The origins of the Tate conjecture

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Motivation I:
Ш over global function fields and the Tate conjecture

1. Finiteness of Ш

k: global field
A: abelian variety over k

$$0 \longrightarrow Ш \longrightarrow H^1(k, A) \longrightarrow \prod_v H^1(k_v, A)$$

Conjecture (Shafarevich? Tate? 1962)

Ш is finite.

Later, this was proved in many cases, such as
$E/\mathbb{Q}$ with $\operatorname{ord}_{s=1} L(E,s) \leq 1$ (Rubin 1987, Kolyvagin 1988, ...).

Tate 1994:

> *"If Ш, or at least its ℓ-primary part, were not finite, then the Galois cohomology of the abelian variety would be a mess..."*

2. Ш vs. Br

Fibered surface setting:

$X \xrightarrow{\pi} B$ (assume X, B nice[1] of dimensions $2, 1$ over $\mathbb{F}_q$)

$X_\eta \to \operatorname{Spec} k$: generic fiber (assume X_η is a nice curve over k)

$J :=$ Jacobian of X_η

$\operatorname{Br} X := \mathrm{H}^2_{\text{et}}(X, \mathbb{G}_m)$

Theorem (special case of Artin 1960s, Milne 1982)

Ш(J) is finite $\iff$ $\operatorname{Br} X$ is finite.

This suggests...

Conjecture

$\operatorname{Br} X$ is finite for every nice surface X over $\mathbb{F}_q$.

Artin: More generally, is $\operatorname{Br} X$ finite for every scheme X proper over $\mathbb{Z}$?

[1] smooth, projective, geometrically integral

3. Br and cycle classes of divisors

X: nice variety over $\mathbb{F}_q$, $\overline{X} := X \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$, $G := \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$

$\ell \neq \operatorname{char} \mathbb{F}_q$

Take étale cohomology of $1 \to \mu_\ell \to \mathbb{G}_m \xrightarrow{\ell} \mathbb{G}_m \to 1$:

$$0 \longrightarrow (\operatorname{Pic} X) \otimes \frac{\mathbb{Z}}{\ell\mathbb{Z}} \longrightarrow \mathrm{H}^2(X, \mu_\ell) \longrightarrow (\operatorname{Br} X)[\ell] \longrightarrow 0.$$

Do the same for ℓ^n and take $\varprojlim$:

$$0 \longrightarrow (\operatorname{Pic} X) \otimes \mathbb{Z}_\ell \longrightarrow \mathrm{H}^2(X, \mathbb{Z}_\ell(1)) \longrightarrow \underbrace{\varprojlim (\operatorname{Br} X)[\ell^n]}_{\text{torsion-free}} \longrightarrow 0.$$

Theorem (Artin–Tate, announced 1962, details published 1966)

The following conjectures are equivalent:

1. $(\operatorname{Br} X)[\ell^\infty]$ *is finite.*
2. $\varprojlim (\operatorname{Br} X)[\ell^n] = 0$.
3. $(\operatorname{Pic} X) \otimes \mathbb{Z}_\ell \longrightarrow \mathrm{H}^2(X, \mathbb{Z}_\ell(1))$ *is surjective.*
4. $(\operatorname{Pic} X) \otimes \mathbb{Q}_\ell \longrightarrow \mathrm{H}^2(X, \mathbb{Q}_\ell(1))$ *is surjective.*

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$$= \mathrm{H}^2(\overline{X}, \mathbb{Q}_\ell(1))^G.$$

Why $=$? Hochschild–Serre & Frobenius eigenvalues on $\mathrm{H}^1(\overline{X}, \mathbb{Q}_\ell(1))$ are $\neq 1$.

3. Br and cycle classes of divisors

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Take étale cohomology of $1 \to \mu_\ell \to \mathbb{G}_m \xrightarrow{\ell} \mathbb{G}_m \to 1$:

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3. $(\operatorname{Pic} X) \otimes \mathbb{Z}_\ell \longrightarrow \mathrm{H}^2(X, \mathbb{Z}_\ell(1))$ *is surjective.*
4. $(\operatorname{Div} X) \otimes \mathbb{Q}_\ell \longrightarrow \mathrm{H}^2(X, \mathbb{Q}_\ell(1))$ *is surjective.*

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Why $=$? Hochschild–Serre & Frobenius eigenvalues on $\mathrm{H}^1(\overline{X}, \mathbb{Q}_\ell(1))$ are $\neq 1$.

Statements:

The Tate conjecture
and related conjectures

The Tate conjecture (1962 for divisors on surface/$\mathbb{F}_q$, 1965 in general)

k: finitely generated field, $\overline{k}$: separable closure, $G := \mathrm{Gal}(\overline{k}/k)$
X: nice variety of dimension d over k, $\overline{X} := X \times_k \overline{k}$

$\mathcal{Z}^r(X)$:= free abelian group on
{codimension r closed integral subvarieties of X}
(Example: $\mathcal{Z}^1(X) = \operatorname{Div} X$.)

Conjecture T^r

For each $\ell \neq \operatorname{char} k$, the cycle class map

$$\mathcal{Z}^r(X) \otimes \mathbb{Q}_\ell \longrightarrow \underbrace{\mathrm{H}^{2r}(\overline{X}, \mathbb{Q}_\ell(r))^G}_{\text{Tate classes}}$$

is surjective (onto the Galois-invariant classes).

Compare with the Hodge conjecture: For a nice variety X over $\mathbb{C}$,

$$\mathcal{Z}^r(X) \otimes \mathbb{Q} \longrightarrow \mathrm{H}^{2r}(X, \mathbb{Q}) \cap \mathrm{H}^{r,r}$$

is surjective.

The integral Hodge conjecture ($\mathbb{Z}$ in place of $\mathbb{Q}$) and integral Tate conjecture ($\mathbb{Z}_\ell$ in place of $\mathbb{Q}_\ell$) are false for $r > 1$ (Atiyah–Hirzebruch, Kollár, Totaro, Hassett–Tschinkel, . . .)

Define subgroups

$$\text{rat} \subset \text{alg} \subset \text{hom}_\ell \subset \text{num} \subset \mathcal{Z}^r(X):$$

- rat = rationally equivalent to 0
  (coming from rational functions on $V \subset X$ of codim $r-1$)
- alg = algebraically equivalent to 0
  ($\exists$ family of cycles over a connected base connecting it to 0)
- hom_ℓ = homologically equivalent to 0
  (maps to 0 in $\mathrm{H}^{2r}(\overline{X}, \mathbb{Q}_\ell(r))$)
- num = numerically equivalent to 0
  (Y satisfies $Y.Z = 0$ for all $Z \in \mathcal{Z}^{d-r}(X)$)

By rat $\subset$ alg $\subset$ hom_ℓ and the definition of hom_ℓ,

$$\mathcal{Z}^r(X) \xrightarrow{\text{cl}} \mathrm{H}^{2r}(\overline{X}, \mathbb{Q}_\ell(r))$$

factors through any of

$$\underbrace{\mathcal{Z}^r(X)/\text{rat}}_{\mathrm{CH}^r(X)}, \quad \mathcal{Z}^r(X)/\text{alg}, \quad \underbrace{\mathcal{Z}^r(X)/\text{hom}_\ell}_{\text{cl}(\mathcal{Z}^r(X))}.$$

When $r = 1$: $\quad \operatorname{Pic} X, \quad \operatorname{NS} X, \quad \operatorname{NS} X/\text{torsion}.$

Conjecture E^r

$\mathrm{hom}_\ell = \mathrm{num}$ for every $\ell \neq \mathrm{char}\, k$.

E^1 is known.

Conjecture I^r

$(\mathcal{Z}^r(X)/\mathrm{hom}_\ell) \otimes \mathbb{Q}_\ell \longrightarrow \mathrm{H}^{2r}(\overline{X}, \mathbb{Q}_\ell(r))^G$ is injective.

I^r says: $\mathbb{Z}$-independent elements of $\mathrm{cl}(\mathcal{Z}^r(X))$ are $\mathbb{Q}_\ell$-independent.
Beilinson injectivity conjecture for $\mathbb{F}_q$:

$\mathrm{CH}^r(X) \otimes \mathbb{Q}_\ell \to \mathrm{H}^{2r}(\overline{X}, \mathbb{Q}_\ell(r))^G$ is injective.

Conjecture S^r

Let $V = \mathrm{H}^{2r}(\overline{X}, \mathbb{Q}_\ell(r))$. Then $V^G \to V_G$ is an isomorphism.

S^r is partial semisimplicity: When $k = \mathbb{F}_q$, S^r is equivalent to

$$\text{1-generalized eigenspace of Frob} = \text{1-eigenspace}$$

$$\text{multiplicity of the eigenvalue } 1 = \dim \mathrm{H}^{2r}(\overline{X}, \mathbb{Q}_\ell(r))^G.$$

Folklore (Tate,Katz,Messing,Jannsen,Milne): $T^r + E^r \implies I^r, S^r$.
Proof: Formal consequence of Poincaré duality & hard Lefschetz.

Motivation II:
BSD over global function fields and the Tate conjecture

4. Tate conjecture and poles of zeta functions

X: finite type $\mathbb{Z}$-scheme
x: closed point of X
$k(x)$: residue field
$q_x := \#k(x)$

$$\zeta_X(s) := \prod_{\text{closed } x \in X} (1 - q_x^{-s})^{-1}.$$

The product converges when $\operatorname{Re} s > \dim X$,
and conjecturally has a meromorphic continuation to all of $\mathbb{C}$.

X is a nice variety of dimension d over $\mathbb{F}_q$.
$F\colon X \to X$ is the relative Frobenius morphism.
$P_i(T) := \det\big(1 - TF^* | \mathrm{H}^i(\overline{X}, \mathbb{Q}_\ell)\big)$

Theorem (some of the Weil conjectures; ..., Deligne 1974)

1. $P_i(T) \in \mathbb{Z}[T]$
2. $\zeta_X(s) = \dfrac{P_1(q^{-s}) \cdots P_{2d-1}(q^{-s})}{P_0(q^{-s})\, P_2(q^{-s}) \cdots P_{2d}(q^{-s})}$
3. *All complex roots α of $P_i(T)$ satisfy $|\alpha| = q^{i/2}$.*

Corollary: For $r \in \{0, 1, \ldots, d\}$,

$$
\begin{aligned}
-\operatorname{ord}_{s=r} \zeta_X(s) &= \operatorname{ord}_{s=r}\ P_{2r}(q^{-s}) \\
&= \operatorname{ord}_{T=q^{-r}}\ P_{2r}(T) \\
&= \text{multiplicity of eigenvalue } q^r \text{ of } F^* | \mathrm{H}^{2r}(\overline{X}, \mathbb{Q}_\ell) \\
&= \text{multiplicity of eigenvalue 1 of } F^* | \mathrm{H}^{2r}(\overline{X}, \mathbb{Q}_\ell(r)) \\
&\geq \dim_{\mathbb{Q}_\ell}\ \underbrace{\mathrm{H}^{2r}(\overline{X}, \mathbb{Q}_\ell(r))^G}_{\text{1-eigenspace}} \quad (\text{equality} \iff S^r) \\
&\geq \operatorname{rank}_{\mathbb{Z}}(\mathcal{Z}^r(X)/\mathrm{num}) \quad (\text{equality} \iff T^r + I^r + E^r).
\end{aligned}
$$

Summary:

Theorem (Tate 1965, modulo Weil conjectures at the time)

For a nice variety X over $\mathbb{F}_q$, the following are equivalent:

(a) $T^r + E^r$ *for any one* ℓ

(b) $\operatorname{rank}_{\mathbb{Z}}(\mathcal{Z}^r(X)/\mathrm{num}) \;=\;$ order of the pole of $\zeta_X(s)$ at r.

Tate also conjectured a generalization of (b)
for a nice variety over a finitely generated field k instead of $\mathbb{F}_q$.

5. The BSD conjecture (rank part)

k: global field
J: abelian variety over k

BSD conjecture (Birch and Swinnerton-Dyer 1965 for elliptic curves over number fields, Tate 1965 in general)

$$\operatorname{rank} J(k) = \operatorname{ord}_{s=1} L(J, s)$$

The Artin–Tate program:

For a fibered surface $X \xrightarrow{\pi} B$ over $\mathbb{F}_q$
(with irreducible fibers, for simplicity),
express what BSD for $J := \operatorname{Jac} X_\eta$ says in terms of X.

6. $\mathrm{NS}(X)$ vs. $J(k)$

Lemma

$$\operatorname{rank} \mathrm{NS}(X) = 2 + \operatorname{rank} J(k).$$

Proof.

Algebraic geometry gives an exact sequence

$$0 \longrightarrow \operatorname{Pic} B \xrightarrow{\pi^*} \operatorname{Pic} X \longrightarrow \operatorname{Pic} X_\eta \longrightarrow 0.$$

Ranks are $1, \quad \operatorname{rank} \mathrm{NS}(X), \quad 1 + \operatorname{rank} J(k).$ □

7. Poles of zeta functions vs. zeros of L-functions

Lemma

$$-\operatorname{ord}_{s=1} \zeta_X(s) \;=\; 2 + \operatorname{ord}_{s=1} L(J,s).$$

Proof.

For closed $b \in B$, let $X_b = \pi^{-1}(b)$.

Take the product over b of $\quad \zeta_{X_b}(s) = \dfrac{P_{1,X_b}(q_b^{-s})}{(1-q_b^{-s})(1-q_b^{1-s})}$

to get $\quad \zeta_X(s) = \dfrac{\zeta_B(s)\;\zeta_B(s-1)}{L(J,s)}.$

Here $\quad \zeta_B(s) = \dfrac{P_{1,B}(q^{-s})}{(1-q^{-s})\,(1-q^{1-s})},$

so both $\zeta_B(s)$ and $\zeta_B(s-1)$ contribute a simple pole at 1. □

Putting 4–7 together: BSD for $J \iff T^1$ for X

$$2 + \operatorname{rank} J(k) \overset{\text{BSD}}{=\!=} 2 + \operatorname{ord}_{s=1} L(J,s)$$

geometry: $2 + \operatorname{rank} J(k) = \operatorname{rank} \mathrm{NS}(X)$

$$2 + \operatorname{ord}_{s=1} L(J,s) = -\operatorname{ord}_{s=1} \zeta_X(s)$$

Weil conjectures: $-\operatorname{ord}_{s=1} \zeta_X(s) =$ multiplicity of eigenvalue 1 of $F^*|\mathrm{H}^2(\overline{X}, \mathbb{Q}_\ell(1))$

$$\operatorname{rank} \mathrm{NS}(X) \overset{T^1}{=\!=} \text{multiplicity of eigenvalue 1 of } F^*|\mathrm{H}^2(\overline{X}, \mathbb{Q}_\ell(1))$$

Artin and Tate also translated

full BSD for J (the leading coefficient statement)

into a conjectural formula about X.
To prove this conditionally on finiteness of Ш took four decades (mainly to finish dealing with the power of p in the formula):

Theorem (Tate, Milne, Schneider, Bauer, Kato–Trihan 2003)

For a fibered surface $X \to B$ over $\mathbb{F}_q$, the following are equivalent:

- T^1 *for* X *for all* ℓ *(or for one* ℓ*)*
- $\operatorname{Br} X$ *is finite (or* $(\operatorname{Br} X)[\ell^\infty]$ *is finite for one* ℓ*, possibly* p*)*
- $Ш(J)$ *is finite (or* $Ш(J)[\ell^\infty]$ *is finite for one* ℓ*, possibly* p*)*
- *rank part of BSD holds for* J*,*
- *full BSD holds for* J*.*

Moreover, T^1 for all surfaces $\Longrightarrow$ T^1 for all varieties
(de Jong unpublished, Morrow 2019, Kahn 2023).
Also, T^1 over $\mathbb{F}_p$ and $\mathbb{Q}$ $\Longrightarrow$ T^1 over any f.g. k (Ambrosi 2018).

Motivation III:

Hom between abelian varieties and the Tate conjecture

8. Homomorphisms between abelian varieties

k: finitely generated field

A: abelian variety over k

$\ell \neq \operatorname{char} k$, $\quad T_\ell A := \varprojlim A(\bar{k})[\ell^n]$, $\quad V_\ell A := (T_\ell A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$

Theorem (Tate 1966 for $\mathbb{F}_q$ with an idea from Lichtenbaum; Zarhin 1974 for char $k = p$; Faltings 1983–84 for char $k = 0$)

For any abelian varieties A, B over k,

$$\operatorname{Hom}(A, B) \otimes \mathbb{Q}_\ell \longrightarrow \operatorname{Hom}_{\mathbb{Q}_\ell}(V_\ell A, V_\ell B)^G$$

is an isomorphism.

- Theorem for $k \iff T^1$ for all abelian varieties over k.
- Theorem for k implies that for all nice varieties X, Y over k,

$$T^1(X \times Y) \iff T^1(X) + T^1(Y).$$

- Theorem for $\mathbb{F}_q$ + Honda's construction (taking CM abelian varieties mod p) yields the Honda–Tate explicit description of the category of abelian varieties over $\mathbb{F}_q$ "up to isogeny".
- Theorem for number fields $\implies$ Mordell conjecture (Faltings).

Proof of Hom theorem (simplified using Zarhin 1974)

Lemma 1

For fixed g and $\mathbb{F}_q$,

$$\{g\text{-dimensional abelian varieties over } \mathbb{F}_q\}/\text{isom}$$

is finite.

Sketch of proof.

1. Each g-dimensional abelian variety A is a direct factor of an $8g$-dimensional principally polarized abelian variety P. (Zarhin's trick: $A^4 \times \tilde{A}^4$ has a principal polarization.)
2. There are only finitely many possible P. ($\mathbb{F}_q$-points of a finite-type moduli space, up to twists)
3. Each P has only finitely many isom types of direct factors. □

Proof of Hom theorem, page 2

Using $\mathrm{Hom}(A, B) \subset \mathrm{End}(A \times B)$, reduce to End case.
Let $T = T_\ell A$ and $V = V_\ell A$ and $E = (\mathrm{End}\, A) \otimes \mathbb{Q}_\ell$.

Lemma 2

Each G-stable subspace $V' \subset V$ is $u(V)$ for some $u \in E$.

Proof.

1. Let $T' = V' \cap T$.
2. Each sublattice $T' + \ell^n T \subset T$ corresponds to an isogeny $\phi_n \colon B_n \to A$.
3. By Lemma 1, infinitely many B_n are isomorphic to one B.
4. In the compact group $\mathrm{Hom}(B, A) \otimes \mathbb{Z}_\ell$, some subsequence of (ϕ_n) converges, say to ϕ.
5. Then $\phi(T_\ell B) = T'$, so $\phi(V_\ell B) = V'$.
6. Let u be the composition of an isogeny $A \to B$ with ϕ. □

Proof of Hom theorem, page 3 (end)

Recall: $E = (\operatorname{End} A) \otimes \mathbb{Q}_\ell$ and $V = V_\ell A$.

Theorem

$E \hookrightarrow (\operatorname{End} V)^G$ is surjective.

Proof.

Let $f \in (\operatorname{End} V)^G$.

Lemma 2 produces $u \in \operatorname{End}(A \times A) \otimes \mathbb{Q}_\ell = \mathrm{M}_2(E)$ such that

$$u(V \times V) = \operatorname{graph}(f).$$

If $c \in \operatorname{End} V$ commutes with all elements of E, then

- $\begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$ commutes with u.
- $\begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$ maps $\operatorname{graph}(f)$ into itself.
- f commutes with c.

That is,

$$f \in (\text{double commutant of } E \text{ in } \operatorname{End} V) = E,$$

since E is semisimple. □

Algorithms:

Computing Néron–Severi groups using the Tate conjecture

Algorithmic aspects

Theorem (P., Testa, van Luijk 2015)

There is an algorithm attempting to compute
$\operatorname{NS}\overline{X}$ *as a finitely generated abelian group with G-action; it terminates with success if and only if T^1 holds.*

- The main challenge is to compute $\rho := \operatorname{rank}\operatorname{NS}(\overline{X})$.
- Can similarly compute $(\mathcal{Z}^r(\overline{X})/\mathrm{num}) \otimes \mathbb{Q}$ if $T^r + E^r$ holds.

Obstacle to an easy proof:
The image of $G \to \operatorname{Aut} \mathrm{H}^2(\overline{X}, \mathbb{Q}_\ell(1))$ is usually infinite.
What would it mean to compute it?

Let's set up notation for a different approach.

Let $\mathrm{H}^2_{\mathrm{Tate}} = \bigcup_{\text{finite-index } H \leq G} \mathrm{H}^2(\overline{X}, \mathbb{Q}_\ell(1))^H$. Let $\tau = \dim \mathrm{H}^2_{\mathrm{Tate}}$.

Then $\rho \leq \tau$, with equality if T^1 holds.

Theorem (P., Testa, van Luijk 2015)

There is an algorithm attempting to compute $\rho := \operatorname{rank} \mathrm{NS}(\overline{X})$*;*
it terminates with success if and only if T^1 *holds.*

Sketch of proof: Use $H_n := \mathrm{H}^2\left(\overline{X}, \frac{\mathbb{Z}}{\ell^n\mathbb{Z}}(1)\right)$ instead of $\mathrm{H}^2(\overline{X}, \mathbb{Q}_\ell)$.

1. H_n is computable (PTvL 2015, Madore–Orgogozo 2015)
2. Enlarge k to assume that G acts trivially on H_1 (use H_2 if $\ell = 2$). Then G acts trivially on $\mathrm{H}^2_{\mathrm{Tate}}$ (Minkowski-type argument).
3. There exist $c, C > 0$ with c computable such that, for all n,
$$c\ell^{\tau n} \leq \#H_n^G \leq C\ell^{\tau n}.$$
4. By day, search for divisors and compute intersection numbers, to get eventually sharp lower bounds on ρ.
5. By night, compute $\#H_n^G$ for larger and larger n to get eventually sharp upper bounds on τ.
6. If T^1 holds, the lower and upper bounds eventually match!