehat{\mathbf{X}}_{\alpha})^{\mathbf{K}_{\alpha}} \longrightarrow (\widehat{\mathbf{X}}_{\alpha'})^{\mathbf{K}_{\alpha}}$$

(where $\sigma_*: \widehat{\mathbf{X}}_{\alpha} \rightarrow \widehat{\mathbf{X}}_{\alpha'}$) and the projection of the map ψ on the factor indexed by $\sigma: \alpha \rightarrow \alpha'$ is the composition of a natural projection from the product with the map

$$(1_{\widehat{\mathbf{X}}_{\alpha'}})^{\mathbf{K}_{\sigma_*}}: (\widehat{\mathbf{X}}_{\alpha'})^{\mathbf{K}_{\alpha'}} \longrightarrow (\widehat{\mathbf{X}}_{\alpha'})^{\mathbf{K}_{\alpha}}$$

(where $\sigma_*: \mathbf{K}_{\alpha} \rightarrow \mathbf{K}_{\alpha'}$), and so $\mathcal{M}(W, \text{hom}_{\widehat{\mathbf{X}}}^{\mathcal{C}}(\mathbf{K}, \mathbf{X}))$ is naturally isomorphic to the limit of the diagram

$$\prod_{\alpha \in \text{Ob}(\mathcal{C})} \mathcal{M}(W, (\widehat{\mathbf{X}}_{\alpha})^{\mathbf{K}_{\alpha}}) \begin{array}{c} \xrightarrow{\phi_*} \\ \xrightarrow{\psi_*} \end{array} \prod_{(\sigma: \alpha \rightarrow \alpha') \in \mathcal{C}} \mathcal{M}(W, (\widehat{\mathbf{X}}_{\alpha'})^{\mathbf{K}_{\alpha}}).$$

This is naturally isomorphic to the limit of the diagram

$$\prod_{\alpha \in \text{Ob}(\mathcal{C})} \text{SS}(\mathbf{K}_{\alpha}, \text{map}_{\widehat{\mathbf{X}}}(W, \widehat{\mathbf{X}}_{\alpha})) \begin{array}{c} \xrightarrow{\hat{\phi}_*} \\ \xrightarrow{\hat{\psi}_*} \end{array} \prod_{(\sigma: \alpha \rightarrow \alpha') \in \mathcal{C}} \text{SS}(\mathbf{K}_{\alpha}, \text{map}_{\widehat{\mathbf{X}}}(W, \widehat{\mathbf{X}}_{\alpha'}))$$

(see Proposition 7.7) which is the definition of $\text{SS}^{\mathcal{C}}(\mathbf{K}, \text{map}_{\widehat{\mathbf{X}}}(W, \mathbf{X}))$. \square

8. TOTAL OBJECTS

We define the total object of a cosimplicial object in Definition 8.1, the total object of a multicosimplicial object in Definition 8.5, and show in Theorem 8.10 that the total object of a multicosimplicial object is isomorphic to the total object of its diagonal cosimplicial object.

8.1. The total object of a cosimplicial object.

Definition 8.1. If \mathcal{M} is a model category, \mathbf{X} is a cosimplicial object in \mathcal{M} , and $\widehat{\mathbf{X}}$ is a Reedy simplicial frame on \mathbf{X} (see Definition 7.1), then the *total object* $\text{Tot } \mathbf{X}$ of \mathbf{X} is the object of \mathcal{M} that is the end (see [6, Def. 18.3.2] or [7, pages 218–223] or [3, page 329]) of the functor $\widehat{\mathbf{X}}^\Delta: \Delta \times \Delta^{\text{op}} \rightarrow \mathcal{M}$. This is a subobject of the product

$$\prod_{k \geq 0} (\widehat{\mathbf{X}}^k)^{\Delta[k]}$$

and is denoted $\text{hom}_{\widehat{\mathbf{X}}}^\Delta(\Delta, \mathbf{X})$ in [6, Def. 19.2.2] and $\int_{[k]} (\widehat{\mathbf{X}}^k)^{\Delta[k]}$ in [7, pages 218–223].

Proposition 8.2. *If \mathcal{M} is a model category, \mathbf{X} is a Reedy fibrant cosimplicial object in \mathcal{M} , and $\widehat{\mathbf{X}}$ and $\widehat{\mathbf{X}}'$ are two Reedy simplicial frames on \mathbf{X} , then there is an essentially unique zig-zag of weak equivalences connecting $\text{Tot } \mathbf{X}$ defined using $\widehat{\mathbf{X}}$ and $\text{Tot } \mathbf{X}$ defined using $\widehat{\mathbf{X}}'$.*

Proof. This follows from Proposition 7.2 and [6, Cor. 19.7.4]. \square

Example 8.3. If \mathcal{M} is the category of simplicial sets, \mathbf{X} is a cosimplicial object in \mathcal{M} , and $\widehat{\mathbf{X}}$ is the standard simplicial frame on \mathbf{X} , then $\text{Tot } \mathbf{X}$ is the simplicial set of maps of cosimplicial simplicial sets from Δ to \mathbf{X} , i.e., a subset of the product simplicial set $\prod_{k \geq 0} (\mathbf{X}^k)^{\Delta[k]}$.

If \mathcal{M} is the category of topological spaces, \mathbf{X} is a cosimplicial object in \mathcal{M} , and $\widehat{\mathbf{X}}$ is the standard simplicial frame on \mathbf{X} , then $\text{Tot } \mathbf{X}$ is the topological space of maps of cosimplicial spaces from $|\Delta|$ to \mathbf{X} , i.e., a subset of the product space $\prod_{k \geq 0} (\mathbf{X}^k)^{|\Delta[k]|}$.

Proposition 8.4. *If \mathcal{M} is a model category and \mathbf{X} is a Reedy fibrant cosimplicial object in \mathcal{M} , and $\widehat{\mathbf{X}}$ is a Reedy simplicial frame on $\widehat{\mathbf{X}}$, then $\text{Tot } \mathbf{X}$ is a fibrant object of \mathcal{M} .*

Proof. See [6, Thm. 19.8.2]. \square

8.2. The total object of a multicosimplicial object.

Definition 8.5. If n is a positive integer, \mathcal{M} is a model category, \mathbf{X} is an n -cosimplicial object in \mathcal{M} , and $\widehat{\mathbf{X}}$ is a Reedy simplicial frame on \mathbf{X} (see Definition 7.1), then the *total object* $\text{Tot } \mathbf{X}$ of \mathbf{X} is the object of \mathcal{M} that is the end (see [6, Def. 18.3.2], [7, pages 218–223], or [3, page 329]) of the functor $\widehat{\mathbf{X}}^{\Delta^{(n)}}: \Delta^n \times (\Delta^n)^{\text{op}} \rightarrow \mathcal{M}$. This is a subobject of the product

$$\prod_{k_1 \geq 0, k_2 \geq 0, \dots, k_n \geq 0} (\widehat{\mathbf{X}}^{(k_1, k_2, \dots, k_n)})^{\Delta[k_1] \times \Delta[k_2] \times \dots \times \Delta[k_n]}$$

and is denoted $\text{hom}_{\widehat{\mathbf{X}}}^{\Delta^n}(\Delta^{(n)}, \mathbf{X})$ in [6, Def. 19.2.2] and

$$\int_{([k_1], [k_2], \dots, [k_n])} (\widehat{\mathbf{X}}^{(k_1, k_2, \dots, k_n)})^{\Delta[k_1] \times \Delta[k_2] \times \dots \times \Delta[k_n]}$$

in [7, pages 218–223].

Proposition 8.6. *If \mathcal{M} is a model category, \mathbf{X} is a Reedy fibrant multic simplicial object in \mathcal{M} , and $\widehat{\mathbf{X}}$ and $\widehat{\mathbf{X}}'$ are two Reedy simplicial frames on \mathbf{X} , then there is an essentially unique zig-zag of weak equivalences connecting $\text{Tot } \mathbf{X}$ defined using $\widehat{\mathbf{X}}$ and $\text{Tot } \mathbf{X}$ defined using $\widehat{\mathbf{X}}'$.*

Proof. This follows from Proposition 7.2 and [6, Cor. 19.7.4]. \square

Example 8.7. If n is a positive integer, \mathcal{M} is the category of simplicial sets, \mathbf{X} is an n -cosimplicial object in \mathcal{M} , and $\widehat{\mathbf{X}}$ is the standard simplicial frame on \mathbf{X} , then $\text{Tot } \mathbf{X}$ is the simplicial set of maps of n -cosimplicial simplicial sets from $\Delta^{(n)}$ to \mathbf{X} , i.e., a subset of the product simplicial set

$$\prod_{([k_1], [k_2], \dots, [k_n])} (\mathbf{X}^{(k_1, k_2, \dots, k_n)})_{(\Delta[k_1] \times \Delta[k_2] \times \dots \times \Delta[k_n])} .$$

If n is a positive integer, \mathcal{M} is the category of topological spaces, \mathbf{X} is an n -cosimplicial object in \mathcal{M} , and $\widehat{\mathbf{X}}$ is the standard simplicial frame on \mathbf{X} , then $\text{Tot } \mathbf{X}$ is the topological space of maps of n -cosimplicial spaces from $|\Delta^{(n)}|$ to \mathbf{X} , i.e., a subspace of the product space $\prod_{([k_1], [k_2], \dots, [k_n])} (\mathbf{X}^{(k_1, k_2, \dots, k_n)})_{(|\Delta[k_1] \times \Delta[k_2] \times \dots \times \Delta[k_n]|)}$.

Proposition 8.8. *If n is a positive integer, \mathcal{M} is a model category, \mathbf{X} is a Reedy fibrant n -cosimplicial object in \mathcal{M} , and $\widehat{\mathbf{X}}$ is a Reedy simplicial frame on \mathbf{X} , then $\text{Tot } \mathbf{X}$ is a fibrant object of \mathcal{M} .*

Proof. Since $\Delta^{(n)}$ is Reedy cofibrant, this follows from [6, Cor. 19.7.3]. \square

8.3. The total object of the diagonal. In Theorem 8.10 we use Theorem 3.9 to show that the total object of an n -cosimplicial object in an arbitrary model category is isomorphic to the total object of its diagonal cosimplicial object (see also [5, Remark on p. 172] and [9, Prop. 8.1]).

Proposition 8.9. *If n is a positive integer, \mathcal{M} is a model category, \mathbf{X} is an n -cosimplicial object in \mathcal{M} , and $\widehat{\mathbf{X}}$ is a Reedy simplicial frame on \mathbf{X} , then $\text{diag } \widehat{\mathbf{X}}$ is a Reedy simplicial frame on $\text{diag } \mathbf{X}$.*

Proof. Since a Reedy simplicial frame on a Reedy fibrant n -cosimplicial object is a Reedy fibrant diagram in $(\mathcal{M}^{\Delta^n})^{\Delta^{\text{op}}} \approx (\mathcal{M}^{\Delta^{\text{op}}})^{\Delta^n}$, and the different Reedy model category structures on that category coincide (see [6, Thm. 15.5.2]), this follows from Corollary 3.4. \square

Theorem 8.10. *If n is a positive integer, \mathcal{M} is a model category, \mathbf{X} is an n -cosimplicial object in \mathcal{M} , and $\widehat{\mathbf{X}}$ is a Reedy simplicial frame on \mathbf{X} , then $\text{diag } \widehat{\mathbf{X}}$ is a Reedy simplicial frame on $\text{diag } \mathbf{X}$ (see Proposition 8.9) (which, by abuse of notation, we will also denote by $\widehat{\mathbf{X}}$) and there is a natural isomorphism*

$$\text{Tot } \mathbf{X} \approx \text{Tot}(\text{diag } \mathbf{X})$$

from the total object of \mathbf{X} to the total object of the diagonal cosimplicial object of \mathbf{X} .

Proof. For every object W of \mathcal{M} there are natural isomorphisms of sets

$$\begin{aligned}
\mathcal{M}(W, \text{Tot } \mathbf{X}) &= \mathcal{M}(W, \text{hom}_{\widehat{\mathbf{X}}}^{\Delta^n}(\Delta^{(n)}, \mathbf{X})) \\
&\approx \text{SS}^{\Delta^n}(\Delta^{(n)}, \text{map}_{\widehat{\mathbf{X}}}(W, \mathbf{X})) && \text{(see Theorem 7.8)} \\
&\approx \text{SS}^{\Delta}(\Delta, \text{diag map}_{\widehat{\mathbf{X}}}(W, \mathbf{X})) && \text{(see Theorem 3.9)} \\
&\approx \text{SS}^{\Delta}(\Delta, \text{map}_{\widehat{\mathbf{X}}}(W, \text{diag } \mathbf{X})) \\
&\approx \mathcal{M}(W, \text{hom}_{\widehat{\mathbf{X}}}^{\Delta}(\Delta, \text{diag } \mathbf{X})) && \text{(see Theorem 7.8)} \\
&= \mathcal{M}(W, \text{Tot}(\text{diag } \mathbf{X}))
\end{aligned}$$

and the Yoneda lemma implies that the composition of those is induced by a natural isomorphism $\text{Tot } \mathbf{X} \approx \text{Tot}(\text{diag } \mathbf{X})$. \square

9. HOMOTOPY LIMITS AND TOTAL OBJECTS

We show that the Bousfield-Kan map from the total object of a multicosimplicial object to its homotopy limit is a weak equivalence for Reedy fibrant multicosimplicial objects. We also show that this behaves well with respect to passing to the diagonal cosimplicial object of a multicosimplicial object.

9.1. Cosimplicial objects.

Definition 9.1. The *Bousfield-Kan map* is the map of cosimplicial simplicial sets $\phi: \mathbf{B}(\Delta \downarrow -) \rightarrow \Delta$ that for $k \geq 0$ and $n \geq 0$ takes the n -simplex

$$\left(([i_0] \xrightarrow{\sigma_0} [i_1] \xrightarrow{\sigma_1} \dots \xrightarrow{\sigma_{n-1}} [i_n]), \tau: [i_n] \rightarrow [k] \right)$$

of $\mathbf{B}(\Delta \downarrow [k])$ to the n -simplex

$$[\tau\sigma_{n-1}\sigma_{n-2}\dots\sigma_0(i_0), \tau\sigma_{n-1}\sigma_{n-2}\dots\sigma_1(i_1), \dots, \tau\sigma_{n-1}(i_{n-1}), \tau(i_n)]$$

of $\Delta[k]$ (see [6, Def. 18.7.1]). It is a weak equivalence of Reedy cofibrant cosimplicial simplicial sets (see [6, Prop. 18.7.2]).

Theorem 9.2. *If \mathcal{M} is a simplicial model category, \mathbf{X} is a Reedy fibrant cosimplicial object in \mathcal{M} , and $\widehat{\mathbf{X}}$ is a Reedy simplicial frame on \mathbf{X} , then the map*

$$\text{Tot } \mathbf{X} \approx \text{hom}_{\widehat{\mathbf{X}}}^{\Delta}(\Delta, \mathbf{X}) \xrightarrow{\text{hom}_{\widehat{\mathbf{X}}}^{\Delta}(\phi, 1_{\mathbf{X}})} \text{hom}_{\widehat{\mathbf{X}}}^{\Delta}(\mathbf{B}(\Delta \downarrow -), \mathbf{X}) \approx \text{holim } \mathbf{X}$$

(where $\phi: \mathbf{B}(\Delta \downarrow -) \rightarrow \Delta$ is the Bousfield-Kan map of cosimplicial simplicial sets; see Definition 9.1) is a natural weak equivalence of fibrant objects $\text{Tot } \mathbf{X} \cong \text{holim } \mathbf{X}$, which we will also call the Bousfield-Kan map.

Proof. Since the Bousfield-Kan map of cosimplicial simplicial sets is a weak equivalence of Reedy cofibrant cosimplicial sets, this follows from [6, Cor. 19.7.5]. \square

9.2. Multicosimplicial objects.

Lemma 9.3. *Let n be a positive integer, let \mathcal{C}_i be a small category for $1 \leq i \leq n$, and let $\mathcal{C} = \prod_{1 \leq i \leq n} \mathcal{C}_i$. If $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is an object of \mathcal{C} , then the overcategory $(\mathcal{C} \downarrow \alpha) \approx \prod_{1 \leq i \leq n} (\mathcal{C}_i \downarrow \alpha_i)$, and its classifying space (or nerve) is $\mathbf{B}(\mathcal{C} \downarrow \alpha) \approx \prod_{1 \leq i \leq n} \mathbf{B}(\mathcal{C}_i \downarrow \alpha_i)$.*

Proof. This follows directly from the definitions. \square

Definition 9.4. If n is a positive integer, then the *product Bousfield-Kan map* of n -cosimplicial simplicial sets $\phi^n: \mathbf{B}(\Delta^n \downarrow -) \rightarrow \Delta^{(n)}$ is the composition

$$\mathbf{B}(\Delta^n \downarrow -) \approx \prod_{1 \leq i \leq n} \mathbf{B}(\Delta \downarrow -) \xrightarrow{\phi^n} \Delta^{(n)}$$

(see Lemma 9.3) where ϕ is the Bousfield-Kan map of cosimplicial simplicial sets (see Definition 9.1).

Theorem 9.5. *If n is a positive integer, \mathcal{M} is a simplicial model category, \mathbf{X} is a Reedy fibrant n -cosimplicial object in \mathcal{M} , and $\widehat{\mathbf{X}}$ is a Reedy simplicial frame on \mathbf{X} , then the map*

$$\mathrm{Tot} \mathbf{X} \approx \mathrm{hom}_{\widehat{\mathbf{X}}}^{\Delta^n}(\Delta^{(n)}, \mathbf{X}) \xrightarrow{\mathrm{hom}_{\widehat{\mathbf{X}}}^{\Delta^n}(\phi^n, 1_{\mathbf{X}})} \mathrm{hom}_{\widehat{\mathbf{X}}}^{\Delta^n}(\mathbf{B}(\Delta^n \downarrow -), \mathbf{X}) \approx \mathrm{holim} \mathbf{X}$$

(where $\phi^n: \mathbf{B}(\Delta^n \downarrow -) \rightarrow \Delta^{(n)}$ is the product Bousfield-Kan map of n -cosimplicial simplicial sets; see Definition 9.4) is a natural weak equivalence of fibrant objects $\mathrm{Tot} \mathbf{X} \cong \mathrm{holim} \mathbf{X}$, which we will also call the product Bousfield-Kan map.

Proof. Since the product Bousfield-Kan map of n -cosimplicial simplicial sets is a weak equivalence of Reedy cofibrant n -cosimplicial sets (see Lemma 3.7), this follows from [6, Cor. 19.7.5]. \square

9.3. The homotopy limit and total object of the diagonal. We first show that for an objectwise fibrant multicosplicial object the canonical map from the homotopy limit to the homotopy limit of the diagonal cosimplicial object is a weak equivalence, and then we show that all the maps we've defined between the homotopy limits and total objects of a multicosplicial object and its diagonal cosimplicial object commute.

Proposition 9.6. *If n is a positive integer, then the diagonal embedding $D: \Delta \rightarrow \Delta^n$ (see Definition 2.5) is homotopy left cofinal (see [6, Def. 19.6.1]).*

Proof. For an object $([p_1], [p_2], \dots, [p_n])$ of Δ^n , an object of $(\Delta \downarrow ([p_1], [p_2], \dots, [p_n]))$ is a map

$$(\alpha_1, \alpha_2, \dots, \alpha_n): ([k], [k], \dots, [k]) \longrightarrow ([p_1], [p_2], \dots, [p_n])$$

in Δ^n , where each $\alpha_i: [k] \rightarrow [p_i]$ is a map in Δ , i.e., a k -simplex of $\Delta[p_i]$. Thus, $(\alpha_1, \alpha_2, \dots, \alpha_n)$ is a k -simplex of $\Delta[p_1] \times \Delta[p_2] \times \dots \times \Delta[p_n]$, and the category $(\Delta \downarrow ([p_1], [p_2], \dots, [p_n]))$ is the category of simplices of $\Delta[p_1] \times \Delta[p_2] \times \dots \times \Delta[p_n]$ (see Lemma 3.8). Since the nerve of the category of simplices of a simplicial set is weakly equivalent to that simplicial set (see [6, Thm. 18.9.3]), $\mathbf{B}(\Delta \downarrow ([p_1], [p_2], \dots, [p_n]))$ is weakly equivalent to $\Delta[p_1] \times \Delta[p_2] \times \dots \times \Delta[p_n]$, and is thus contractible. \square

Theorem 9.7. *If n is a positive integer, \mathcal{M} is a model category, and \mathbf{X} is an objectwise fibrant n -cosimplicial object in \mathcal{M} , then the natural map $\mathrm{holim}_{\Delta^n} \mathbf{X} \rightarrow \mathrm{holim}_{\Delta} \mathrm{diag} \mathbf{X}$ induced by the diagonal embedding $D: \Delta \rightarrow \Delta^n$ is a weak equivalence.*

Proof. This follows from Proposition 9.6 and [6, Thm. 19.6.7]. \square

Theorem 9.8. *If n is a positive integer, \mathcal{M} is a model category, \mathbf{X} is an n -cosimplicial object in \mathcal{M} , and $\widehat{\mathbf{X}}$ is a Reedy simplicial frame on \mathbf{X} , then the diagram*

$$\begin{array}{ccc} \text{Tot } \mathbf{X} & \longrightarrow & \text{holim}_{\Delta^n} \mathbf{X} \\ \downarrow & & \downarrow \\ \text{Tot diag } \mathbf{X} & \longrightarrow & \text{holim}_{\Delta} \text{diag } \mathbf{X} \end{array}$$

(where the upper horizontal map is the product Bousfield-Kan map (see Theorem 9.5), the lower horizontal map is the Bousfield-Kan map (see Theorem 9.2), the left vertical map is the isomorphism of Theorem 8.10, and the right vertical map is the natural map induced by the diagonal embedding $D: \Delta \rightarrow \Delta^n$ (see [6, Prop. 19.1.8])) commutes. If \mathbf{X} is objectwise fibrant, then the vertical maps in that diagram are weak equivalences. If \mathbf{X} is Reedy fibrant, then all of the maps in that diagram are weak equivalences.

Proof. It is sufficient to show that if W is an object of \mathcal{M} , then the diagram

$$\begin{array}{ccc} \mathcal{M}(W, \text{Tot } \mathbf{X}) & \longrightarrow & \mathcal{M}(W, \text{holim}_{\Delta^n} \mathbf{X}) \\ \downarrow & & \downarrow \\ \mathcal{M}(W, \text{Tot diag } \mathbf{X}) & \longrightarrow & \mathcal{M}(W, \text{holim}_{\Delta} \text{diag } \mathbf{X}) \end{array}$$

commutes. Theorem 7.8 gives us natural isomorphisms

$$\begin{aligned} \mathcal{M}(W, \text{Tot } \mathbf{X}) &= \mathcal{M}(W, \text{hom}_{\widehat{\mathbf{X}}}^{\Delta^n}(\Delta^{(n)}, \mathbf{X})) \\ &\approx \text{SS}^{\Delta^n}(\Delta^{(n)}, \text{map}_{\widehat{\mathbf{X}}}(W, \mathbf{X})) \end{aligned}$$

$$\begin{aligned} \mathcal{M}(W, \text{holim } \mathbf{X}) &= \mathcal{M}(W, \text{hom}_{\widehat{\mathbf{X}}}^{\Delta^n}(\text{B}(\Delta^n \downarrow -), \mathbf{X})) \\ &\approx \text{SS}^{\Delta^n}(\text{B}(\Delta^n \downarrow -), \text{map}_{\widehat{\mathbf{X}}}(W, \mathbf{X})) \end{aligned}$$

$$\begin{aligned} \mathcal{M}(W, \text{Tot diag } \mathbf{X}) &= \mathcal{M}(W, \text{hom}_{\widehat{\mathbf{X}}}^{\Delta}(\Delta, \text{diag } \mathbf{X})) \\ &\approx \text{SS}^{\Delta}(\Delta, \text{map}_{\widehat{\mathbf{X}}}(W, \text{diag } \mathbf{X})) \end{aligned}$$

$$\begin{aligned} \mathcal{M}(W, \text{holim diag } \mathbf{X}) &= \mathcal{M}(W, \text{hom}_{\widehat{\mathbf{X}}}^{\Delta}(\text{B}(\Delta \downarrow -), \text{diag } \mathbf{X})) \\ &\approx \text{SS}^{\Delta}(\text{B}(\Delta \downarrow -), \text{map}_{\widehat{\mathbf{X}}}(W, \text{diag } \mathbf{X})) \end{aligned}$$

and so this is equivalent to showing that the diagram

$$\begin{array}{ccc} \text{SS}^{\Delta^n}(\Delta^{(n)}, \text{map}_{\widehat{\mathbf{X}}}(W, \mathbf{X})) & \longrightarrow & \text{SS}^{\Delta^n}(\text{B}(\Delta^n \downarrow -), \text{map}_{\widehat{\mathbf{X}}}(W, \mathbf{X})) \\ \downarrow & & \downarrow \\ \text{SS}^{\Delta}(\Delta, \text{map}_{\widehat{\mathbf{X}}}(W, \text{diag } \mathbf{X})) & \longrightarrow & \text{SS}^{\Delta}(\text{B}(\Delta \downarrow -), \text{map}_{\widehat{\mathbf{X}}}(W, \text{diag } \mathbf{X})) \end{array}$$

commutes. If $f \in \text{SS}^{\Delta^n}(\Delta^{(n)}, \text{map}_{\widehat{\mathbf{X}}}(W, \mathbf{X}))$, then the image of f under the counterclockwise composition is the composition of

$$\text{B}(\Delta \downarrow -) \xrightarrow{\phi} \Delta \xrightarrow{\alpha} \text{diag}(\Delta^{(n)}) \xrightarrow{\text{diag } f} \text{diag } \mathbf{X}$$

(where ϕ is the Bousfield-Kan map of Definition 9.1 and α is as in Theorem 3.9) and the image of f under the clockwise composition is

$$\mathbf{B}(\Delta \downarrow -) \xrightarrow{D_*} \text{diag}(\mathbf{B}(\Delta^n \downarrow -)) \xrightarrow{\phi^n} \text{diag}(\Delta^{(n)}) \xrightarrow{\text{diag } f} \text{diag } \mathbf{X}$$

(where D_* is the map induced by the diagonal embedding $D: \Delta \rightarrow \Delta^n$ and ϕ^n is the product Bousfield-Kan map of Definition 9.4). Since the diagram

$$\begin{array}{ccc} \mathbf{B}(\Delta \downarrow -) & \xrightarrow{D_*} & \text{diag } \mathbf{B}(\Delta^n \downarrow -) \\ \phi \downarrow & & \downarrow \text{diag } \phi^n \\ \Delta & \xrightarrow{\alpha} & \text{diag } \Delta^{(n)} \end{array}$$

commutes, these two compositions are equal, and so our diagram commutes. \square

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