Bertini irreducibility theorems via statistics

Bjorn Poonen
(joint work with Kaloyan Slavov of ETH Zürich)

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Bertini irreducibility theorem

k: algebraically closed field

$\mathbb{P}^n$: projective space over k

$\check{\mathbb{P}}^n$: the dual projective space ($H \in \check{\mathbb{P}}^n$ means H is a hyperplane)

$X \subset \mathbb{P}^n$: irreducible subvariety of dimension ≥ 2

Bertini irreducibility theorem (vague form)

$H \cap X$ is irreducible for "most" hyperplanes H.

$\mathcal{M}_{\text{good}} := \{H \in \check{\mathbb{P}}^n : H \cap X \text{ is irreducible}\}$

$\mathcal{M}_{\text{bad}} := \{H \in \check{\mathbb{P}}^n : H \cap X \text{ is not irreducible}\}$

Bertini irreducibility theorem (precise form)

$\mathcal{M}_{\text{good}}$ contains a dense open subvariety of $\check{\mathbb{P}}^n$.

Equivalently, $\dim \mathcal{M}_{\text{bad}} \leq n - 1$.

How big is $\mathcal{M}_{\text{bad}}$, *really*?

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Example

For a hypersurface $X \subset \mathbb{P}^n$, it turns out that $\dim \mathcal{M}_{\text{bad}} \leq 2$!

Benoist's theorem

$X \subset \mathbb{P}^n$: irreducible subvariety ~~of dimension ≥ 2~~
$\mathcal{M}_{\text{bad}} := \{H \in \check{\mathbb{P}}^n : H \cap X \text{ is not irreducible}\}$

Theorem (Benoist 2011)

$\dim \mathcal{M}_{\text{bad}} \leq \operatorname{codim} X + 1.$

The bound $\operatorname{codim} X + 1$ is best possible:

Example (warmup)

curve $C \subset \mathbb{P}^m$, not a line $\qquad \dim \mathcal{M}_{\text{bad}} = m = \operatorname{codim} C + 1.$

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Example (~~warmup~~)

Take inverse images under a linear projection:

$$\begin{tikzcd} \pi^{-1}C \subset \mathbb{P}^n \arrow[d, dashed, "\pi"] \\ \text{curve } C \subset \mathbb{P}^m, \text{ not a line} \end{tikzcd}$$

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Benoist's theorem: two proof strategies

$X \subset \mathbb{P}^n$: irreducible subvariety
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Theorem (Benoist 2011)

$\dim \mathcal{M}_{\mathrm{bad}} \leq \operatorname{codim} X + 1.$

Benoist's proof is purely geometric, but tricky:

1. reduce to the case of a hypersurface;
2. reduce further to the case of a cone over a plane curve;
3. degenerate to a union of hyperplanes;
4. use normalization and the EGA IV_4 form of the Ramanujam–Samuel criterion for a divisor to be Cartier.

We will give a new proof based on counting over finite fields, partly inspired by Tao's 2012 blog post on the Lang–Weil bound.

Irreducible vs. geometrically irreducible

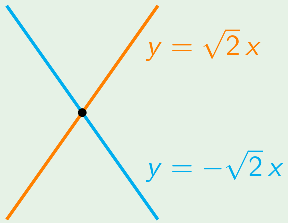
Let X be a variety over an *arbitrary* field F.
Call X geometrically irreducible if $X \underset{F}{\times} \overline{F}$ is irreducible.

Example

Suppose that 2 is not a square in $\mathbb{F}_p$. Let

$$X := \operatorname{Spec} \mathbb{F}_p[x, y]/(y^2 - 2x^2).$$

Then X is irreducible, but not geometrically irreducible:



We have $X(\mathbb{F}_p) = \{(0, 0)\}$.

Lang–Weil bound

Theorem (Lang–Weil 1954)

Let X be an r-dimensional variety over $\mathbb{F}_q$. Let $|X| = |X(\mathbb{F}_q)|$.

1. *General crude upper bound:*

$$|X| = O(q^r).$$

2. *If X is geometrically irreducible, then*

$$|X| = q^r + O(q^{r-1/2}).$$

3. *More generally, if a is the number of irreducible components of X that are geometrically irreducible of dimension r, then*

$$|X| = aq^r + O(q^{r-1/2}).$$

Reduction to the case of a finite field

$X \subset \mathbb{P}^n$: geometrically irreducible subvariety over a field F
$\mathcal{M}_{\text{bad}} := \{H \in \check{\mathbb{P}}^n : H \cap X \text{ is not geometrically irreducible}\}$

Theorem (Benoist 2011)

$\dim \mathcal{M}_{\text{bad}} \leq \operatorname{codim} X + 1.$

Standard specialization argument for reducing to the case $F = \mathbb{F}_q$:

1. $X \subset \mathbb{P}^n_F$ is the base change of some $\mathscr{X} \subset \mathbb{P}^n_R$, for some finitely generated $\mathbb{Z}$-subalgebra $R \subset F$, such that $\mathscr{X} \to \operatorname{Spec} R$ has geometrically irreducible fibers all of the same dimension.
2. There is a big bad $\mathscr{M}_{\text{bad}} \subset \check{\mathbb{P}}^n_R$ such that for any R-field k, the fiber $(\mathscr{M}_{\text{bad}})_k$ is the little $\mathcal{M}_{\text{bad}}$ for $\mathscr{X}_k \subset \mathbb{P}^n_k$.
3. If each little $\mathcal{M}_{\text{bad}}$ over a closed point has dimension $\leq \operatorname{codim} X + 1$, then the same holds for $X \subset \mathbb{P}^n_F$.
4. The residue field at each closed point of $\operatorname{Spec} R$ is a finite field.

Upper bound on variance for hyperplane sections

$X \subset \mathbb{P}^n$: geom. irreducible subvariety over $\mathbb{F}_q$. Let $m = \dim X$.
$H \subset \mathbb{P}^n$: random hyperplane over $\mathbb{F}_q$
Z: the random variable $|(H \cap X)(\mathbb{F}_q)|$

Proposition

The mean μ of Z is $\sim |X|/q \sim q^{m-1}$,

The variance σ^2 of Z is $O(|X|/q) = O(q^{m-1})$.

Sketch of proof.

$$Z = \sum_{x \in X} 1_{x \in H}, \text{ so}$$

$$\mu = \mathbb{E}Z = \frac{\sum_H \sum_{x \in X} 1_{x \in H}}{\sum_H 1} = \frac{\sum_{x \in X} \sum_H 1_{H \ni x}}{\sum_H 1} = \cdots$$

$$\sigma^2 = \mathbb{E}((Z-\mu)^2) = \mathbb{E}(Z^2) - \mu^2 = \cdots,$$

where $\mathbb{E}(Z^2)$ can be similarly computed in terms of the easy sums $\sum_H 1_{H \ni x,y}$ for $x, y \in X(\mathbb{F}_q)$. $\square$

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Proposition

The mean μ of Z is $\sim |X|/q \sim q^{m-1}$,
The variance σ^2 of Z is $O(|X|/q) = o(q^{m-1})$. Very small!

Sketch of proof.

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Lower bound on variance

From previous slide: $\mu \sim q^{m-1}$, and $\sigma^2 = O(q^{m-1})$.

On the other hand:

- Each $H \in \mathcal{M}_{\mathrm{bad}}(\mathbb{F}_q)$ contributes a lot to the variance:

$$\Big| |H \cap X| - \mu \Big| \gtrsim q^{m-1}.$$

- Let $b := \dim \mathcal{M}_{\mathrm{bad}}$. If b is large, then there are many bad H:

$$|\mathcal{M}_{\mathrm{bad}}(\mathbb{F}_q)| \gtrsim q^b.$$

Thus

$$\sigma^2 \ \gtrsim \ \frac{q^b (q^{m-1})^2}{\textit{total}\text{ number of } H} \ \sim \ q^{b+2(m-1)-n}.$$

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On the other hand, after replacing $\mathbb{F}_q$ by a finite extension:

- A positive fraction of $H \in \mathcal{M}_{\text{bad}}(\mathbb{F}_q)$ contribute a lot to the variance:
$$\Big| |H \cap X| - \mu \Big| \gtrsim q^{m-1}.$$
- Let $b := \dim \mathcal{M}_{\text{bad}}$. If b is large, then there are many bad H:
$$|\mathcal{M}_{\text{bad}}(\mathbb{F}_q)| \gtrsim q^b.$$

Thus
$$\sigma^2 \;\gtrsim\; \frac{q^b (q^{m-1})^2}{\textit{total} \text{ number of } H} \;\sim\; q^{b+2(m-1)-n}.$$

End of proof

Combine the lower and upper bounds on the variance:

$$q^{b+2(m-1)-n} \lesssim \sigma^2 \lesssim q^{m-1}.$$

If q is sufficiently large, this implies

$$\begin{aligned} b + 2(m-1) - n &\leq m - 1 \\ b &\leq (n-m) + 1 \\ \dim \mathcal{M}_{\mathrm{bad}} &\leq \operatorname{codim} X + 1. \qquad \square \end{aligned}$$

Jouanolou's Bertini irreducibility theorem

X: geometrically irreducible variety
$\phi \colon X \to \mathbb{P}^n$: an arbitrary morphism (previously was an *immersion*)
$\mathcal{M}_{\text{good}} := \{H \in \check{\mathbb{P}}^n : \phi^{-1}H \text{ is geometrically irreducible}\}$
$\mathcal{M}_{\text{bad}} := \{H \in \check{\mathbb{P}}^n : \phi^{-1}H \text{ is not geometrically irreducible}\}$

Theorem (Jouanolou 1983)

If $\dim \phi(X) \geq 2$, *then* $\dim \mathcal{M}_{\text{bad}} \leq n-1$.

Theorem (P.–Slavov 2020)

If the nonempty fibers of ϕ *all have the same dimension, then* $\dim \mathcal{M}_{\text{bad}} \leq \operatorname{codim} \phi(X) + 1$.

The proof is the same, but using the random variable $|\phi^{-1}(H)|$.

Counterexample (without the fiber dimension hypothesis)

If $X \to \mathbb{P}^n$ is the blow-up of a point P, then $\mathcal{M}_{\text{bad}}$ consists of the H containing P, so $\dim \mathcal{M}_{\text{bad}} = n-1$, but $\operatorname{codim} \phi(X) + 1 = 1$.

Application to monodromy

k: algebraically closed field
$\phi \colon X \to Y$: generically étale morphism of integral k-varieties
$k(X)'$: the Galois closure of $k(X)/k(Y)$
$\mathrm{Mon}(\phi)$: the monodromy group $\mathrm{Gal}(k(X)'/k(Y))$
(There is also a definition not requiring X to be integral.)

Now suppose in addition that $Y \subset \mathbb{P}^n$.
For each $H \subset \mathbb{P}^n$, restrict ϕ to obtain $\phi_H \colon \phi^{-1}(H \cap Y) \to (H \cap Y)$.
The following says that $\mathrm{Mon}(\phi_H) \simeq \mathrm{Mon}(\phi)$ for "most" H:

Theorem (P.–Slavov 2020)

Let $\mathcal{M}_{\mathrm{good}}$ be the set of $H \in \check{\mathbb{P}}^n$ such that

1. *$H \cap Y$ is irreducible;*
2. *the generic point of $H \cap Y$ has a neighborhood U in Y such that U is normal and $\phi^{-1}U \to U$ is finite étale; and*
3. *the inclusion $\mathrm{Mon}(\phi_H) \hookrightarrow \mathrm{Mon}(\phi)$ is an isomorphism.*

Let $\mathcal{M}_{\mathrm{bad}} := \check{\mathbb{P}}^n - \mathcal{M}_{\mathrm{good}}$. Then $\dim \mathcal{M}_{\mathrm{bad}} \leq \operatorname{codim} Y + 1$.

Motivic version?

Let us return to our finite field proof of Benoist's theorem.

Question

Is there a motivic version?

More specifically, can one make the variance argument work when one replaces finite field point counts with classes in some version of the Grothendieck ring of varieties?

“This is not the last slide!”