ERRATA FOR
MODEL CATEGORIES AND THEIR LOCALIZATIONS

PHILIP S. HIRSCHHORN

Contents

1. Proposition 4.2.5

Proposition 4.2.5 is incorrect. It was correct for early drafts of the book, in which the definition of a cellular model category required that the domains of the elements of the set J of generating trivial cofibrations be cofibrant. The current definition of a cellular model category, however, requires only that the domains of the elements of J be small with respect to I (see Definition 12.1.1).

To correct the error:

- In the third line of the statement of Proposition 4.2.5, change "cofibrant domains" to "domains that are small relative to I",

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- in the proof of Proposition 4.2.5, in lines 4–5 of page 74, change "with cofibrant domain" to "whose domain is small relative to I", and
- in the proof of Theorem 4.3.1 on page 75, in the first line of that proof, change "Theorem 12.4.3" to "Proposition 11.2.3".

2. Theorem 7.6.5

Part 3 of Theorem 7.6.5 is incorrect as written. That part claims that if A and B are objects of a model category $\mathcal{M}$, then the category $(A\downarrow\mathcal{M}\downarrow B)$ of objects of $\mathcal{M}$ under A and over B is a model category. The problem with that statement is that $(A\downarrow\mathcal{M}\downarrow B)$ is usually neither complete nor cocomplete. This is because if you have two objects of $(A\downarrow\mathcal{M}\downarrow B)$, $A \xrightarrow{s} X \xrightarrow{t} B$ and $A \xrightarrow{u} Y \xrightarrow{v} B$, there can be no morphisms between them unless $ts = vu$.

The correction is to choose a fixed map $f\colon A \to B$ in $\mathcal{M}$ and to let $(A\downarrow\mathcal{M}\downarrow B)_f$ be the full subcategory of $(A\downarrow\mathcal{M}\downarrow B)$ with objects the diagrams $A \xrightarrow{s} X \xrightarrow{t} B$ such that $ts = f$ (that is, $(A\downarrow\mathcal{M}\downarrow B)_f$ is the category of factorizations of f). To form the limit of a diagram in $(A\downarrow\mathcal{M}\downarrow B)_f$, you form its limit in the category $(\mathcal{M}\downarrow B)$; there is then a natural map from A to that limit, and this forms the limit in $(A\downarrow\mathcal{M}\downarrow B)_f$. Dually, to form the colimit of a diagram in $(A\downarrow\mathcal{M}\downarrow B)_f$, you form its colimit in $(A\downarrow\mathcal{M})$; there is then a natural map from that colimit to B, and this forms the colimit in $(A\downarrow\mathcal{M}\downarrow B)_f$.

The corrected statement of part 3 is then

(3) For every map $f\colon A \to B$ in $\mathcal{M}$ the full subcategory $(A\downarrow\mathcal{M}\downarrow B)_f$ of $(A\downarrow\mathcal{M}\downarrow B)$ with objects equal to the diagrams $A \xrightarrow{s} X \xrightarrow{t} B$ for which $ts = f$ is a model category in which a map is a weak equivalence, fibration, or cofibration if it is one in $\mathcal{M}$. This is the *model category of factorizations of* f.

There are several theorems and proofs that make reference to part 3 of Theorem 7.6.5, and those references need to be rephrased to take this change into account:

2.1. **Proof of Proposition 3.3.15.** In the fifth line from the bottom of page 62, the sentence that begins "Thus, in the category $(A\downarrow\mathcal{M}\downarrow Z)$ of objects of $\mathcal{M}$ under A and over Z" should instead begin "Thus, in the category $(A\downarrow\mathcal{M}\downarrow Z)_{ui}$ of factorizations of $ui : A \to Z$".

2.2. **Statement and proof of Proposition 7.6.13.**

- In both of the diagrams (one in the statement and one in the proof), the map from B to Y should be labelled "s".
- The last line of the statement of the proposition is "as maps in $(A\downarrow\mathcal{M}\downarrow Y)$, the category of objects of $\mathcal{M}$ under A and over Y.", and that should be changed to "as maps in $(A\downarrow\mathcal{M}\downarrow Y)_{si}$, the category of factorizations of $si\colon A \to Y$.".
- In the part of the proof that follows the diagram displayed in the proof, the sentence that begins on the second line begins "In the category $(A\downarrow\mathcal{M}\downarrow Y)$ of objects of $\mathcal{M}$ under A and over Y", and that should be changed to "In the category $(A\downarrow\mathcal{M}\downarrow Y)_{si}$ of factorizations of $si\colon A \to Y$".
- In the last two lines of that proof there are two appearances of "$(A\downarrow\mathcal{M}\downarrow Y)$", and both of those should be changed to "$(A\downarrow\mathcal{M}\downarrow Y)_{si}$".

2.3. **Statement of Proposition 7.6.14.** There are two appearances of "$(A\downarrow\mathcal{M}\downarrow Y)$", and they should both be changed to "$(A\downarrow\mathcal{M}\downarrow Y)_{qi}$".

2.4. **Proof of Proposition 13.2.1.** In both of the diagrams in that proof (on page 243), the map $B \to Y$ should be labelled as "s". The third line from the bottom of page 243 begins "The category of objects under A and over Y", and that should be changed to "The category $(A\downarrow\mathcal{M}\downarrow Y)_{sg}$ of factorizations of $sg\colon A \to Y$".

3. Proposition 10.2.6

The fourth itemized point in the statement of Proposition 10.2.6 should be

- for every $\beta \le \gamma$ the map $\operatorname{colim}_{\alpha<\beta} X_\alpha \to X_\beta$ is an element of $\mathcal{D}$,

4. Theorem 14.5.4

The proof of Theorem 14.5.4 has two errors.

1. The object $(\mathrm{F}\boldsymbol{X}, p_{\boldsymbol{X}})$ may not be a terminal object of $\widetilde{\mathcal{D}}$ because there may be distinct objects $(\widetilde{\boldsymbol{X}}, i)$ and $(\widetilde{\boldsymbol{X}}', i')$ of $\mathcal{D}$ such that $(\mathrm{F}\widetilde{\boldsymbol{X}}, i\circ p_{\widetilde{\boldsymbol{X}}})$ and $(\mathrm{F}\widetilde{\boldsymbol{X}}', i'\circ p_{\widetilde{\boldsymbol{X}}'})$ are the same object of $\widetilde{\mathcal{D}}$.
2. The definition of $\widetilde{\mathrm{F}}$ may not work because $\widetilde{\mathrm{F}}$ is defined to be the identity on $\widetilde{\mathcal{D}}$ but is not the identity on any objects of $\mathcal{D}$. This is a problem because those two subcategories of $\mathcal{D}'$ might have objects in common.

The proof we present here uses a different definition of $\mathcal{D}'$ and does not use the subcategory $\widetilde{\mathcal{D}}$. We are indebted to Fabian Lenhardt and Alberto Vezzani for help in identifying the problems and correcting the proof.

Proof of Theorem 14.5.4. Let $\mathcal{D}$ be a small category of functors over $\boldsymbol{X}$ relative to $\mathcal{W}$. Let $\mathcal{D}'$ be the smallest subcategory of functors over $\boldsymbol{X}$ relative to $\mathcal{W}$ containing

1. $\mathcal{D}$,
2. the object $(\mathrm{F}\boldsymbol{X}, p_{\boldsymbol{X}})$,
3. for each object $(\widetilde{\boldsymbol{X}}, i)$ in $\mathcal{D}'$
   - the object $(\mathrm{F}\widetilde{\boldsymbol{X}}, i\circ p_{\widetilde{\boldsymbol{X}}})$,
   - the map $p_{\widetilde{\boldsymbol{X}}}\colon (\mathrm{F}\widetilde{\boldsymbol{X}}, i\circ p_{\widetilde{\boldsymbol{X}}}) \to (\widetilde{\boldsymbol{X}}, i)$, and
   - the map $\mathrm{F}(i)\colon (\mathrm{F}\widetilde{\boldsymbol{X}}, i\circ p_{\widetilde{\boldsymbol{X}}}) \to (\mathrm{F}\boldsymbol{X}, p_{\boldsymbol{X}})$,

   and
4. for each map $g\colon (\widetilde{\boldsymbol{X}}, i) \to (\widetilde{\boldsymbol{X}}', i')$ in $\mathcal{D}'$ the map $\mathrm{F}(g)\colon (\mathrm{F}\widetilde{\boldsymbol{X}}, i\circ p_{\widetilde{\boldsymbol{X}}}) \to (\mathrm{F}\widetilde{\boldsymbol{X}}', i'\circ p_{\widetilde{\boldsymbol{X}}'})$.

The category $\mathcal{D}'$ is small because it can be constructed as the union of an increasing sequence of small categories, where each category in the sequence is obtained by "applying F to everything in the preceding category" (i.e., adding the objects and maps described in the last two items above).

We will show that $\mathrm{B}\mathcal{D}'$ is contractible by constructing a functor $\mathrm{G}\colon \mathcal{D}' \to \mathcal{D}'$ together with

- a natural transformation ϕ from G to the identity functor $1_{\mathcal{D}'}$ and
- a natural transformation ψ from G to a constant functor.

Proposition 14.3.10 will then imply that the identity map of $\mathrm{B}\mathcal{D}'$ is homotopic to a constant map.

If $(\widetilde{\boldsymbol{X}}, i)$ is an object in $\mathcal{D}'$ then we let $\mathrm{G}(\widetilde{\boldsymbol{X}}, i) = (\mathrm{F}\widetilde{\boldsymbol{X}}, i \circ p_{\widetilde{\boldsymbol{X}}})$, and if $g\colon (\widetilde{\boldsymbol{X}}, i) \to (\widetilde{\boldsymbol{X}}', i')$ is a map in $\mathcal{D}'$ then we let $\mathrm{G}(g) = \mathrm{F}(g)$.

We define a natural transformation ϕ from G to the identity functor of $\mathcal{D}'$ by letting $\phi(\widetilde{\boldsymbol{X}}, i) = p_{\widetilde{\boldsymbol{X}}}\colon (\mathrm{F}\widetilde{\boldsymbol{X}}, i \circ p_{\widetilde{\boldsymbol{X}}}) \to (\widetilde{\boldsymbol{X}}, i)$.

We define a natural transformation ψ from G to the constant functor that takes every object of $\mathcal{D}'$ to $(\mathrm{F}\boldsymbol{X}, p_{\boldsymbol{X}})$ by letting $\psi(\widetilde{\boldsymbol{X}}, i) = \mathrm{F}(i)\colon (\mathrm{F}\widetilde{\boldsymbol{X}}, i \circ p_{\widetilde{\boldsymbol{X}}}) \to (\mathrm{F}\boldsymbol{X}, p_{\boldsymbol{X}})$. □

5. Theorem 19.8.2

In part 2 of Theorem 19.8.2, the simplicial object $\boldsymbol{X}$ must be Reedy fibrant. That is, "$\boldsymbol{X}$ is a simplicial object in $\mathcal{M}$" should be "$\boldsymbol{X}$ is a Reedy fibrant simplicial object in $\mathcal{M}$".

6. Theorem 19.8.4

In part 2 of Theorem 19.8.4, the simplicial object $\boldsymbol{X}$ must be Reedy fibrant. That is, "$\boldsymbol{X}$ is a simplicial object in $\mathcal{M}$" should be "$\boldsymbol{X}$ is a Reedy fibrant simplicial object in $\mathcal{M}$".

In addition, there is no proof provided for Theorem 19.8.4. Theorem 19.8.4 is superseded by Theorem 19.8.7, from which it follows immediately.

7. Theorem 19.8.7

Part 1 of Theorem 19.8.7 should begin with "Let $(\boldsymbol{\Delta}^{\mathrm{op}}, \mathcal{M})$ be a Reedy framed diagram category structure on the category of simplicial objects in $\mathcal{M}$.", and part 2 should begin with "Let $(\boldsymbol{\Delta}, \mathcal{M})$ be a Reedy framed diagram category structure on the category of cosimplicial objects in $\mathcal{M}$.".

8. Chapter 3

Page 51, line 35:	"$\mathcal{C}$-local objects and $\mathcal{C}$-local" should be "$\mathcal{C}$-colocal objects and $\mathcal{C}$-colocal".
Page 59, line 5:	"Proposition 9.1.9" should be "Proposition 8.1.23".
Page 59, line 11:	"Proposition 8.3.26" should be "Theorem 7.5.10".
Page 59, line 14:	"Proposition 8.3.20" should be "Proposition 7.3.4".
Page 59, line 14:	"Proposition 8.4.4" should be "Proposition 7.6.8".
Page 59, line 16:	"Definition 9.6.2" should be "Definition 8.3.2".
Page 59, line 16:	"Lemma 9.6.3" should be "Lemma 8.3.4".
Page 59, line 17:	"Theorem 9.6.9" should be "Theorem 8.3.10".
Page 63, line 15:	"Theorem 4.1.10" should be "Theorem 3.2.13".
Page 67, line 3:	"Corollary 8.5.4" should be "Corollary 7.7.4".
Page 67, line 7:	"Proposition 8.3.7" should be "Proposition 7.3.10".
Page 67, line 23:	"is an trivial cofibration" should be "is a trivial cofibration".
Page 68, line 5:	"Definition 11.2.12" should be "Definition 13.3.12".

9. Chapter 4

Page 72, line 6: "$\mathrm{L}_{\mathcal{C}}\mathcal{M}$" should be "$\mathrm{L}_{S}\mathcal{M}$".
Page 72, line 6: "$\mathcal{C}$-local" should be "S-local".
Page 72, line 8: "$\mathrm{L}_{\mathcal{C}}\mathcal{M}$" should be "$\mathrm{L}_{S}\mathcal{M}$".
Page 72, line 10: "$\mathrm{L}_{\mathcal{C}}\mathcal{M}$" should be "$\mathrm{L}_{S}\mathcal{M}$".
Page 72, line 12: "$\mathcal{C}$-local" should be "S-local".

10. Chapter 5

Page 83, line 11: "K-local equivalences" should be "K-colocal equivalences".
Page 83, line 15: "$\mathcal{C}$-colocal" should be "K-colocal".
Page 83, line 20: "$\mathcal{C}$-colocal" should be "K-colocal".
Page 83, line 21: "$\mathcal{C}$-local" should be "K-colocal".

11. Chapter 8

Page 155, Proposition 8.5.7, part (2): "object" should be "objects".
Page 155, Theorem 8.5.8, part (2): "$\mathbf{R}\mathrm{F}$" should be "$\mathbf{R}\mathrm{U}$".

12. Chapter 9

Page 163, line 4: "$X \times K^{+}$" should be "$X \wedge K^{+}$".

13. Summary of Part 2

Page 103, line 4 from bottom: "$\mathbf{R}\mathrm{U}\colon \mathcal{N} \to \mathcal{M}$" should be "$\mathbf{R}\mathrm{U}\colon \operatorname{Ho}\mathcal{N} \to \operatorname{Ho}\mathcal{M}$"

14. Chapter 14

Page 270, Lemma 14.7.9: "$\mathrm{B}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}$" should be "$\mathrm{B}(\mathcal{C}\downarrow\alpha)$", and "$\mathrm{B}(\mathrm{F}\alpha\downarrow\mathrm{F})$" should be "$\mathrm{B}(\mathrm{F}\downarrow\mathrm{F}\alpha)$".

15. Chapter 15

Page 303, Proposition 15.6.19: The three appearances of "$\partial(\overrightarrow{\mathcal{C}}\downarrow\alpha)^{\mathrm{op}}$" should all be "$\partial(\alpha\downarrow\overleftarrow{\mathcal{C}})^{\mathrm{op}}$" and, on the last line, "be the element" should be "by the element".

16. Chapter 16

Page 342, Definition 16.7.2, last line of the definition: "$\mathrm{cs}_{*}\boldsymbol{Z}_{\alpha}$" should be "$\mathrm{cs}_{*}\boldsymbol{X}_{\alpha}$".

17. Chapter 18

Page 386, Definition 18.3.2, fifth line from the bottom of the page: "$\sigma_{*}\colon (\mathcal{C}\downarrow\alpha) \to (\mathcal{C}\downarrow\alpha')$; see Definition 14.7.8" should be "$\sigma_{*}\colon \boldsymbol{K}_{\alpha} \to \boldsymbol{K}_{\alpha'}$".

Page 388, Proposition 18.3.10: In the displayed line of part (1), "$\mathcal{M}(\boldsymbol{X}, Z)$" should be "$\mathrm{Map}(\boldsymbol{X}, Z)$", and in the displayed line of part (2), "$\mathcal{M}(W, \boldsymbol{X})$" should be "$\mathrm{Map}(W, \boldsymbol{X})$". In the proof, in the last displayed line, the two occurrences of "$\mathcal{M}(\boldsymbol{X}_{\alpha}, Z)$" should both be "$\mathrm{Map}(\boldsymbol{X}_{\alpha}, Z)$", and in the last line of the proof "$\mathcal{M}(\boldsymbol{X}, Z)$" should be "$\mathrm{Map}(\boldsymbol{X}, Z)$".

18. Chapter 19

Page 408, last line of Definition 19.2.2: "$\sigma_* \colon (\mathcal{C}\downarrow\alpha) \to (\mathcal{C}\downarrow\alpha')$; see Definition 14.7.8" should be "$\sigma_* \colon \boldsymbol{K}_\alpha \to \boldsymbol{K}_{\alpha'}$".

Department of Mathematics, Wellesley College, Wellesley, Massachusetts 02481
Email address: psh@math.mit.edu
URL: http://www-math.mit.edu/~psh