
(a) Derive the momentum equation for a fluid using a volume element argument (as opposed to the fluid element argument used in class). You are not required to specify the form of the stress tensor $\sigma$.

(b) The Reynolds Transport Theorem (RTT) states that for a scalar field $f(x, t)$,

$$\frac{d}{dt} \int_{V(t)} f(x, t) dV = \int_{V(t)} \left[ \frac{\partial f}{\partial t} + \nabla \cdot (uf) \right] dV$$

where $V(t)$ indicates a material control volume, i.e., a volume element that moves with the local fluid velocity $u(x, t)$. If the density $\rho$ satisfies the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0,$$

prove the special form of the RTT for another scalar field $g(x, t)$:

$$\frac{d}{dt} \int_{V(t)} \rho g(x, t) dV = \int_{V(t)} \rho \frac{Dg}{Dt} dV,$$

where $D/Dt$ represents the material derivative $D/Dt = \partial/\partial t + u \nabla$.

Problem 2: Solutions of the Navier-Stokes Equations.

The Navier-Stokes equations can be solved analytically for the following flow geometries. In each problem the fluid has viscosity $\mu$ and density $\rho$. Having obtained a solution, sketch the corresponding velocity profile. The solution in (d) is time dependent, so plot the velocity profile at two or three different times. Briefly comment on your solutions (i.e. what would be the effect of changing viscosity? or density? or changing the frequency in (d)?).

(a) Flow of fluid between two rigid boundaries located at $y = \pm h$ under a constant pressure gradient $P = -dp/dx$.

(b) The same as (a) but for flow inside and along the axis of a circular cylinder of radius $h$. (use cylindrical coordinates)
(c) Flow between a cylinder of radius $R_1$ inside another cylinder of radius $R_2$. The inner cylinder rotates at angular velocity $\omega_1$ and the outer cylinder rotates at angular velocity $\omega_2$. (use cylindrical coordinates)

(d) Fluid lying in the plane $0 < y < \infty$, above an oscillating flat plate moving to and fro in the $x$-direction with velocity $U \cos \omega t$. Seek a solution for the velocity field of the form

$$\mathbf{u} = (u(y, t), 0, 0) = (\text{Re}[f(y)e^{i\omega t}], 0, 0).$$

The first two flows are called Poiseuille flows, after the physician who first studied them in connection with blood flow. Their instability at high Reynolds number constitutes one of the most important problems in fluid dynamics. The third flow is cylindrical Couette flow.

**Problem 3: Tea cup experiment.**

(a) Take a cup of water or tea. With a spoon, stir the fluid until the flow is roughly in solid body rotation. Estimate the initial rotational angular velocity of the fluid, and time how long it takes for the fluid to spin down to rest. How does this compare with our calculations in class? Can you explain any discrepancies? How does the spin down time change if you use a more viscous fluid, like honey, or if you change the size of your container? Write a brief description of your experimental method and results.

(b) Our solution to the spin-down of a tea cup was composed of Bessel functions. Use MATLAB to make some plots of the velocity profile for flow in your coffee cup at various stages of the spin down, and describe any difficulties you encounter. Provide a printout of your code for calculating the velocity profile.