IRRATIONALITY OF π

BJORN POONEN

Lambert [4], Hermite [2], Cartwright [3, p. 268], Niven [5], and Bourbaki [1, III.§2, Exercise 5] gave successively simpler versions of a proof that π is irrational. If it is fair game to use e^{ix} instead of $\sin x$ and $\cos x$, we can simplify further:

Theorem. *The number π is irrational.*

Proof. Suppose that $\pi = a/b$ with $a, b \in \mathbb{Z}_{>0}$. Let $n \in \mathbb{Z}_{\geq 0}$ and let

$$f(x) = \frac{b^n x^n (\pi - x)^n}{n!} \in \frac{x^n}{n!}\mathbb{Z}[x],$$

so for all $k \geq 0$, we have $f^{(k)}(0) \in \mathbb{Z}$; also $f^{(k)}(\pi) \in \mathbb{Z}$, since $f(x) = f(\pi - x)$. Let $I = \int_0^\pi f(x) e^{ix}\,dx$. Evaluating I by repeated integration by parts (differentiating $f(x)$ while integrating e^{ix}, and using that $e^{ix} \in \mathbb{Z}$ when $x = 0$ or $x = \pi$) shows that $I \in \mathbb{Z}[i]$, so $\operatorname{im} I \in \mathbb{Z}$. On the other hand,

$$0 < \underbrace{\int_0^\pi f(x) \sin x\,dx}_{\operatorname{im} I} \leq \int_0^\pi \frac{b^n \pi^n \pi^n}{n!}\,dx = \frac{b^n \pi^{2n+1}}{n!} < 1$$

if n is large enough, contradicting $\operatorname{im} I \in \mathbb{Z}$. □

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Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139-4307, USA

Email address: poonen@math.mit.edu

URL: http://math.mit.edu/~poonen/