IRRATIONALITY OF π

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The following is essentially Niven's proof [Niv47], but we simplify by using e^{ix} instead of $\sin x$ and $\cos x$, and by using integration by parts instead of introducing a second auxiliary function out of the blue (Niven's $F(x)$).

Theorem. *The number π is irrational.*

Proof. Suppose that $\pi = a/b$ with $a, b \in \mathbb{Z}_{>0}$. Let $n \in \mathbb{Z}_{\geq 0}$ and let

$$f(x) = \frac{b^n x^n (\pi - x)^n}{n!} \in \frac{x^n}{n!}\mathbb{Z}[x],$$

so for all $k \geq 0$, we have $f^{(k)}(0) \in \mathbb{Z}$; also $f^{(k)}(\pi) \in \mathbb{Z}$, since $f(x) = f(\pi - x)$. Let $I = \int_0^\pi f(x) e^{ix}\, dx$. Evaluating I by repeated integration by parts (differentiating $f(x)$ while integrating e^{ix}, and using that $e^{ix} \in \mathbb{Z}$ when $x = 0$ or $x = \pi$) shows that $I \in \mathbb{Z}[i]$, so $\operatorname{im} I \in \mathbb{Z}$. On the other hand,

$$0 < \operatorname{im} I = \int_0^\pi f(x) \sin x\, dx \leq \int_0^\pi \frac{b^n \pi^n \pi^n}{n!}\, dx = \frac{b^n \pi^{2n+1}}{n!} < 1$$

if n is large enough, contradicting $\operatorname{im} I \in \mathbb{Z}$. □

References

[Niv47] Ivan Niven, *A simple proof that π is irrational*, Bull. Amer. Math. Soc. **53** (1947), 509. MR 21013

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