PRESS RELEASE: TETRAHEDRA WITH RATIONAL DIHEDRAL ANGLES

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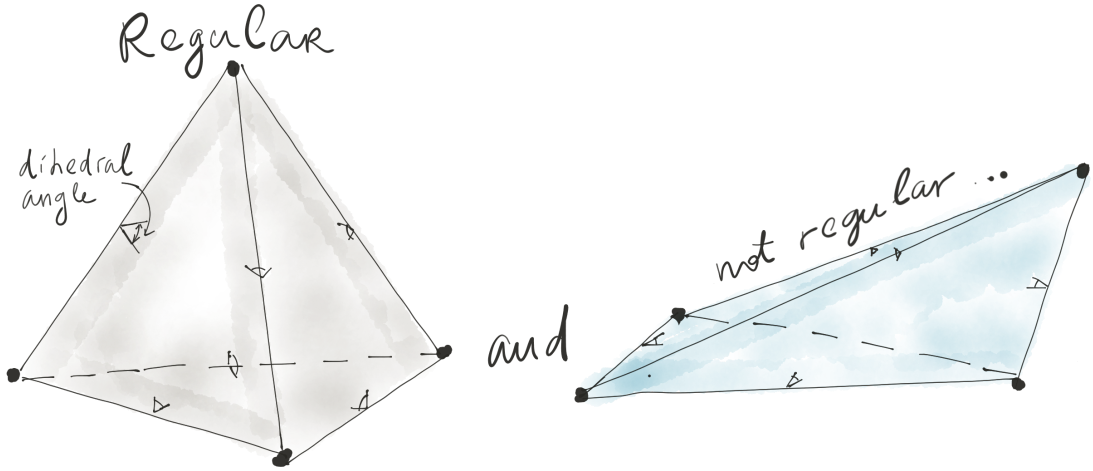
We answer a 44-year-old question of Conway and Jones in 3-dimensional geometry [KKPR20]. The question concerns tetrahedra, which are pyramids with a triangular base.



In fact, the study of tetrahedra is ancient, and there are three major problems concerning them.

(1) **Given a tetrahedron, does it tile space?** That is, can one fill space with copies of the tetrahedron?

Aristotle claimed that one could tile space with copies of a *regular* tetrahedron, but 1800 years later he was proved wrong [Sen81]. In fact, it is not even possible to fill the space around one edge. If the *dihedral angle* formed by two sides of a regular tetrahedron were $60°$, then it would be possible to arrange six regular tetrahedra around an edge, but the actual dihedral angle is $\cos^{-1}(1/3) = 70.528779\ldots°$, which is not even a rational number of degrees (rational means an integer such as 60 or ratio of integers such as 7/3). One can arrange five regular tetrahedra around an edge, but then there is a gap that is too small to fit a sixth.

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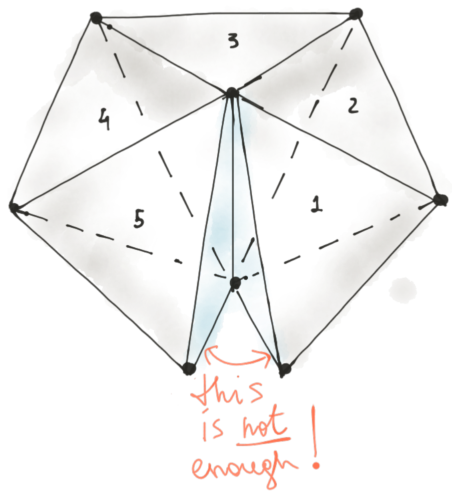


Placing the regular tetrahedra so that their sides do not line up does not help either.

On the other hand, some *nonregular* tetrahedra *can* tile space. It is still not known how to find them all, or even how to test whether a given tetrahedron can tile space.

(2) **Given a tetrahedron, is it scissors-congruent to a cube?** That is, can it be cut into pieces that can be reassembled into a cube?

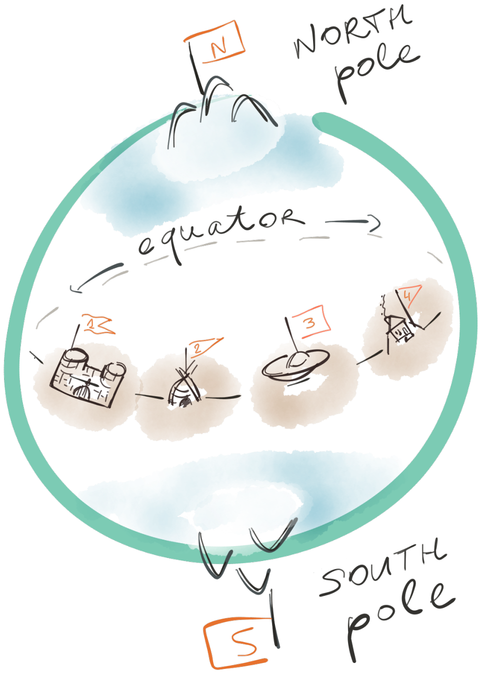
Hilbert realized that if the answer were always yes, then one could explain the formula for the volume of a tetrahedron without resorting to calculus. But he suspected that the answer was not always yes, and Hilbert's third problem (in a famous list he published in 1900) asked for a counterexample. His former student Dehn solved the problem and proved in particular that the regular tetrahedron is not scissors-congruent to a cube [Deh01]. Today, work of Dehn and Sydler [Syd65] lets one easily test whether a *given* tetrahedron is scissors-congruent to a cube, but an explicit description of *all* such tetrahedra is still not known. Such a description might also help with problem (1), since Debrunner proved that any tetrahedron that tiles space is scissors-congruent to a cube [Deb80].

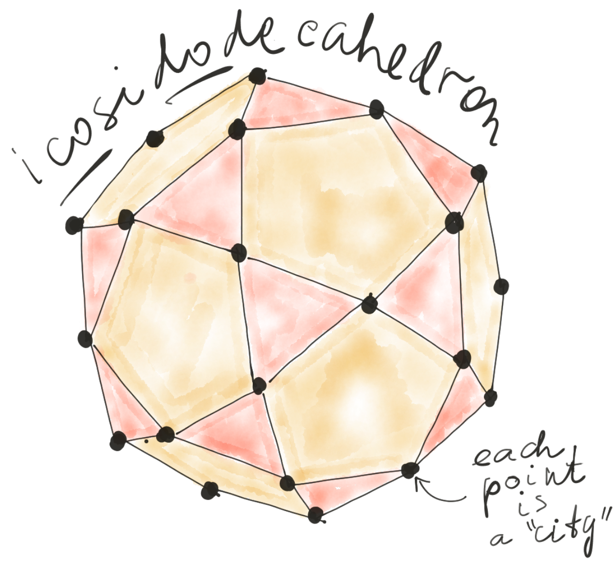
(3) **Can one describe all tetrahedra for which all six dihedral angles are a rational number of degrees?** This was asked explicitly by Conway and Jones in 1976 [CJ76]. These tetrahedra are of interest because any such tetrahedron is scissors-congruent to a cube.

In 1895, Hill discovered an infinite family of tetrahedra that are scissors-congruent to a cube, and among them are infinitely many with rational dihedral angles [Hil95]. Between then and 1974, fifteen more tetrahedra with rational dihedral angles were discovered [Hil95; Cox48, p. 192; Syd56; Gol58; Len62; Gol74]. We discovered a second infinite family and 44 more, bringing the total to two infinite families plus 59 sporadic tetrahedra. In addition, and this was the real challenge, we proved that there are no more beyond these [KKPR20, Theorem 1.8]. Thus problem (3) above is now solved!

Solving the problem required a mix of theoretical and computational techniques that we developed over the last 25 years. In a triangle, the three angles sum to 180°, but in a tetrahedron the six dihedral angles satisfy a more complicated equation involving 17 terms, each a product of cosines of some of the six angles. We solve this equation by applying ideas from algebraic number theory to reduce the problem to a series of computer calculations. Our argument builds upon an approach described already in [CJ76], but significant extra effort is needed to make the theoretical and computational results cover all cases.

Our solution to the Conway–Jones problem led, with more work, to the answer to another question: **How many cities can one place on a sphere such that the distance between any two cities is a rational number times the circumference of the sphere?** Distance here means distance *along the sphere* — no burrowing! One way is to place any number of cities equally spaced along the Equator, and two more cities at the North and South Poles; see the image at left below. We prove that, excluding configurations contained in one like this, the maximum number of cities is 30, and 30 cities are possible only if they are arranged at the vertices of an icosidodecahedron, shown at right below.





A team of MIT undergraduates will be exploring some of the questions left unanswered by our work, by classifying the tetrahedra in (1) and (2) above and investigating analogues in four dimensions and higher.

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