

18.03 LECTURE NOTES, SPRING 2025

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These are an approximation of what was covered in lecture. (Please clear your browser's cache before reloading this file to make sure you are getting the current version.) This PDF file is divided into sections; the instructions for viewing the table of contents depend on which PDF viewer you are using.

If your PDF viewer is linked to a browser, you should be able to click on URLs, such as the ones below to go to the online mathlets:

<http://mathlets.org/mathlets/>

<https://web-cert.mit.edu/jorloff/www/jmoapplets/applets.html>

Small text contains a technical explanation that you might want to ignore when reading for the first time.

February 3

1. SOLUTIONS TO DIFFERENTIAL EQUATIONS

1.1. Introduction. A **differential equation (DE)** is an equation relating an unknown function and some of its derivatives. DEs arise in engineering, chemistry, biology, physics, economics, etc., because many laws of nature describe the instantaneous rate of change of a quantity in terms of current conditions.

Overall goals: to learn how to

- model real-world problems with DEs;
- solve DEs exactly when possible, or else solve numerically (get an approximate solution);
- extract qualitative information from a DE, whether or not it can be solved.

(There is ongoing research on these questions. For instance, there is a \$1,000,000 prize for understanding the solutions to the **Navier–Stokes equations** modeling fluid flow.)

1.2. Notation and units. (To be done in recitation on Feb. 4.)

Notation for higher derivatives of a function $y(t)$:

$$\begin{aligned}
\text{first derivative: } & \dot{y}, y', \frac{dy}{dt} \\
\text{second derivative: } & \ddot{y}, y'', \frac{d^2y}{dt^2} \\
\text{third derivative: } & \dddot{y}, y^{(3)}, \frac{d^3y}{dt^3} \\
& \vdots \\
n^{\text{th}} \text{ derivative: } & y^{(n)}
\end{aligned}$$

Warning: The notation $\dot{y}$ is a standard abbreviation for $\frac{dy}{dt}$; use it only for the derivative with respect to *time*. If y is a function of x , write y' or $\frac{dy}{dx}$ instead.

If y has units m (meters) and t has units s (seconds), then $\frac{dy}{dt}$ has the same units as $\frac{y}{t}$ would, namely m/s (meters per second). Similarly, $\ddot{y}$ would have units m/s².

1.3. A secret function.

Example: Can you guess my secret function $y(t)$?

Clue: It satisfies the DE

$$\dot{y} = 3y. \tag{1}$$

(This might model population growth in some biological system.)

Maybe you guessed $y = e^{3t}$. This is *a* solution to the differential equation (1), because substituting it into the DE gives $3e^{3t} = 3e^{3t}$. But it's not the function I was thinking of! Some other solutions are $y = 7e^{3t}$, $y = -5e^{3t}$, $y = 0$, etc. Later we'll explain why the **general solution** to (1) is

$$y = ce^{3t}, \quad \text{where } c \text{ is a parameter;}$$

saying this means that

- for each number c , the function $y = ce^{3t}$ is a solution, and
- there are no other solutions besides these.

So there is a 1-parameter family of solutions to (1).

You still haven't guessed my secret function.

Clue 2: My function satisfies the **initial condition** $y(0) = 6$.

Solution: There is a number c such that $y(t) = ce^{3t}$ holds for all t ; we need to find c . Plugging in $t = 0$ shows that $6 = ce^0$, so $c = 6$. Thus, among the infinitely many solutions to the DE, the **particular solution** satisfying the initial condition is $y(t) = 6e^{3t}$. $\square$

Important: Checking a solution to a DE is usually easier than finding the solution in the first place, so it is often worth doing. Just plug in the function to both sides, and also check that it satisfies the initial condition.

In (1) only one initial condition was needed, since only one parameter c needed to be recovered.

1.4. **Classification of differential equations.** There are two kinds:

- **Ordinary differential equation (ODE):** involves derivatives of a function of *only one* variable.

$$\ddot{y} = -9y \quad (\text{solve for } y(t))$$

- **Partial differential equation (PDE):** involves *partial derivatives* of a *multivariable* function.

$$\frac{\partial u}{\partial t} = 9 \frac{\partial^2 u}{\partial x^2} \quad (\text{solve for } u(x, t))$$

Order of an ODE: the highest n such that the n^{th} derivative of the function appears.

(The definition of order for a PDE is similar; just know that a term like $\frac{\partial^2 u}{\partial x^2}$ or $\frac{\partial^2 u}{\partial x \partial t}$ counts as a 2nd derivative since it is a partial derivative of a partial derivative.)

Example 1.1. Is

$$\begin{aligned} & 707099375 \cos(t^5) \ddot{y}^4 + 3487980982(y + t^3)^7 \dot{y} \\ & - 389750387y^{(3)}y^{(4)} + 2ty^{(5)} + 8453723054723985730987 \\ & = 80970874y^6 - 2809754087 \sin(t/y) + 8957092 \ln(1 - t^7) \\ & + 64893745723786e^{y^8 - t^3} + 987t^6 + 543y^2 + 18.03 ? \end{aligned}$$

an ODE or a PDE? **ODE.**

Question: *What is its order?*

The order is **5**, because the highest derivative that appears is the 5th derivative, $y^{(5)}$.

2. MODELING

I sometimes tell people that I have a career in modeling. We're going to talk about **mathematical modeling**, which is converting a real-world problem into mathematical equations.

Guidelines:

1. Identify relevant quantities, both known and unknown, and give them symbols. Find the units for each.
2. Identify the independent variable(s). The other quantities will be functions of them, or constants. Often, *time* is the only independent variable.

3. Write down equations expressing how the functions change in response to small changes in the independent variable(s). Also write down any “laws of nature” relating the variables. As a check, make sure that each summand in an equation has the same units.

Often simplifying assumptions need to be made; the challenge is to simplify the equations so that they can be solved but so that they still describe the real-world system well.

2.1. Example: savings account.

Problem 2.1. I have a savings account earning interest compounded daily, and I make frequent deposits or withdrawals into the account. Find an ODE with initial condition to model the balance.

Simplifying assumptions: Daily compounding is almost the same as continuous compounding, so let’s assume that interest is paid continuously instead of at the end of each day. Similarly, let’s assume that my deposits/withdrawals are frequent enough that they can be approximated by a continuous money flow at a certain rate, the **net deposit rate** (which is negative when I am withdrawing). Finally, let’s assume that the interest rate and net deposit rate vary continuously with time, but do not depend on the balance.

Variables and functions (with units): Define the following:

P : the **principal**, the initial amount that the account starts with (dollars)

t : time from the start (years)

x : balance (dollars)

I : the interest rate (year^{-1}) (e.g., $4\%/\text{year} = 0.04 \text{ year}^{-1}$)

q : the net deposit rate (dollars/year).

Here t is the independent variable, P is a constant, and x , I , q are functions of t .

Equations: During a time interval $[t, t + dt]$ for an “infinitesimally small” increment dt , the following hold (technically speaking, dt is a differential; if dt were replaced by a positive number Δt , then the equations below would be only approximations, but when we divide by Δt and take a limit, the end result is the same):

$$\text{interest earned per dollar} = I(t) dt$$

$$\text{interest earned} = I(t)x(t) dt \quad (\text{asked as question to class})$$

$$\text{amount deposited into the account} = q(t) dt$$

so

$$dx = \text{change in balance} = I(t)x(t) dt + q(t) dt$$

$$\frac{dx}{dt} = I(t)x(t) + q(t).$$

(Check: the units in each of the three terms are dollars/year.) Also, there is the initial condition $x(0) = P$. Thus we have an ODE with initial condition:

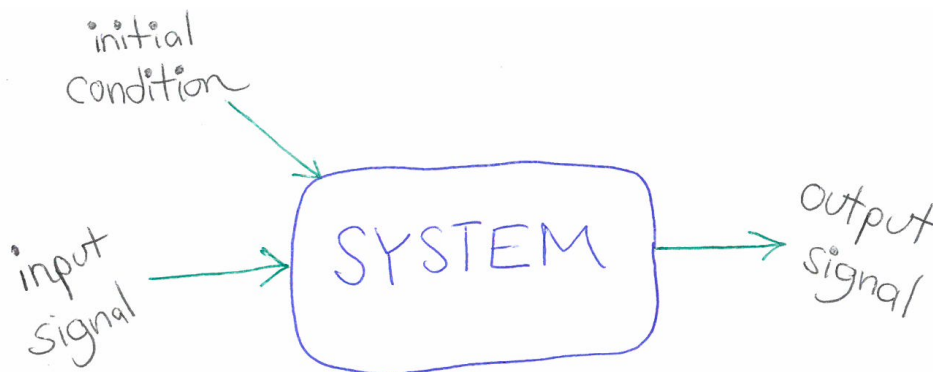
$$\dot{x} = I(t)x + q(t), \quad x(0) = P. \quad \square \quad (2)$$

Now that the modeling is done, the next step might be to *solve* (2) for the function $x(t)$, but we won't do that yet.

2.2. Systems and signals. Maybe for financial planning I am interested in testing different saving strategies (different functions q) to see what balances x they result in. To help with this, rewrite the ODE as

$$\underbrace{\dot{x} - I(t)x}_{\text{controlled by bank}} = \underbrace{q(t)}_{\text{controlled by me}}.$$

In the “systems and signals” language of engineering, q is called the *input signal*, the bank is the *system*, and x is the *output signal*. These terms do not have a mathematical meaning dictated by the DE alone; their interpretation is guided by what is being modeled. But the general picture is this:



- The **input signal** is a function of the independent variable alone, a function that enters into the DE somehow (usually the right side of the DE, or part of the right side).
- The **system** processes the input signal by solving the DE with the given initial condition.
- The **output signal** (also called **system response**) is the solution to the DE.

3. SEPARATION OF VARIABLES FOR FIRST-ORDER ODES

(To be done in recitation on Feb. 4.)

Separation of variables is a technique that quickly solves some simple first-order ODEs. Here is how it works:

1. Check that the DE is a *first-order ODE*. (If not, give up and try another method.)
Suppose that the function to be solved for is $y = y(t)$.
2. Rewrite $\dot{y}$ as $\frac{dy}{dt}$.

3. Add and/or subtract to move terms to the other side of the DE so that the term with $\frac{dy}{dt}$ is on the left and all other terms are on the right.
4. Try to separate the y 's and t 's. Specifically, try to multiply and/or divide (and in particular move the dt to the right side) so that it ends up as an equality of differentials of the form

$$f(y) dy = g(t) dt.$$

Note: If there are factors involving both variables, such as $y + t$, then it is impossible to separate variables; in this case, give up and try a different method.

Warning: Dividing the equation by an expression invalidates the calculation if that expression is 0, so at the end, check what happens if the expression is 0; this may add to the list of solutions.

5. Integrate both sides to get an equation of the form

$$F(y) = G(t) + C.$$

These are implicit equations for the solutions, in terms of a parameter C .

6. If possible (and if desired), solve for y in terms of t .
7. Check for extra solutions coming from the warning in Step 4. The solutions in the previous step and this step comprise the general solution.
8. (Optional, but recommended) Check your work by verifying that the general solution actually satisfies the original DE.

Problem 3.1. Solve $\dot{y} - 2ty = 0$.

Solution:

Step 1. This involves only the *first* derivative of a *one-variable* function $y(t)$, so it is a first-order ODE. Thus we can *attempt* separation of variables.

Step 2. Rewrite as $\frac{dy}{dt} - 2ty = 0$.

Step 3. Isolate the $\frac{dy}{dt}$ term: $\frac{dy}{dt} = 2ty$.

Step 4. We can separate variables! Namely, $\frac{1}{y} dy = 2t dt$. (**Warning:** We divided by y , so at some point we will have to check $y = 0$ as a potential solution.)

Step 5. Integrate: $\ln |y| = t^2 + C$.

Step 6. Solve for y :

$$\begin{aligned} |y| &= e^{t^2+C} \\ y &= \pm e^C e^{t^2}. \end{aligned}$$

As C runs over all real numbers, and as the $\pm$ sign varies, the coefficient $\pm e^C$ runs over all nonzero real numbers. Thus these solutions are $y = ce^{t^2}$ for all nonzero c .

Step 7. Because of Step 4, we need to check also the constant function $y = 0$; it turns out that it is a solution too. It can be considered as the function ce^{t^2} for $c = 0$.

Conclusion: The general solution to $\dot{y} - 2ty = 0$ is

$$y = ce^{t^2}, \quad \text{where } c \text{ is an arbitrary real number. } \square$$

Step 8. Plugging in $y = ce^{t^2}$ to $\dot{y} - 2ty = 0$ gives $ce^{t^2}(2t) - 2tce^{t^2} = 0$, which is true, as it should be.

4. LINEAR ODES VS. NONLINEAR ODES

4.1. Linear ODEs.

4.1.1. *Building a homogeneous linear ODE.* One way to build a DE is as follows:

1. Start with a list like

$$\ddot{y} \quad \dot{y} \quad y$$

in which each term is one of $y, \dot{y}, \ddot{y}, \dots$ (it's OK to skip some).

2. Multiply each term by a function of t (possibly a *constant* function):

$$e^t \ddot{y} \quad 5 \dot{y} \quad t^9 y.$$

3. Add them up and set the result equal to 0:

$$e^t \ddot{y} + 5 \dot{y} + t^9 y = 0.$$

Any DE that arises in this way is called a **homogeneous linear ODE**.

(“Homogeneous” has an e after the n, and the e is pronounced!)

The functions e^t , 5, and t^9 used in Step 2 are called the **coefficients**.

Most general n^{th} order homogeneous linear ODE:

$$p_n(t) y^{(n)} + \dots + p_1(t) \dot{y} + p_0(t) y = 0$$

for some functions $p_n(t), \dots, p_0(t)$.

4.1.2. *Building an inhomogeneous linear ODE.* If you start with a homogeneous linear ODE, and replace the 0 on the right by **a function of t only**, the result is called an **inhomogeneous linear ODE**. The function of t could be a constant function, but it is not allowed to involve y . For example,

$$e^t \ddot{y} + 5 \dot{y} + t^9 y = 7 \sin t + 2$$

is an inhomogeneous linear ODE. So is

$$e^t \ddot{y} + 5 \dot{y} + t^9 y = 2.$$

Most general n^{th} order inhomogeneous linear ODE:

$$p_n(t) y^{(n)} + \cdots + p_1(t) \dot{y} + p_0(t) y = q(t)$$

for some functions $p_n(t), \dots, p_0(t), q(t)$.

Imagine feeding different “input signals” $q(t)$ into the right hand side of an inhomogeneous linear ODE to see what “output signals” $y(t)$ the system responds with.

4.1.3. *Both kinds together.* In testing whether an ODE is a homogeneous linear ODE or inhomogeneous linear ODE, you are allowed to rearrange the terms. A **linear ODE** is an ODE that can be rearranged into one of these two types.

Remark 4.1. If you already know that an ODE is linear, there is an easy test to decide if it is homogeneous or not: plug in the constant function $y = 0$.

- If $y = 0$ is a solution, the ODE is homogeneous.
- If $y = 0$ is not a solution, the ODE is inhomogeneous.

4.1.4. *Standard linear form.* Dividing the DE

$$e^t \ddot{y} + 5 \dot{y} + t^9 y = 0.$$

by the leading coefficient e^t gives an equivalent DE

$$\ddot{y} + \frac{5}{e^t} \dot{y} + \frac{t^9}{e^t} y = 0.$$

The same can be done for any linear ODE, to put it in **standard linear form**

$$y^{(n)} + p_{n-1}(t) y^{(n-1)} + \cdots + p_1(t) \dot{y} + p_0(t) y = q(t)$$

for some functions $p_{n-1}(t), \dots, p_0(t), q(t)$ (not the same ones as before the division).

For now, we assume that we are looking for a solution $y(t)$ defined on an open interval I , and that the functions $p_{n-1}(t), \dots, p_0(t), q(t)$ are continuous (or at least piecewise continuous) on I . **Open interval** means a connected set of real numbers without endpoints, i.e., one of the following: (a, b) , $(-\infty, b)$, (a, ∞) , or $(-\infty, \infty) = \mathbb{R}$.

4.2. **Nonlinear ODEs.** For an ODE to be nonlinear, the functions $y, \dot{y}, \dots$ must enter the equation in a more complicated way: raised to powers, multiplied by each other, or with nonlinear functions applied to them.

Question: Which of the following ODEs is linear?

$$\ddot{y} - 7t \dot{y} = 0$$

$$\ddot{y} = e^t(y + t^2)$$

$$\dot{y} - y^2 = 0$$

$$\dot{y}^2 - ty = \sin t$$

$$\dot{y} = \cos(y + t).$$

Answer: The second one is linear since it can be rearranged into

$$\ddot{y} + (-e^t)y = t^2 e^t.$$

The others are nonlinear (the nonlinear portion is **highlighted in red**).

5. SOME ADVICE FOR SUCCESS IN 18.03

- Read the Information pages on Canvas.
- Do the reading *before* lecture! (See Canvas, Modules.)
- Come to office hours!
- Start the homework now.
- Work together (psetpartners.mit.edu).
- By the time you submit an assignment, you should be pretty sure that it is complete and correct.

February 5

6. SOLVING A FIRST-ORDER LINEAR ODE

Every *first-order* linear ODE in standard linear form is as follows:

$$\text{Homogeneous: } \dot{y} + p(t)y = 0$$

$$\text{Inhomogeneous: } \dot{y} + p(t)y = q(t).$$

6.1. Homogeneous equations: separation of variables. Homogeneous first-order linear ODEs can always be solved by separation of variables:

$$\begin{aligned}\dot{y} + p(t)y &= 0 \\ \frac{dy}{dt} + p(t)y &= 0 \\ \frac{dy}{dt} &= -p(t)y \\ \frac{dy}{y} &= -p(t)dt \quad (\text{assume for now that } y \text{ is not } 0).\end{aligned}$$

Choose *any* antiderivative $P(t)$ of $p(t)$. Integrating gives

$$\begin{aligned}\ln |y| &= -P(t) + C \\ |y| &= e^{-P(t)+C} \\ y &= \pm e^C e^{-P(t)} \\ y &= ce^{-P(t)},\end{aligned}$$

where c is any number (we brought back the solution $y = 0$ corresponding to $c = 0$).

If you choose a different antiderivative, it will have the form $P(t) + d$ for some constant d , and then the new $e^{-P(t)}$ is just a constant e^{-d} times the old one, so the set of *all* scalar multiples of the function $e^{-P(t)}$ is the same as before.

Conclusion:

Theorem 6.1 (General solution to first-order homogeneous linear ODE). *Let $p(t)$ be a continuous function on an open interval I (this ensures that $p(t)$ has an antiderivative). Let $P(t)$ be any antiderivative of $p(t)$. Then the general solution to $\dot{y} + p(t)y = 0$ is $y = ce^{-P(t)}$, where c is a parameter.*

6.2. Inhomogeneous equations: variation of parameters. **Variation of parameters** is a method for solving inhomogeneous linear ODEs. Given a first-order inhomogeneous linear ODE

$$\dot{y} + p(t)y = q(t), \quad (3)$$

follow these steps:

1. Find a nonzero solution, say y_h , of the associated homogeneous ODE

$$\dot{y} + p(t)y = 0.$$

(You need just *one* nonzero solution. If instead you found the *general* solution to the homogeneous ODE, set the parameter c equal to 1, say, to get one solution.)

2. For an undetermined function $u(t)$, substitute

$$y = u(t)y_h(t) \quad (4)$$

into the *inhomogeneous* equation (3) and solve for $u(t)$, to find all choices of $u(t)$ that make this y a solution to the inhomogeneous equation.

3. Now that the general $u(t)$ has been found, plug it back into $y = u(t)y_h(t)$ to get the general solution to the inhomogeneous equation. $\square$

The reason for considering uy_h in Step 2 is this: we know that if we try cy_h for a *constant* c in the inhomogeneous equation

$$\dot{y} + p(t)y = q(t),$$

it won't work since the left side will evaluate to 0 (the function cy_h is a solution to the *homogeneous* equation). Therefore instead we try uy_h for a *function* $u(t)$, and try to figure out *which* functions u will make the left side evaluate to $q(t)$. (That's why it's called variation of parameters: the parameter c has been replaced by something varying.)

Problem 6.2. Solve $t\dot{y} + 2y = t^5$ on the interval $(0, \infty)$.

Solution:

Step 1. The associated homogeneous equation is $t\dot{y} + 2y = 0$, or equivalently, $\dot{y} + \frac{2}{t}y = 0$. Solve by separation of variables:

$$\begin{aligned}\frac{dy}{dt} &= -\frac{2}{t}y \\ \frac{dy}{y} &= -\frac{2}{t} dt \\ \ln |y| &= -2 \ln t + C \quad (\text{since } t > 0) \\ y &= ce^{-2 \ln t} \\ y &= ct^{-2}.\end{aligned}$$

Choose *one* nonzero solution, say $y_h = t^{-2}$.

Step 2. Substitute $y = ut^{-2}$ into the inhomogeneous equation: the left side is

$$t\dot{y} + 2y = t(\dot{u}t^{-2} + u(-2t^{-3})) + 2ut^{-2} = t^{-1}\dot{u},$$

so the inhomogeneous equation becomes

$$\begin{aligned}t^{-1}\dot{u} &= t^5 \\ \dot{u} &= t^6 \\ u &= \frac{t^7}{7} + c.\end{aligned}$$

Step 3. The general solution to the inhomogeneous equation is

$$y = ut^{-2} = \left(\frac{t^7}{7} + c\right)t^{-2} = \frac{t^5}{7} + ct^{-2}.$$

(If you want, check by direct substitution that this really is a solution.) $\square$

6.3. Inhomogeneous equations: integrating factor. (Done in recitation.)

Another approach to solving

$$\dot{y} + p(t)y = q(t) \tag{5}$$

is to use an **integrating factor**:

1. Find an antiderivative $P(t)$ of $p(t)$.

2. Multiply both sides of the ODE by the integrating factor $e^{P(t)}$ in order to make the left side the derivative of something:

$$\begin{aligned} e^{P(t)} \dot{y} + e^{P(t)} p(t) y &= q(t) e^{P(t)} \\ \frac{d}{dt} (e^{P(t)} y) &= q(t) e^{P(t)} \\ e^{P(t)} y &= \int q(t) e^{P(t)} dt \\ y &= e^{-P(t)} \int q(t) e^{P(t)} dt. \end{aligned}$$

Here $\int q(t) e^{P(t)} dt$ represents all possible antiderivatives of $q(t) e^{P(t)}$, so there are infinitely many solutions.

If you fix one antiderivative, say $R(t)$, then the others are $R(t) + c$ for a constant c , so the general solution is

$$y = R(t) e^{-P(t)} + c e^{-P(t)}. \quad \square$$

6.4. Linear combinations. A **linear combination** of a list of functions is any function that can built from them by scalar multiplication and addition.

linear combinations of $f(t)$:	the functions $cf(t)$, where c is any number
linear combinations of $f_1(t)$ and $f_2(t)$:	the functions of the form $c_1 f_1(t) + c_2 f_2(t)$, where c_1 and c_2 are any numbers.
	⋮

Examples:

- $2 \cos t + 3 \sin t$ is a linear combination of the functions $\cos t$ and $\sin t$.
- $9t^5 + 3$ is a linear combination of the functions t^5 and 1 .

Question: One of the functions below is **not** a linear combination of $\cos^2 t$ and 1 . Which one?

1. $3 \cos^2 t - 4$
2. $\sin^2 t$
3. $\sin(2t)$
4. $\cos(2t)$
5. 5
6. 0

Answer: 3.

All the others are linear combinations:

$$3 \cos^2 t - 4 = 3 \cos^2 t + (-4) \cdot 1$$

$$\sin^2 t = (-1) \cos^2 t + 1 \cdot 1$$

$$\sin(2t) = ???$$

$$\cos(2t) = 2 \cos^2 t + (-1) \cdot 1$$

$$5 = 0 \cos^2 t + 5 \cdot 1$$

$$0 = 0 \cos^2 t + 0 \cdot 1.$$

Could there be some fancy identity that expresses $\sin(2t)$ as a linear combination of $\cos^2 t$ and 1? No; here's one way to see this: Every linear combination of $\cos^2 t$ and 1 has the form

$$c_1 \cos^2 t + c_2$$

for some numbers c_1 and c_2 . All such functions are *even* functions, but $\sin(2t)$ is an *odd* function. (Warning: This trick might not work in other situations.)

6.5. Superposition.

Problem 6.3 (Multiplying an input signal by 9). Fill in the blank:

Given that $t^5/7$ is one solution to $t\dot{y} + 2y = t^5$,

it follows that _____ is one solution to $t\dot{y} + 2y = 9t^5$.

You probably guessed that to get a right hand side that is 9 times as large, the solution needs to be 9 times as large: $9t^5/7$. This is a correct possible answer!

Why does this work? Imagine plugging in $y = 9t^5/7$ instead of $y = t^5/7$ into the left side of the DE; then each of $\dot{y}$ and y will be 9 times larger, so $t\dot{y} + 2y$ will be 9 times larger too; that is, it will equal $9t^5$ instead of t^5 .

What special property of $t\dot{y} + 2y$ ensured that it would be 9 times larger? It is that each summand ($t\dot{y}$ and $2y$) is a function of t times one of y , $\dot{y}$, $\dots$, so that when y is multiplied by 9, each summand gets multiplied by 9. In other words, this worked precisely because the DE was *linear*!

In the language of systems and signals, if the input signal (right hand side) is multiplied by 9, the output signal (the solution) is multiplied by 9.

Problem 6.4 (Adding two input signals). Fill in the blank:

Given that $t^5/7$ is one solution to $t\dot{y} + 2y = t^5$,

and $1/2$ is one solution to $t\dot{y} + 2y = 1$,

it follows that _____ is one solution to $t\dot{y} + 2y = t^5 + 1$.

As you probably guessed, $t^5/7 + 1/2$ is a possible answer. Again, this works because the DE is *linear*.

Adding input signals is also called superimposing them; this explains the name of the general principle:

Superposition principle (works for linear DEs only).

1.

Multiplying a solution to $p_n(t)y^{(n)} + \cdots + p_0(t)y = q(t)$ by a number a gives a solution to $p_n(t)y^{(n)} + \cdots + p_0(t)y = aq(t)$.

2.

Adding a solution of $p_n(t)y^{(n)} + \cdots + p_0(t)y = q_1(t)$
to a solution of $p_n(t)y^{(n)} + \cdots + p_0(t)y = q_2(t)$
gives a solution of $p_n(t)y^{(n)} + \cdots + p_0(t)y = q_1(t) + q_2(t)$.

Using both parts gives a principle for linear combinations, as in the following example.

Problem 6.5 (Linear combination of input signals). Fill in the blank:

Given that $t^5/7$ is one solution to $t\dot{y} + 2y = t^5$,
and $1/2$ is one solution to $t\dot{y} + 2y = 1$,
it follows that _____ is one solution to $t\dot{y} + 2y = 9t^5 + 3$.

One possible answer: $9t^5/7 + 3/2$.

6.6. Consequence of superposition for an inhomogeneous linear DE.

Problem 6.6. Fill in the blank:

Given that ct^{-2} is the *general* solution to the homogeneous DE $t\dot{y} + 2y = 0$,
and $t^5/7$ is a particular solution to the **inhomogeneous** DE $t\dot{y} + 2y = t^5$,
it follows that _____ is the *general* solution to the **inhomogeneous** DE $t\dot{y} + 2y = t^5$.

Answer: $t^5/7 + ct^{-2}$. This is great news: if you already solved the homogeneous DE, you just have to find *one* solution to the inhomogeneous DE to build *all* solutions to the inhomogeneous DE!

Why does this work? For any number c , superposition says that adding ct^{-2} and $t^5/7$ will give a solution to the inhomogeneous DE with right side $0 + t^5 = t^5$, but why do *all* solutions of that inhomogeneous DE arise this way? It is because the process can be reversed: given

any solution to the inhomogeneous DE $t\dot{y} + 2y = t^5$ we can *subtract* the particular solution $y_p = t^5/7$ to the same DE to get a solution to the homogeneous DE $t\dot{y} + 2y = 0$.

The strategy suggested by Problem 6.6 can help you find the general solution y_i to any

$$\text{inhomogeneous linear DE: } p_n(t) y^{(n)} + \cdots + p_0(t) y = q(t) :$$

1. List all solutions to the associated

$$\text{homogeneous linear DE: } p_n(t) y^{(n)} + \cdots + p_0(t) y = 0;$$

i.e., write down *its* general solution y_h .

2. Find (in some way) any *one* particular solution y_p to the *inhomogeneous* DE.
3. Add y_p to all the solutions of the homogeneous DE to get all the solutions to the inhomogeneous DE.

Summary:

general solution to homogeneous equation



add one particular inhomogeneous solution y_p

general solution to inhomogeneous equation

As an equation:

$$\begin{array}{ccccc} y_i & = & y_p & + & y_h \\ \text{general inhomogeneous solution} & & \text{particular inhomogeneous solution} & & \text{general homogeneous solution} \end{array} .$$

6.7. Newton's law of cooling.

Problem 6.7. My minestrone soup is in an insulating thermos. Model its temperature as a function of time.

Simplifying assumptions:

- The insulating ability of the thermos does not change with time.
- The rate of cooling depends only on the *difference* between the soup temperature and the external temperature.

Variables and functions (with units): Define the following:

t : time (minutes)

x : external temperature ($^{\circ}\text{C}$)

y : soup temperature ($^{\circ}\text{C}$)

Here t is the independent variable, and x and y are functions of t .

Equation:

$$\dot{y} = f(y - x)$$

for some function f . Another simplifying assumption: $f(z) = -kz + \ell$ for some constants k and ℓ (any reasonable function be approximated on small inputs by its linearization at 0); this leads to

$$\dot{y} = -k(y - x) + \ell.$$

Common sense says

- If $y = x$, then $\dot{y} = 0$. Thus ℓ should be 0.
- If $y > x$, then y is decreasing. This is why we wrote $-k$ instead of just k .

So the equation becomes

$$\dot{y} = -k(y - x)$$

This is **Newton's law of cooling**: the rate of cooling of an object is proportional to the difference between its temperature and the external temperature. The (positive) constant k is called the **coupling constant**, in units of minutes^{-1} ; smaller k means better insulation, and $k = 0$ is perfect insulation. This ODE can be rearranged into standard form:

$$\dot{y} + ky = kx.$$

It's a first-order inhomogeneous linear ODE! The input signal is x , the system is the thermos, and the output signal is y .

Wed Feb 7 lecture actually ended here.

Special case: suppose that the external temperature x is constant. Then

- One particular solution to the inhomogeneous ODE: $y_p = x$
(the solution in which the soup is already in equilibrium with the exterior)
- General solution to the *homogeneous* ODE: $y_h = ce^{-kt}$.
- General solution to the inhomogeneous ODE: $y = x + ce^{-kt}$.
(As $t \rightarrow \infty$, the soup temperature approaches x ; this makes sense.)

6.8. Things to remember when solving first-order ODEs.

1. Find the general solution first!

- Try **separation of variables**, since this is often easier than the other methods, if it works.

If the ODE is linear:

- **Homogeneous**: Use **separation of variables**. The general solution will be $cf(t)$ for some function f .
- **Inhomogeneous**: Choose one of the following methods.
 - (i) Feeling lucky? Guess (and check) a particular solution y_p and write down

$$y_p + (\text{general homogeneous solution}).$$

- (ii) **Variation of parameters**: Find *one* homogeneous solution y_h and plug in $u y_h$.

- (iii) **Integrating factor** (to be discussed in recitation).
2. Use the initial conditions. This should always be the very last step.

February 7

7. EXISTENCE AND UNIQUENESS OF SOLUTIONS

Using separation of variables (in the homogeneous case) and variation of parameters (in the inhomogeneous case), we showed that every 1st order linear ODE has a 1-parameter family of solutions. To nail down a specific solution in this family, we need 1 initial condition, such as $y(0)$.

It will turn out that every 2nd order linear ODE has a 2-parameter family of solutions. To nail down a specific solution, we need 2 initial conditions at the same starting time, $y(0)$ and $\dot{y}(0)$. The starting time could also be some number a other than 0.

Example 7.1 (Height of a ball). The height y of a thrown ball satisfies the 2nd order linear ODE $\ddot{y} = -9.8 \text{ m/s}^2$. To determine $y(t)$, it is enough to know the initial position $y(0)$ and initial velocity $y'(0)$.

Here is the general result:

Existence and uniqueness theorem for a linear ODE. *Let $p_{n-1}(t), \dots, p_0(t), q(t)$ be continuous functions on an open interval I . Let $a \in I$, and let $b_0, \dots, b_{n-1}$ be given numbers. Then there is exactly one solution to the n^{th} order linear ODE*

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \dots + p_1(t)\dot{y} + p_0(t)y = q(t)$$

satisfying the n initial conditions

$$y(a) = b_0, \quad \dot{y}(a) = b_1, \quad \dots, \quad y^{(n-1)}(a) = b_{n-1}.$$

Existence means that there is *at least* one solution.

Uniqueness means that there is *only* one solution.

Remark 7.2. For a linear ODE as above, the solution $y(t)$ is defined on the whole interval I where the functions $p_{n-1}(t), \dots, p_0(t), q(t)$ are continuous. In particular, if $p_{n-1}(t), \dots, p_0(t), q(t)$ are continuous on all of $\mathbb{R}$, then the solution $y(t)$ will be defined on all of $\mathbb{R}$.

8. COMPLEX NUMBERS

Complex numbers are expressions of the form $a + bi$, where a and b are real numbers, and i is a new symbol. Multiplication of complex numbers will eventually be defined so that $i^2 = -1$. (Electrical engineers sometimes write j instead of i , because they want to reserve i for current, but everybody else thinks that's weird.)

Just as the set of all real numbers is denoted $\mathbb{R}$, the set of all complex numbers is denoted $\mathbb{C}$. The notation " $\alpha \in \mathbb{C}$ " means literally that α is an element of the set of complex numbers, so it is a short way of saying " α is a complex number".

Question: Is 9 a real number or a complex number?

Possible answers:

1. real number
2. complex number
3. both
4. neither

Answer: Both, because 9 can be identified with $9 + 0i$.

8.1. Operations on complex numbers.

real part $\operatorname{Re}(a + bi) := a$

imaginary part $\operatorname{Im}(a + bi) := b$ (**Note:** It is b , not bi , so $\operatorname{Im}(a + bi)$ is real!)

complex conjugate $\overline{a + bi} := a - bi$ (negate the imaginary component)

Question: What is $\operatorname{Im}(17 - 83i)$?

Possible answers:

1. 17
2. $17i$
3. 83
4. -83
5. $83i$
6. $-83i$

Answer: 4. The imaginary part is -83 , without the i .

(In lecture there was a joke about the Greek letter Ξ ; sorry, you had to be there.)

One can **add**, **subtract**, **multiply**, and **divide** complex numbers (except for division by 0). Addition, subtraction, and multiplication are defined as for polynomials, except that after

multiplication one simplifies by using $i^2 = -1$; for example,

$$\begin{aligned}(2 + 3i)(1 - 5i) &= 2 - 7i - 15i^2 \\ &= 17 - 7i.\end{aligned}$$

To divide z by w , multiply z/w by $\overline{w}/\overline{w}$ so that the denominator becomes real; for example,

$$\frac{2 + 3i}{1 - 5i} = \frac{2 + 3i}{1 - 5i} \cdot \frac{1 + 5i}{1 + 5i} = \frac{2 + 13i + 15i^2}{1 - 25i^2} = \frac{-13 + 13i}{26} = -\frac{1}{2} + \frac{1}{2}i.$$

The arithmetic operations on complex numbers satisfy the same properties as for real numbers ($zw = wz$ and so on). The mathematical jargon for this is that $\mathbb{C}$, like $\mathbb{R}$, is a **field**. In particular, for any complex number z and integer n , the **n^{th} power** z^n can be defined in the usual way (need $z \neq 0$ if $n < 0$); e.g., $z^3 := zzz$, $z^0 := 1$, $z^{-3} := 1/z^3$. (**Warning:** Although there is a way to define z^n also for a *complex* number n , when $z \neq 0$, it turns out that z^n has more than one possible value for non-integral n , so it is ambiguous notation. Anyway, the most important cases are e^z , and z^n for integers n ; the other cases won't even come up in this class.)

If you change every i in the universe to $-i$ (that is, take the complex conjugate everywhere), then all true statements remain true. For example, $i^2 = -1$ becomes $(-i)^2 = -1$. Another example: If $z = v + w$, then $\overline{z} = \overline{v} + \overline{w}$; in other words,

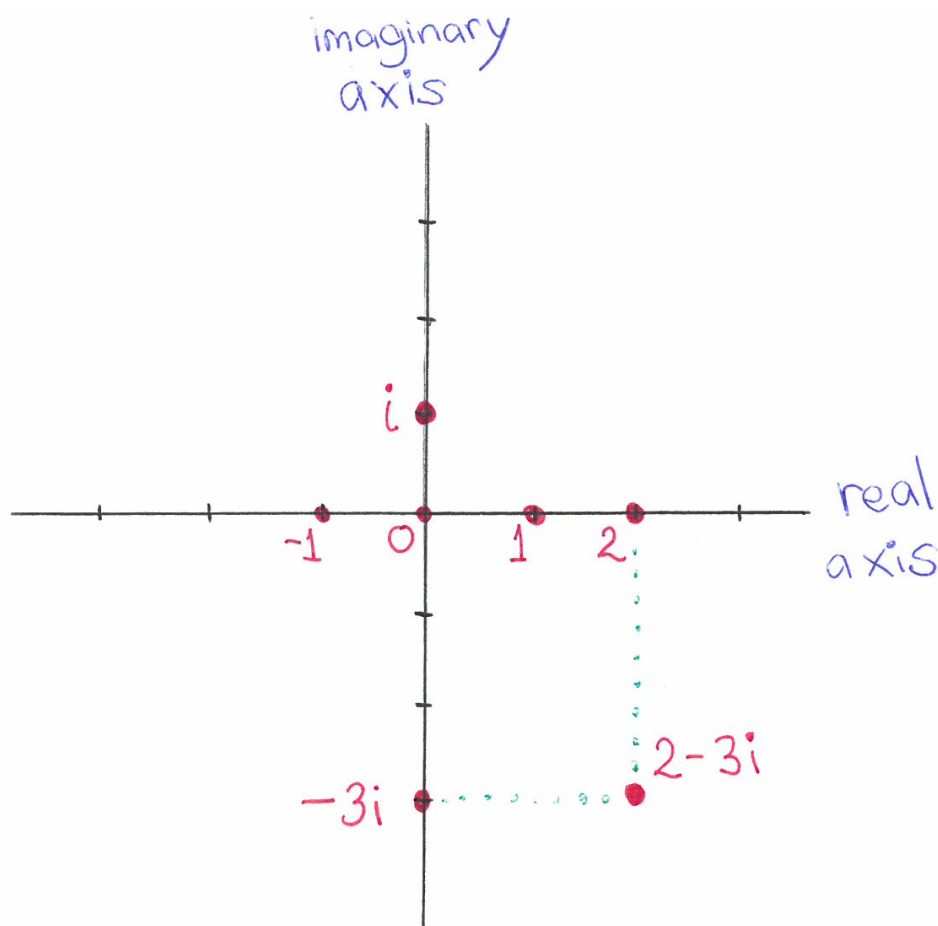
$$\overline{v + w} = \overline{v} + \overline{w}$$

for any complex numbers v and w . Similarly,

$$\overline{v w} = \overline{v} \overline{w}.$$

(These two identities say that “complex conjugation respects addition and multiplication”.)

8.2. The complex plane. Just as real numbers can be plotted on a line, complex numbers can be plotted on a plane: plot $a + bi$ at the point (a, b) in $\mathbb{R}^2$.

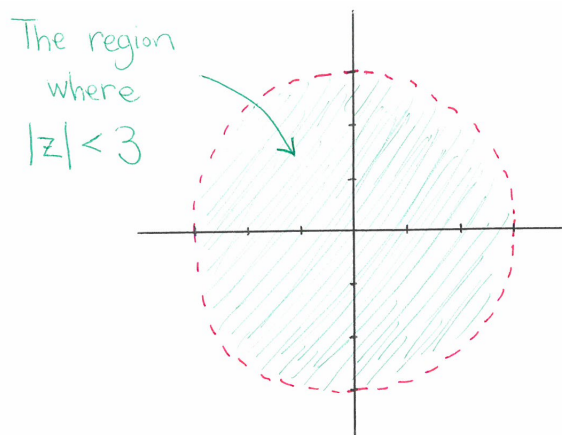


Addition and subtraction of complex numbers has the same geometric interpretation as for vectors. The same holds for scalar multiplication by a *real* number. (The geometric interpretation of multiplication by a *complex* number is different; we'll explain it soon.) Complex conjugation reflects a complex number in the real axis.

The **absolute value** (also called **magnitude** or **modulus**) $|z|$ of a complex number $z = a + bi$ is its distance to the origin:

$$|a + bi| := \sqrt{a^2 + b^2} \quad (\text{this is a real number}).$$

For a complex number z , inequalities like $z < 3$ do not make sense, but inequalities like $|z| < 3$ do, because $|z|$ is a real number. The complex numbers satisfying $|z| < 3$ are those in the open disk of radius 3 centered at 0 in the complex plane. (**Open** disk means the disk without its boundary.)



8.3. Some useful identities. The following are true for all complex numbers z :

$$\operatorname{Re} z = \frac{z + \bar{z}}{2}, \quad \operatorname{Im} z = \frac{z - \bar{z}}{2i}, \quad \overline{\bar{z}} = z, \quad z\bar{z} = |z|^2.$$

Also, for any *real* number c and complex number z ,

$$\operatorname{Re}(cz) = c \operatorname{Re} z, \quad \operatorname{Im}(cz) = c \operatorname{Im} z.$$

(These can fail if c is not real.)

Proof of the first identity: Write z as $a + bi$. Then

$$\begin{aligned} \operatorname{Re} z &= a, \\ \frac{z + \bar{z}}{2} &= \frac{(a + bi) + (a - bi)}{2} = a, \end{aligned}$$

so $\operatorname{Re} z = \frac{z + \bar{z}}{2}$. $\square$

The proofs of the others are similar.

Many identities have a geometric interpretation too. For example, $\operatorname{Re} z = \frac{z + \bar{z}}{2}$ says that $\operatorname{Re} z$ is the midpoint between z and its reflection $\bar{z}$.

8.4. Complex roots of polynomials.

real polynomial: polynomial with real coefficients

complex polynomial: polynomial with complex coefficients

Example 8.1. How many roots does the polynomial $z^3 - 3z^2 + 4$ have? It factors as $(z - 2)(z - 2)(z + 1)$, so it has only two distinct roots (2 and -1). But if we count 2 twice, then the number of roots *counted with multiplicity* is 3, equal to the degree of the polynomial.

Some real polynomials, like $z^2 + 9$, cannot be factored completely into degree 1 real polynomials, but do factor into degree 1 complex polynomials: $(z + 3i)(z - 3i)$. In fact, *every* complex polynomial factors completely into degree 1 complex polynomials — this is proved in advanced courses in complex analysis. This implies the following:

Fundamental theorem of algebra. *Every degree n complex polynomial $f(z)$ has exactly n complex roots, if counted with multiplicity.*

Since real polynomials are special cases of complex polynomials, the fundamental theorem of algebra applies to them too. For real polynomials, the non-real roots can be paired off with their complex conjugates.

Example 8.2. The degree 3 polynomial $z^3 + z^2 - z + 15$ factors as $(z + 3)(z - 1 - 2i)(z - 1 + 2i)$, so it has three distinct roots: -3 , $1 + 2i$, and $1 - 2i$. Of these roots, -3 is real, and $1 + 2i$ and $1 - 2i$ form a complex conjugate pair.

Example 8.3. Want a fourth root of i ? The fundamental theorem of algebra guarantees that $z^4 - i = 0$ has a complex solution (in fact, four of them). We'll soon learn how to find them.

The fundamental theorem of algebra will be useful for constructing solutions to higher order linear ODEs with constant coefficients, and for discussing eigenvalues.

8.5. Real and imaginary parts of complex-valued functions. Suppose that $y(t)$ is a complex-valued function of a real variable t . Then

$$y(t) = f(t) + i g(t)$$

for some *real-valued* functions of t . Here $f(t) := \operatorname{Re} y(t)$ and $g(t) := \operatorname{Im} y(t)$. Differentiation and integration can be done component-wise:

$$\begin{aligned} y'(t) &= f'(t) + i g'(t) \\ \int y(t) dt &= \int f(t) dt + i \int g(t) dt. \end{aligned}$$

Example 8.4. Suppose that $y(t) = \frac{2 + 3i}{1 + it}$. Then

$$y(t) = \frac{2 + 3i}{1 + it} = \frac{2 + 3i}{1 + it} \cdot \frac{1 - it}{1 - it} = \frac{(2 + 3t) + i(3 - 2t)}{1 + t^2} = \underbrace{\left(\frac{2 + 3t}{1 + t^2} \right)}_{f(t)} + i \underbrace{\left(\frac{3 - 2t}{1 + t^2} \right)}_{g(t)}.$$

The functions in parentheses labelled $f(t)$ and $g(t)$ are real-valued, so these are the real and imaginary parts of the function $y(t)$. $\square$

8.6. Defining the complex exponential function: introduction. Raising e to a complex number has no *a priori* meaning; it needs to be defined. People long ago tried to define it so that the key properties of the function e^t for real numbers t would be true for complex numbers too. They succeeded, and we will too!

The most important property of e^t is that it satisfies

$$\dot{y} = y, \quad y(0) = 1,$$

so we use this to guide the definition for complex numbers.

8.7. The function e^{it} .

Definition 8.5. The function e^{it} of a real number t is defined to be the solution to

$$\dot{y} = iy, \quad y(0) = 1.$$

(The existence and uniqueness theorem for linear ODEs guarantees that there is exactly one solution.)

Remark 8.6. Strictly speaking, we need a *vector-valued* variant of the existence and uniqueness theorem, since a complex-valued function of t is equivalent to a pair of real-valued functions of t . Anyway, it is true.

Euler's formula. We have $\boxed{e^{it} = \cos t + i \sin t}$ for every real number t .

Proof. The claim is that $y := \cos t + i \sin t$ is the solution to

$$\dot{y} = iy, \quad y(0) = 1.$$

Check this by plugging it in:

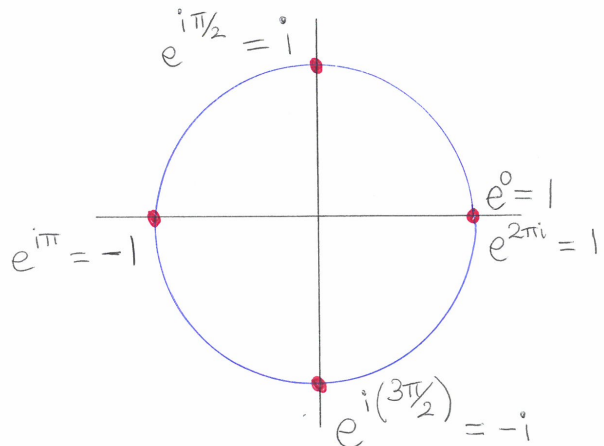
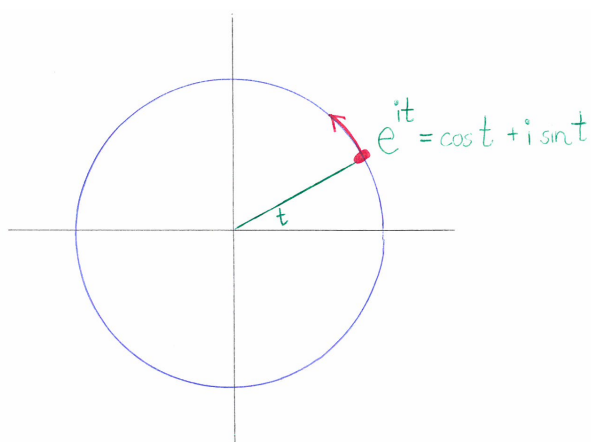
$$\begin{aligned} \dot{y} &= \frac{d}{dt} (\cos t + i \sin t) \\ &= -\sin t + i \cos t \\ &= i(\cos t + i \sin t) \\ &= iy, \end{aligned}$$

and

$$y(0) = \cos 0 + i \sin 0 = 1 + i0 = 1. \quad \square$$

Remark 8.7. Some older books use the awful abbreviation $\text{cis } t := \cos t + i \sin t$, but this belongs in a cispool [sic], since e^{it} is a more useful expression for the same thing.

As t increases, the complex number $e^{it} = \cos t + i \sin t$ travels counterclockwise around the unit circle in $\mathbb{C}$, because that is what $(\cos t, \sin t)$ does in $\mathbb{R}^2$.



8.8. The complex exponential function.

Definition 8.8. For each complex constant α , the function $e^{\alpha t}$ is defined to be the unique solution to

$$\dot{y} = \alpha y, \quad y(0) = 1.$$

For each nonzero α , the value of αt traces out the line through 0 and α as t ranges over real numbers, so the function $e^{\alpha t}$ specifies the value of e^z for every z on this line. These lines for varying α cover the whole complex plane, so defining the function $e^{\alpha t}$ for every α assigns a value to e^z for every complex number z .

Remark 8.9. Each value of e^z has been defined multiple times: for example, if $\beta = 2\alpha$, then e^β has been defined as both the value of $e^{\beta t}$ at $t = 1$ and the value of $e^{\alpha t}$ at $t = 2$. Fortunately, these multiple definitions are consistent: in the example, $e^{\alpha(2t)}$ satisfies the same DE and initial condition that specified $e^{\beta t}$, so $e^{\alpha(2t)} = e^{\beta t}$; now set $t = 1$.

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8.9. Properties of the complex exponential function. Let's show how the definition implies that the function e^z for complex z satisfies many of the same properties as e^t for real t , as well as some new properties.

Lemma 8.10. For any complex numbers α and β , the functions $e^{(\alpha+\beta)t}$ and $e^{\alpha t}e^{\beta t}$ are equal.

Proof. By definition, the first function, $e^{(\alpha+\beta)t}$, satisfies

$$\dot{y} = (\alpha + \beta)y, \quad y(0) = 1.$$

The second function, $y(t) := e^{\alpha t} e^{\beta t}$, satisfies the same DE and initial condition:

$$\begin{aligned}\dot{y} &= (\alpha e^{\alpha t})e^{\beta t} + e^{\alpha t}(\beta e^{\beta t}) && \text{(product rule)} \\ &= (\alpha + \beta)e^{\alpha t}e^{\beta t} \\ &= (\alpha + \beta)y\end{aligned}$$

and $y(0) = e^{\alpha 0}e^{\beta 0} = 1 \cdot 1 = 1$. But the uniqueness part of the existence and uniqueness theorem says that there is only one solution! So these two functions must be the same. $\square$

Theorem 8.11.

- (a) $e^0 = 1$.
- (b) $e^{z+w} = e^z e^w$ for all complex numbers z and w .
- (c) $\frac{1}{e^z} = e^{-z}$ for every complex number z .
- (d) $(e^z)^n = e^{nz}$ for every complex number z and integer n .
- (e) $e^{a+bi} = e^a(\cos b + i \sin b)$ for all real numbers a and b .
- (f) $e^{-it} = \cos t - i \sin t = \overline{e^{it}}$ for every real number t .
- (g) $|e^{it}| = 1$ for every real number t .

Proof.

- (a) True by definition.
- (b) The previous lemma, with variables renamed, says that $e^{(z+w)t} = e^{zt}e^{wt}$. Evaluating at $t = 1$ gives $e^{z+w} = e^z e^w$.
- (c) (The proofs of (c) and (d) were skipped in lecture.) We have

$$e^z e^{-z} \stackrel{(b)}{=} e^{z+(-z)} = e^0 \stackrel{(a)}{=} 1,$$

so e^{-z} is the inverse of e^z .

- (d) If $n = 0$, then this is $1 = 1$ by definition. If $n = 3$,

$$(e^z)^3 = e^z e^z e^z \stackrel{(b) \text{ repeatedly}}{=} e^{z+z+z} = e^{3z};$$

the same argument works for any positive integer n . If $n = -3$, then

$$(e^z)^{-3} = \frac{1}{(e^z)^3} \stackrel{(\text{just shown})}{=} \frac{1}{e^{3z}} \stackrel{(c)}{=} e^{-3z};$$

the same argument works for any negative integer n .

- (e) We have $e^{a+bi} = e^a e^{ib} = e^a(\cos b + i \sin b)$.
- (f) Changing every i in the universe to $-i$ transforms $e^{it} = \cos t + i \sin t$ into $e^{-it} = \cos t - i \sin t$. (Substituting $-t$ for t would do it too.) On the other hand, applying complex conjugation to both sides of $e^{it} = \cos t + i \sin t$ gives $\overline{e^{it}} = \cos t - i \sin t$.
- (g) By Euler's formula, $|e^{it}| = |\cos t + i \sin t| = \sqrt{\cos^2 t + \sin^2 t} = \sqrt{1} = 1$. $\square$

Of lesser importance is the power series representation

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots. \quad (6)$$

This formula can be deduced by using Taylor's theorem with remainder, or by showing that the right hand side satisfies the DE and initial condition. Some books use $e^{a+bi} = e^a(\cos b + i \sin b)$ or the power series $e^z = 1 + z + \frac{z^2}{2!} + \cdots$ as the *definition* of the complex exponential function, but the DE definition we gave is less contrived and focuses on what makes the function useful.

8.10. Polar forms of a complex number. Given a nonzero complex number $z = x + yi$, we can express the point (x, y) in polar coordinates r and θ :

$$x = r \cos \theta, \quad y = r \sin \theta.$$

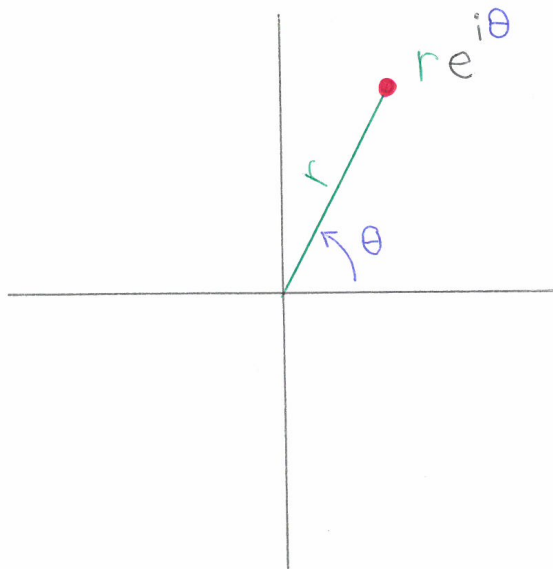
Then

$$x + yi = (r \cos \theta) + (r \sin \theta)i = r(\cos \theta + i \sin \theta).$$

In other words,

$$\boxed{z = re^{i\theta}}.$$

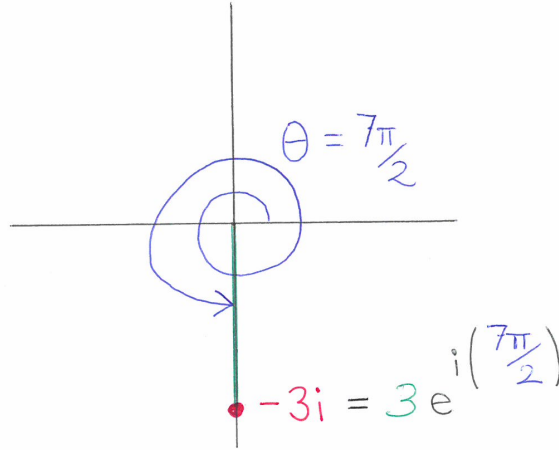
Any such expression $re^{i\theta}$ is called a **polar form** of the complex number z . It is required that r and θ be real numbers, with $r > 0$ (assuming $z \neq 0$), so $r = |z|$.



Any possible θ for z (a possible value for the **angle** or **argument** of z) may be called **arg z** , but this is dangerously ambiguous notation since there are many values of θ for the same z : this means that **arg z** is not a function.

Example 8.12. Suppose that $z = -3i$. So z corresponds to the point $(0, -3)$. Then $r = |z| = 3$, but there are infinitely many possibilities for the angle θ . One possibility is $-\pi/2$; all the others are obtained by adding integer multiples of 2π :

$$\arg z = \dots, -5\pi/2, -\pi/2, 3\pi/2, 7\pi/2, \dots$$



So z has many polar forms:

$$\dots = 3e^{i(-5\pi/2)} = 3e^{-i\pi/2} = 3e^{i(3\pi/2)} = 3e^{i(7\pi/2)} = \dots \quad \square$$

To specify a *unique* polar form, we would have to restrict the range for θ to some interval of width 2π . The most common choice is to require $-\pi < \theta \leq \pi$. This special θ is called the **principal value of the argument**, and is denoted in various ways:

$$\theta = \text{Arg } z = \underset{\text{Mathematica}}{\text{Arg}[z]} = \underset{\text{Mathematica}}{\text{ArcTan}[x,y]} = \underset{\text{MATLAB}}{\text{atan2}(y,x)}.$$

Warning: Some people require $0 \leq \theta < 2\pi$ instead.

Warning: In MATLAB, be careful to use (y, x) and not (x, y) .

Test for equality of two nonzero complex numbers in polar form:

$$r_1 e^{i\theta_1} = r_2 e^{i\theta_2} \iff r_1 = r_2 \text{ and } \theta_1 = \theta_2 + 2\pi k \text{ for some integer } k.$$

This assumes that r_1 and r_2 are positive real numbers, and that θ_1 and θ_2 are real numbers, as you would expect for polar coordinates.

8.11. Converting from $x + yi$ form to a polar form. This is the same as converting from rectangular coordinates to polar coordinates, so you are supposed to know this already. This section was not covered in lecture.

Problem 8.13. Convert a nonzero complex number $z = x + yi$ to polar form. In other words, given real numbers x and y , find r and a possible θ .

Finding r is easy: $r = |z| = \sqrt{x^2 + y^2}$.

Finding θ is trickier. If $x = 0$ or $y = 0$, then $x + yi$ is on one of the axes and θ will be an appropriate integer multiple of $\pi/2$. So assume that x and y are nonzero. The correct θ satisfies $\tan \theta = y/x$, but there are also other angles that satisfy this equation, namely $\theta + k\pi$ for any integer k . Some of these other angles point in the opposite direction. In particular, $\tan^{-1}(y/x)$ might be in the opposite direction. By definition, the angle $\tan^{-1}(y/x)$ always lies in $(-\pi/2, \pi/2)$, pointing into the right half plane, so it will be wrong when $x + yi$ lies

in the left half plane; in that case, adjust $\tan^{-1}(y/x)$ by adding or subtracting π to get a possible θ . Finally, if desired, add an integer multiple of 2π to get the *principal* value of the argument, which is the θ satisfying $-\pi < \theta \leq \pi$.

The “2-variable arctangent function” in Mathematica and MATLAB mentioned above looks not only at y/x , but also at the point (x, y) , to calculate a correct θ .

Example 8.14. Suppose that $z = -1 - i$. Evaluating $\tan^{-1}(y/x)$ at $(-1, -1)$ gives $\tan^{-1}(1) = \pi/4$, pointing in the direction *opposite* to $(-1, -1)$. Subtracting π gives $-3\pi/4$ as a possible θ . (The other possible angles θ are the numbers $-3\pi/4 + 2\pi k$, where k can be any integer.)

8.12. Operations in polar form. Some arithmetic operations on complex numbers are easy in polar form:

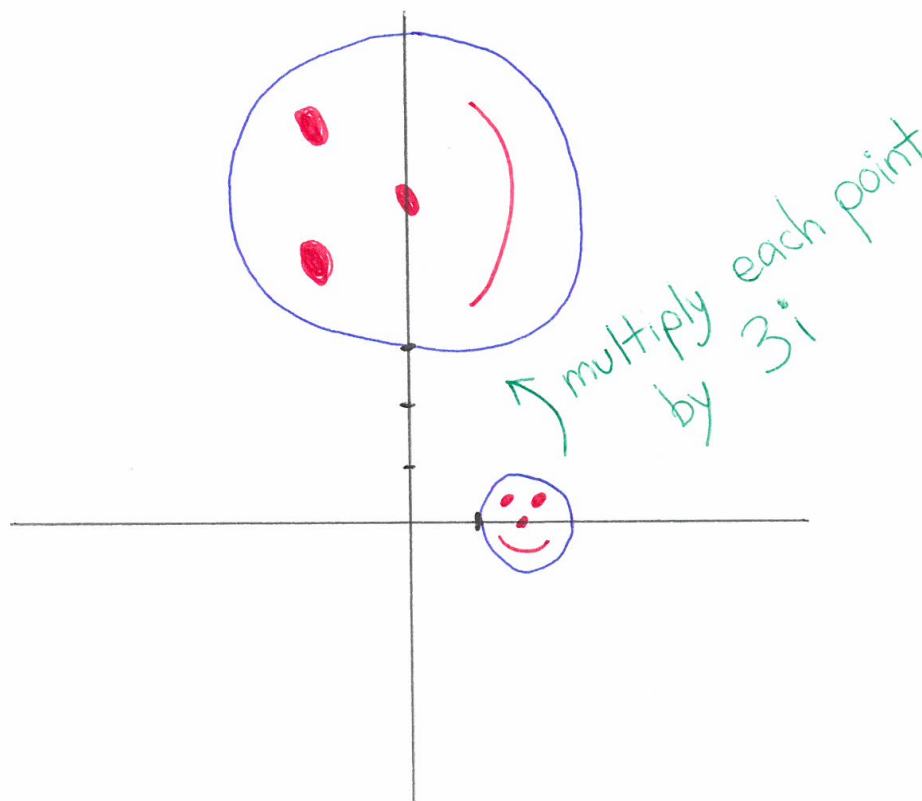
$$\begin{aligned} \text{multiplication: } (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) &= r_1 r_2 e^{i(\theta_1 + \theta_2)} && \text{(multiply absolute values, add angles)} \\ \text{reciprocal: } \frac{1}{r e^{i\theta}} &= \frac{1}{r} e^{-i\theta} \\ \text{division: } \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} &= \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)} && \text{(divide absolute values, subtract angles)} \\ n^{\text{th}} \text{ power: } (r e^{i\theta})^n &= r^n e^{in\theta} && \text{for any integer } n \\ \text{complex conjugation: } \overline{r e^{i\theta}} &= r e^{-i\theta}. \end{aligned}$$

Taking absolute values gives identities:

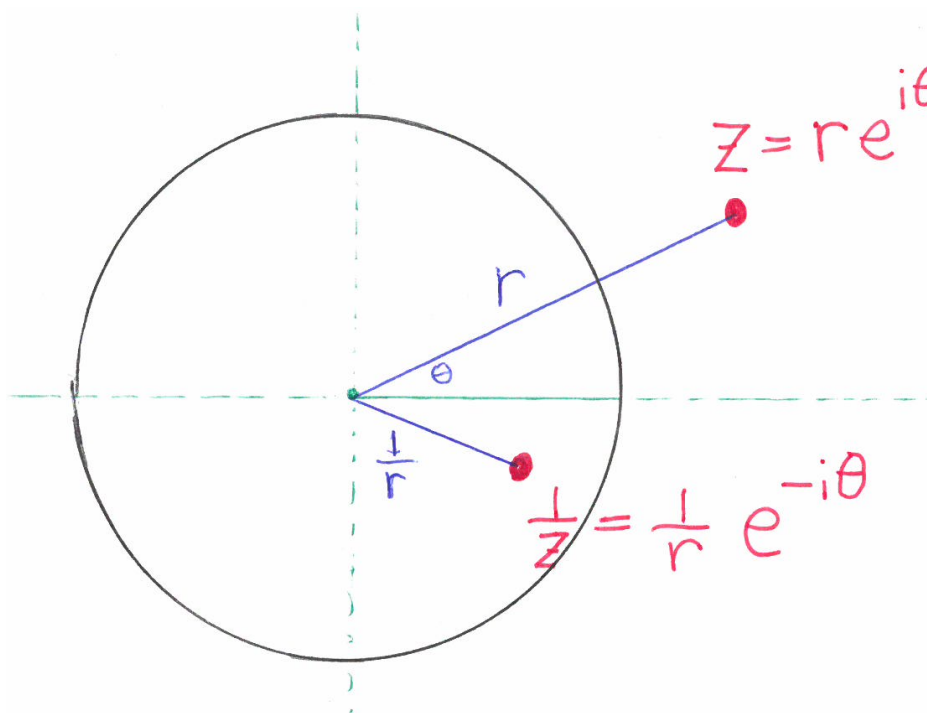
$$|z_1 z_2| = |z_1| |z_2|, \quad \left| \frac{1}{z} \right| = \frac{1}{|z|}, \quad \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}, \quad |z^n| = |z|^n, \quad |\bar{z}| = |z|.$$

Question 8.15. What happens if you take a smiley in the complex plane and multiply each of its points by $3i$?

Solution: Since $i = e^{i\pi/2}$, multiplying by i adds $\pi/2$ to the angle of each point; that is, it rotates counterclockwise by 90° (around the origin). Next, multiplying by 3 does what you would expect: dilate by a factor of 3. Doing both leads to...



For example, the nose was originally on the real line, a little less than 2, so multiplying it by $3i$ produces a big nose close to $(3i)2 = 6i$. $\square$



Question 8.16. How do you trap a lion?

Answer: Build a cage in the shape of the unit circle $|z| = 1$. Get inside the cage. Make sure that the lion is outside the cage. Apply the function $1/z$ to the whole plane. Voilà! The lion is now inside the cage, and you are outside it. (Only problem: There's a lot of other stuff inside the cage too. Also, don't stand too close to $z = 0$ when you apply $1/z$.)

Question 8.17. Why not always write complex numbers in polar form?

Answer: Because addition and subtraction are difficult in polar form!

8.13. The function $e^{(a+bi)t}$. Fix a nonzero complex number $a + bi$. As the real number t increases, the complex number $(a + bi)t$ moves along a line through 0, and $e^{(a+bi)t}$ moves along part of a line, a circle, or a spiral, depending on the value of $a + bi$. Try the “Complex Exponential” mathlet

<http://mathlets.org/mathlets/complex-exponential/>

to see examples of this.

Example 8.18. Consider $e^{(-5-2i)t} = e^{-5t}e^{i(-2t)}$ as $t \rightarrow \infty$. Its absolute value is e^{-5t} , which tends to 0, so the point is moving inward. Its angle is $-2t$, which is decreasing, so the point is moving clockwise. It's spiraling inwards clockwise.

8.14. Finding n^{th} roots.

8.14.1. *An example.*

Problem 8.19. What are the complex solutions to $z^5 = -32$?

Solution: Rewrite the equation in polar form, using $z = re^{i\theta}$:

$$(re^{i\theta})^5 = 32e^{i\pi}$$

$$r^5 e^{i(5\theta)} = 32e^{i\pi}$$

$$\begin{array}{l} r^5 = 32 \\ \text{absolute values} \end{array}$$

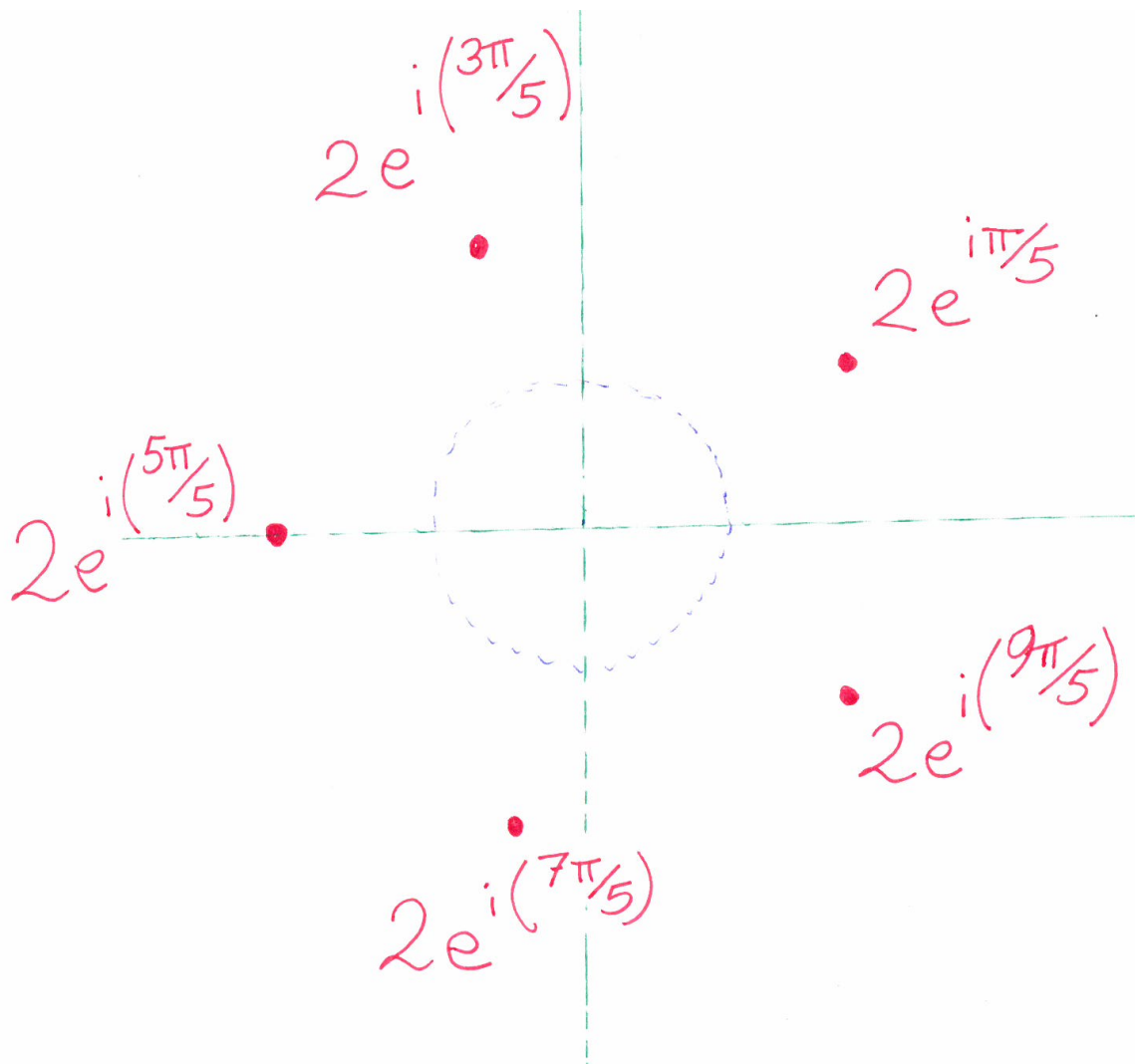
$$\begin{array}{l} 5\theta = \pi + 2\pi k \text{ for some integer } k \\ \text{angles} \end{array}$$

$$r = 2 \quad \text{and} \quad \theta = \frac{\pi}{5} + \frac{2\pi k}{5} \text{ for some integer } k$$

$$z = 2e^{i(\frac{\pi}{5} + \frac{2\pi k}{5})} \text{ for some integer } k.$$

These are numbers on a circle of radius 2; to get from one to the next (increasing k by 1), rotate by $2\pi/5$. Increasing k five times brings the number back to its original position. So it's enough to take $k = 0, 1, 2, 3, 4$. Answer:

$$2e^{i(\pi/5)}, 2e^{i(3\pi/5)}, 2e^{i(5\pi/5)}, 2e^{i(7\pi/5)}, 2e^{i(9\pi/5)}. \quad \square$$



Remark 8.20. The fundamental theorem of algebra predicts that the polynomial $z^5 + 32$ has 5 roots when counted with multiplicity. We found 5 roots, so each must have multiplicity 1.

Remark 8.21. The same approach (write in polar form, solve for absolute value and angle) finds the solutions to $z^n = \alpha$ for any positive integer n and nonzero complex number α .

8.14.2. Roots of unity.

Problem 8.22. Let n be a positive integer. The n^{th} roots of unity are the complex solutions to $z^n = 1$. Find them all.

Solution: Rewrite the equation in polar form, using $z = re^{i\theta}$:

$$\begin{aligned} (re^{i\theta})^n &= 1 \\ r^n e^{i(n\theta)} &= 1 \\ \begin{array}{cc} r^n = 1 & n\theta = 2\pi k \text{ for some integer } k \\ \text{absolute values} & \text{angles} \end{array} \\ r = 1 & \quad \text{and} \quad \theta = \frac{2\pi k}{n} \text{ for some integer } k \\ z = e^{i(\frac{2\pi k}{n})} & \text{ for some integer } k. \end{aligned}$$

As before, it's enough to take $k = 0, 1, 2, \dots, n-1$.

Another way to describe the solutions: Let ζ be the $k = 1$ solution, so $\zeta := e^{2\pi i/n}$. In terms of the number ζ , the complete list of n^{th} roots of unity is

$$\boxed{1, \zeta, \zeta^2, \dots, \zeta^{n-1}}.$$

8.14.3. *Another approach to describing all n^{th} roots of a complex number.* Here is another approach to describing all complex solutions to $z^n = \alpha$, analogous to the $y_i = y_p + y_h$ approach to inhomogeneous linear DEs:

Problem 8.23. Fill in the blank:

If z_0 is one solution to $z^n = \alpha$,
and $1, \zeta, \zeta^2, \dots, \zeta^{n-1}$ are all the solutions to $z^n = 1$,
then _____ are all the solutions to $z^n = \alpha$.

Answer: $\boxed{z_0, \zeta z_0, \zeta^2 z_0, \dots, \zeta^{n-1} z_0}$. Why? These are solutions, since for any integer k ,

$$(\zeta^k z_0)^n = (\zeta^k)^n z_0^n = 1 \cdot \alpha = \alpha;$$

there can't be any more, since a degree n polynomial equation has at most n solutions. $\square$

Remark 8.24. To use this approach to find all solutions to $z^n = \alpha$, you need to know one solution z_0 in advance. If there is no obvious z_0 , you'll still need polar form to solve $z^n = \alpha$.

The answer above says that once you know one solution z_0 , you get the others by repeatedly multiplying by $\zeta = e^{2\pi i/n}$, which rotates by $2\pi/n$; after n rotations, you return to z_0 . Thus the n solutions to $z^n = \alpha$ form the vertices of a regular n -gon (at least if $n \geq 3$).

Try the “Complex Roots” mathlet

<http://mathlets.org/mathlets/complex-roots/>

8.15. e^{it} and e^{-it} as linear combinations of $\cos t$ and $\sin t$, and vice versa.

Example 8.25. The functions e^{it} and e^{-it} are linear combinations of the functions $\cos t$ and $\sin t$:

$$\begin{aligned}e^{it} &= \cos t + i \sin t \\e^{-it} &= \cos t - i \sin t.\end{aligned}$$

If we view e^{it} and e^{-it} as known, and $\cos t$ and $\sin t$ as unknown, then this is a system of two linear equations in two unknowns, and it can be solved for $\cos t$ and $\sin t$. This gives

$$\boxed{\cos t = \frac{e^{it} + e^{-it}}{2}}, \quad \boxed{\sin t = \frac{e^{it} - e^{-it}}{2i}}.$$

Thus $\cos t$ and $\sin t$ are linear combinations of e^{it} and e^{-it} . (Explicitly, $\sin t = \frac{1}{2i}e^{it} + \frac{-1}{2i}e^{-it}$.)

Important: The function e^z has nicer properties than $\cos t$ and $\sin t$, so it is often a good idea to use these formulas to replace $\cos t$ and $\sin t$ by these combinations of e^{it} and e^{-it} , or to view $\cos t$ and $\sin t$ as the real and imaginary parts of e^{it} .

Replacing t by ωt in the identities above leads to

$$\begin{aligned}e^{i\omega t} &= \cos \omega t + i \sin \omega t \\e^{-i\omega t} &= \cos \omega t - i \sin \omega t.\end{aligned}$$

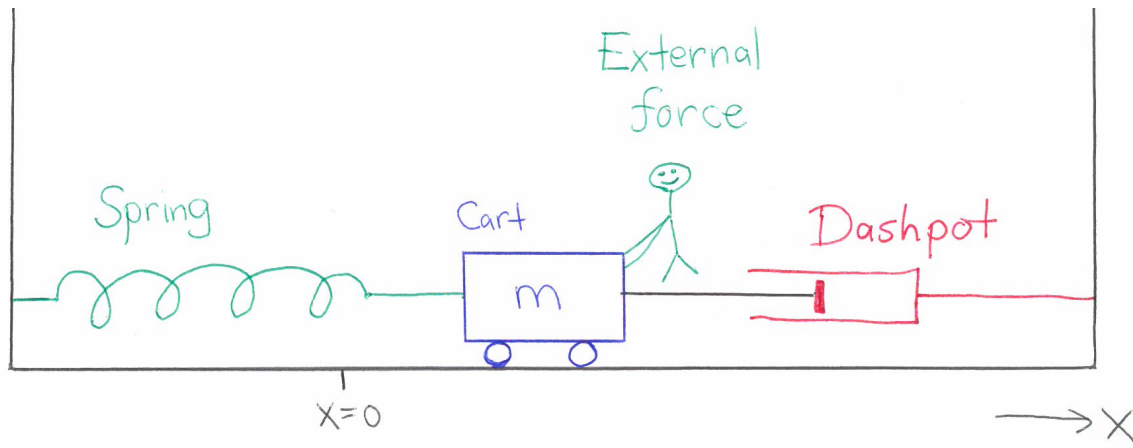
and

$$\boxed{\cos \omega t = \frac{e^{i\omega t} + e^{-i\omega t}}{2}}, \quad \boxed{\sin \omega t = \frac{e^{i\omega t} - e^{-i\omega t}}{2i}}.$$

9. INTRODUCTION TO SECOND-ORDER LINEAR ODEs WITH CONSTANT COEFFICIENTS

9.1. Modeling: a spring-mass-dashpot system.

Problem 9.1. A cart is attached to a spring attached to a wall. The cart is attached also to a dashpot, a damping device. (A dashpot could be a cylinder filled with oil that a piston moves through. Door dampers and car shock absorbers often actually work this way.) Also, there is an external force acting on the cart. Model the motion of the cart.



Solution: Define variables

t : time (s)

x : position of the cart (m), with $x = 0$ being where the spring exerts no force

m : mass of the cart (kg)

F_{spring} : force exerted by the spring on the cart (N)

F_{dashpot} : force exerted by the dashpot on the cart (N)

F_{external} : external force on the cart (N)

F : total force on the cart (N).

The independent variable is t ; everything else is a function of t (well, maybe m is constant).

Physics tells us that

- F_{spring} is a function of x (of opposite sign), and
- F_{dashpot} is a function of $\dot{x}$ (again of opposite sign).

To simplify, approximate these by *linear* functions (probably OK if x and $\dot{x}$ are small):

$$F_{\text{spring}} = -kx, \quad F_{\text{dashpot}} = -b\dot{x},$$

Hooke's law

where k is the **spring constant** (in units N/m) and b is the **damping constant** (in units N s/m); here $k, b > 0$. Substituting these and Newton's second law $F = m\ddot{x}$ into

$$F = F_{\text{spring}} + F_{\text{dashpot}} + F_{\text{external}}$$

gives

$$m\ddot{x} = -kx - b\dot{x} + F_{\text{external}},$$

a second order linear ODE, which we would usually write as

$$\boxed{m\ddot{x} + b\dot{x} + kx = F_{\text{external}}(t)}. \quad \square$$

All this works even if m , b , k are functions of time, but we'll assume from now on that they are constants.

input signal: $F_{\text{external}}(t)$
 system: spring, mass, and dashpot
 output signal: $x(t)$.

Carts attached to springs are not necessarily what interest us. But oscillatory systems arising in all the sciences are governed by the same math, and this physical system lets us visualize their behavior.

9.2. The differential equation $\ddot{x} + x = 0$. Suppose that $m = k = 1$ and there is no dashpot and no external force, only a mass and spring. Then the DE is simply

$$\boxed{\ddot{x} + x = 0}.$$

Each solution $x(t)$ gives rise to a pair of numbers $(x(0), \dot{x}(0))$. Conversely, the existence and uniqueness theorem says that for each pair of numbers (a, b) , there is exactly one solution to $\ddot{x} + x = 0$ satisfying $(x(0), \dot{x}(0)) = (a, b)$. What are these solutions?

Well, $\cos t$ is the solution to $\ddot{x} + x = 0$ such that $x(0) = 1$ and $\dot{x}(0) = 0$,
 and $\sin t$ is the solution to $\ddot{x} + x = 0$ such that $x(0) = 0$ and $\dot{x}(0) = 1$,
 so _____ is the solution to $\ddot{x} + x = 0$ such that $x(0) = a$ and $\dot{x}(0) = b$.

The answer is the function $a \cos t + b \sin t$. In other words, the 2-parameter family of solutions

$$\boxed{a \cos t + b \sin t}$$

is the *general* solution to $\ddot{x} + x = 0$.

There are other ways to construct solutions. For example, for any constant ϕ , the time-shifted function $\cos(t - \phi)$ is a solution, and if A is another constant, then

$$A \cos(t - \phi)$$

is a solution too. It turns out that these functions are the same as the functions $a \cos t + b \sin t$, just written in a different form, so the family of such functions $A \cos(t - \phi)$ is the general solution again!

To explain this and to solve other DEs, we'll need to understand functions like these and learn how to convert between the different forms.

10. SINUSOIDAL FUNCTIONS

10.1. **Construction.** Start with the curve $y = \cos x$. Then

1. Shift the graph ϕ units to the right (ϕ is **phase lag**, measured in radians). (For example, shifting by $\phi = \pi/2$ gives the graph of $\sin x$, which reaches its maximum $\pi/2$ radians after $\cos x$ does.)
2. Compress the result horizontally by *dividing* by a scale factor ω (**angular frequency**, measured in radians/s).
3. Amplify (stretch vertically) by a factor of A (**amplitude**).

(Here $A, \omega > 0$, but ϕ can be any real number.)

Result? The graph of a new function $f(t)$, called a **sinusoidal function** (or just **sinusoid**).

10.2. **Formula.** What is the formula for $f(t)$? Each point (x, y) on $y = \cos x$ is transformed by the Steps 1–3 above as follows:

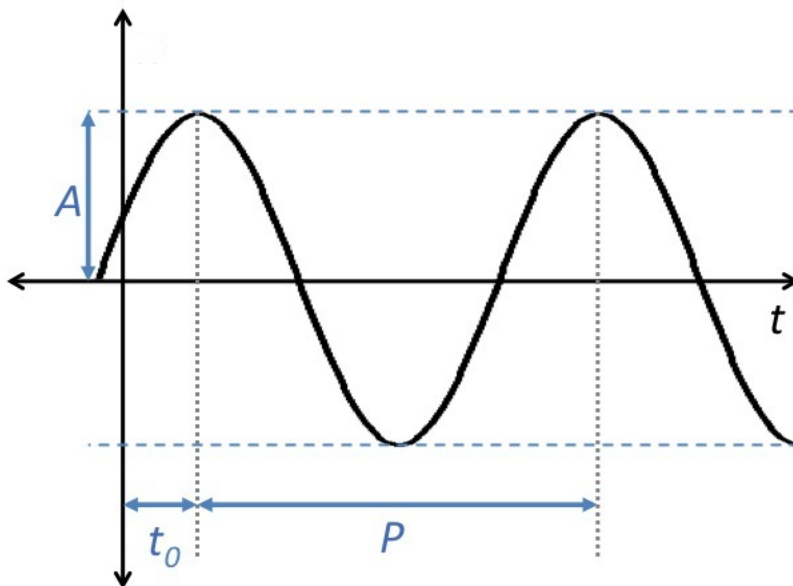
$$(x, y) \xrightarrow{1} (x + \phi, y) \xrightarrow{2} \left(\frac{x + \phi}{\omega}, y \right) \xrightarrow{3} \left(\frac{x + \phi}{\omega}, Ay \right),$$

and the end result is supposed to be a point $(t, f(t))$ on the graph of f , so

$$t = \frac{x + \phi}{\omega}, \quad f = Ay.$$

Solving for x gives $x = \omega t - \phi$; substituting into $f = Ay = A \cos x$ gives

$$\boxed{f(t) = A \cos(\omega t - \phi)}.$$



10.3. **Alternative geometric description.** Alternatively, the graph of $f(t)$ can be described geometrically in terms of

A : its **amplitude**, as above, how high the graph rises above the t -axis at its maximum

t_0 : its **time lag**, also sometimes called τ , a t -value at which a maximum is attained (s)

P : its **period**, the time for one complete oscillation (= width between successive maxima) (s)

How do t_0 and P relate to ω and ϕ ?

- $t_0 = \phi/\omega$, since this is the t -value for which the angle $\omega t - \phi$ becomes 0.
- $P = 2\pi/\omega$, since adding $2\pi/\omega$ to t increases the angle $\omega t - \phi$ by 2π . (To get the right units, 2π should really be 2π *radians*.)

There is also **frequency** $\nu := 1/P$, measured in Hz = cycles/s (the 1 is really 1 cycle). It is the number of complete oscillations per second. To convert from frequency ν to angular frequency ω , multiply by $\frac{2\pi \text{ radians}}{1 \text{ cycle}}$; thus $\omega = 2\pi\nu = 2\pi/P$, which is consistent with the formula $P = 2\pi/\omega$ above.

Question 10.1. What is the difference between phase lag and time lag?

Answer: Phase lag ϕ and time lag t_0 both measure how much a sinusoid $A \cos(\omega t - \phi)$ is shifted relative to the standard sinusoid $\cos(\omega t)$ of the same frequency, but ϕ is measured as a fraction of a cycle (expressed in radians), and t_0 is expressed in time units.

For example, if ϕ is π radians, that is half a cycle, so it means that the sinusoid is completely out of phase, attaining a maximum where $\cos(\omega t)$ has a minimum, and vice versa. The time lag of this same sinusoid, however, will depend on the duration of a cycle: if the angular frequency ω is very high, then there will be many cycles per second, so each cycle represents a very short time period, so the time for half a cycle will also be very short.

To convert between phase lag and time lag, multiply or divide by the angular frequency ω . To remember whether to multiply or divide, compare units. Since ω is measured in radians/s, the conversion is $t_0 = \phi/\omega$.

Another way to think about this: in terms of the construction of a sinusoid, ϕ represents the shift in angle, and then compressing horizontally by dividing by ω gives the shift in time.

□

Try the “Sinusoids” mathlet

<https://mathlets.org/mathlets/sinusoids/>

10.4. **Three forms.** There are three ways to write a sinusoidal function of angular frequency ω :

- **amplitude-phase form:** $A \cos(\omega t - \phi)$, where A and ϕ are real numbers with $A \geq 0$;
- **complex form:** $\operatorname{Re}(ce^{i\omega t})$, where c is a complex number;
- **linear combination:** $a \cos \omega t + b \sin \omega t$, where a and b are real numbers.

Different forms are useful in different contexts, so we'll need to know how to convert between them. The following proposition explains how.

Proposition 10.2. *If constants A, ω, ϕ, a, b, c are set so that the key equations*

$$\begin{array}{ccc}
 Ae^{i\phi} & & A \cos(\omega t - \phi) \\
 \swarrow \quad \searrow & \text{hold, then} & \swarrow \quad \searrow \\
 \bar{c} = a + bi & & \operatorname{Re}(ce^{i\omega t}) = a \cos \omega t + b \sin \omega t.
 \end{array}$$

(The little numbers refer to the proofs below.)

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Warning: Don't forget that it is $\bar{c}$ and not c itself that appears in the key equations.

An equivalent form of the key equations (obtained by taking complex conjugates) is

$$\begin{array}{ccc}
 Ae^{-i\phi} & & \\
 \swarrow \quad \searrow & & \\
 c = a - bi & &
 \end{array}$$

If you ever forget the key equations above, you can do the conversion manually by going through the steps in the proof below.

Proof of Proposition 10.2.

1.

$$\begin{aligned}
 \operatorname{Re}(ce^{i\omega t}) &= \operatorname{Re}(Ae^{-i\phi} e^{i\omega t}) \\
 &= \operatorname{Re}(Ae^{i(\omega t - \phi)}) \\
 &= A \operatorname{Re}(e^{i(\omega t - \phi)}) \\
 &= A \cos(\omega t - \phi).
 \end{aligned}$$

2.

$$\begin{aligned}\operatorname{Re}(ce^{i\omega t}) &= \operatorname{Re}((a - bi)(\cos \omega t + i \sin \omega t)) \\ &= \operatorname{Re}((a \cos \omega t + b \sin \omega t) + i(\cdots)) \\ &= a \cos \omega t + b \sin \omega t.\end{aligned}$$

3. Using $\cos(x - y) = \cos x \cos y + \sin x \sin y$ shows that

$$\begin{aligned}A \cos(\omega t - \phi) &= A \cos \phi \cos \omega t + A \sin \phi \sin \omega t \\ &= a \cos \omega t + b \sin \omega t\end{aligned}$$

since $a = A \cos \phi$ and $b = A \sin \phi$ when (A, ϕ) are polar coordinates of (a, b) .

4.

$$\begin{aligned}a \cos \omega t + b \sin \omega t &= (a, b) \cdot (\cos \omega t, \sin \omega t) \\ &= |(a, b)| |(\cos \omega t, \sin \omega t)| \cos(\text{angle between the vectors}), \\ &\quad (\text{by the geometric interpretation of the dot product}) \\ &= A \cos(\omega t - \phi)\end{aligned}$$

(Actually, it would have been enough to prove equality on two sides of the triangle.) □

Here are two sample problems showing how to use the key equations.

Problem 10.3. Convert $7 \cos(2t - \pi/2)$ into complex form $\operatorname{Re}(ce^{i\omega t})$.

Solution: Given: $A = 7$, $\omega = 2$, $\phi = \pi/2$. Needed: c, ω (actually, we know ω already). The c we need is the one that satisfies the key equation $\bar{c} = Ae^{i\phi}$. This says $\bar{c} = 7e^{i\pi/2} = 7i$, so $c = -7i$. The answer is

$$\operatorname{Re}(ce^{i\omega t}) = \operatorname{Re}(-7ie^{i(2t)}). \quad \square$$

Problem 10.4. Convert $-\cos 5t - \sqrt{3} \sin 5t$ to amplitude-phase form $A \cos(\omega t - \phi)$.

Solution: Given: $a = -1$, $b = -\sqrt{3}$, $\omega = 5$. Wanted: A, ω, ϕ . So we use $Ae^{i\phi} = a + bi$, which says that $Ae^{i\phi} = -1 - i\sqrt{3}$. First, $A = \sqrt{(-1)^2 + (-\sqrt{3})^2} = 2$. The real-part equation $A \cos \phi = -1$ says that $\cos \phi = -1/2$, so the angle of $(-1, -\sqrt{3})$ is $\phi = -2\pi/3$ (or this plus $2\pi k$ for any integer k). Thus the answer is

$$A \cos(\omega t - \phi) = 2 \cos(5t + 2\pi/3). \quad \square$$

Remark 10.5. Sinusoids are always real-valued functions. It would be wrong to write one as $a \cos \omega t + bi \sin \omega t$; the i should not be in there. Even the complex form $\operatorname{Re}(ce^{i\omega t})$ of a sinusoid is a real-valued function, because of the Re on the outside.

Try the “Trigonometric Identity” mathlet

<http://mathlets.org/mathlets/trigonometric-id/>

10.5. **Complex gain, gain, and phase lag.** Later in the class, we'll talk about [LTI systems](#) (LTI stands for linear and time-invariant). These include all systems built of springs, masses, and dashpots, and also all RLC circuits (circuits built of resistors, inductors, and capacitors).

It turns out that such a system, when fed a sinusoidal input signal, usually produces a sinusoidal output signal of the same frequency. How can we compare the input and output sinusoids?

Write each sinusoid in complex form (convert if necessary):

$$\begin{aligned}\text{input signal: } & \operatorname{Re}(ce^{i\omega t}) \\ \text{output signal: } & \operatorname{Re}(Ce^{i\omega t}).\end{aligned}$$

Imagine feeding a corresponding “complex replacement” signal into the system and getting a complex output signal (this probably makes no physical sense, but do it anyway):

$$\begin{aligned}\text{complex input: } & ce^{i\omega t} \\ \text{complex output: } & Ce^{i\omega t}.\end{aligned}$$

Define [complex gain](#) as the factor by which the complex input signal has gotten “bigger”:

$$G := \frac{\text{complex output}}{\text{complex input}} = \frac{Ce^{i\omega t}}{ce^{i\omega t}} = \frac{C}{c}.$$

Complex gain is a complex number.

Question 10.6. What is the physical interpretation of the complex gain G , in terms of amplitudes and phases of the real signals?

The answers are in the two boxes below. The conversion $\operatorname{Re}(ce^{i\omega t}) = A \cos(\omega t - \phi)$ uses the key equation

$$c = Ae^{-i\phi},$$

so multiplying c by the complex scale factor G (to get C) amounts to

- multiplying the amplitude A by $|G|$, and
- increasing ϕ by $-\arg G$.

The amplitude scale factor is called [gain](#):

$$\boxed{\text{gain} := \frac{\text{output amplitude}}{\text{input amplitude}} = |G|}.$$

Gain is a nonnegative real number.

The increase in ϕ is called the [phase lag](#) (of the output *relative to the input*):

$$\boxed{\text{phase lag} := \phi_{\text{output}} - \phi_{\text{input}} = -\arg G}.$$

Phase lag is a real number, measured in radians. **Warning:** This is a *relative* phase lag, different from the absolute phase lag defined earlier comparing a sinusoid to the standard sinusoid $\cos x$.

Example 10.7. If the phase lag is $\pi/2$, that means that the maximum of the output sinusoid occurs $\pi/2$ radians after the maximum of the input signal. (To get instead the relative *time* lag, divide the phase lag by ω .)

Try the “Gain and phase” mathlet

<https://web-cert.mit.edu/jorloff/www/jmoapplets/html5/gainPhase.html>

Remark 10.8. For a fixed LTI system, it turns out that the complex gain depends only on ω . In other words, the complex gain is the same for all sinusoidal input signals having a fixed angular frequency ω . Mathematically, this is because when the complex input signal $ce^{i\omega t}$ is multiplied by a nonzero complex number, linearity implies that the complex output signal is multiplied by the *same* complex number, so the complex gain ($= \frac{\text{complex output}}{\text{complex input}}$) stays the same.

10.6. Beats. Bonus section! Not covered in lecture.

Try the “Beats” mathlet

<http://mathlets.org/mathlets/beats/>

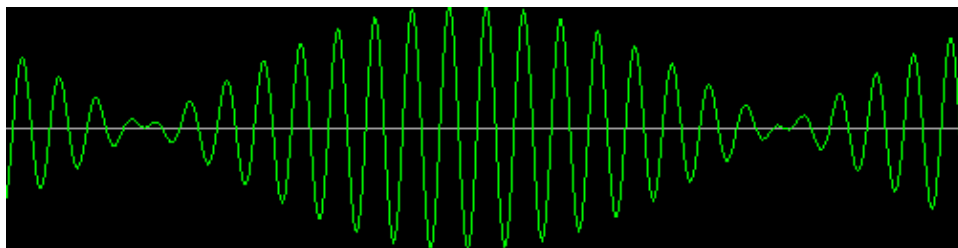
Beats occur when two very nearby pitches are sounded simultaneously.

Problem 10.9. Consider two sinusoid sound waves of angular frequencies $\omega + \epsilon$ and $\omega - \epsilon$, say $\cos((\omega + \epsilon)t)$ and $\cos((\omega - \epsilon)t)$, where ϵ is much smaller than ω . What happens when they are superimposed?

Solution: The sum is

$$\begin{aligned}\cos((\omega + \epsilon)t) + \cos((\omega - \epsilon)t) &= \operatorname{Re}(e^{i(\omega + \epsilon)t}) + \operatorname{Re}(e^{i(\omega - \epsilon)t}) \\ &= \operatorname{Re}(e^{i(\omega + \epsilon)t} + e^{i(\omega - \epsilon)t}) \\ &= \operatorname{Re}(e^{i\omega t}(e^{i\epsilon t} + e^{-i\epsilon t})) \\ &= \operatorname{Re}(e^{i\omega t}(2 \cos \epsilon t)) \\ &= (2 \cos \epsilon t) \operatorname{Re}(e^{i\omega t}) \\ &= 2(\cos \epsilon t)(\cos \omega t).\end{aligned}$$

The function $\cos \omega t$ oscillates rapidly between ± 1 . Multiplying it by the slowly varying function $2 \cos \epsilon t$ produces a rapid oscillation between $\pm 2 \cos \epsilon t$, so one hears a sound wave of angular frequency ω whose amplitude is the slowly varying function $|2 \cos \epsilon t|$. $\square$



11. SECOND-ORDER LINEAR ODEs WITH CONSTANT COEFFICIENTS

11.1. Consequence of superposition for a homogeneous linear ODE; vector spaces.

Problem 11.1 (Multiplying a solution by 9). Fill in the blank:

Given that $\cos t$ is a solution to $\ddot{x} + x = 0$,
it follows that $9 \cos t$ is a solution to $\ddot{x} + x = \underline{\hspace{1cm}}$.

Answer: $9 \cdot 0$, which is 0 again. Thus $9 \cos t$ is a solution to the *same* DE. $\square$

Similarly, adding two solutions to $\ddot{x} + x = 0$ gives another solution to $\ddot{x} + x = 0$.

Conclusion: The set S of solutions to $\ddot{x} + x = 0$ has the following properties:

0. The zero function 0 is in S .
1. Multiplying any one function in S by any scalar gives another function in S .
2. Adding any two functions in S gives another function in S .

A set S of functions satisfying all three properties is called a **vector space** of functions, since these properties say that you can scalar-multiply and add such functions, as you can with vectors. (One can also talk about vector spaces of vectors, or vector spaces of matrices. There is a more abstract notion of **vector space** that includes all these as special cases.)

The three properties above hold not just for the set of all solutions to $\ddot{x} + x = 0$, but also for the set of all solutions of any homogeneous linear DE. In other words:

Theorem 11.2. *For any homogeneous linear DE, the set of all solutions is a vector space.*

Theorem 11.2 is why homogeneous linear DEs are so nice. It says that if you know some solutions, you can form linear combinations to build new solutions, with no extra work! **This is the key point of linearity in the homogeneous case.** We will use it over and over again in applications throughout the course.

11.2. Span. Last week we used the existence and uniqueness theorem to show that the solutions to $\ddot{x} + x = 0$ are exactly the linear combinations of $\cos t$ and $\sin t$:

$$\{\text{all solutions to } \ddot{x} + x = 0\} = \{a \cos t + b \sin t, \text{ where } a \text{ and } b \text{ range over all numbers}\}.$$

Abbreviation for the set on the right:

$$\text{Span}(\cos t, \sin t).$$

Here is what span means in general:

Definition 11.3. Suppose that $f_1, \dots, f_n$ is a list of functions.

The **span** of $f_1, \dots, f_n$ is the set of *all* linear combinations of $f_1, \dots, f_n$:

$$\text{Span}(f_1, \dots, f_n) := \{\text{functions } c_1 f_1 + \dots + c_n f_n, \text{ where } c_1, \dots, c_n \text{ range over all numbers}\}.$$

(To be completely precise, we should say whether $c_1, \dots, c_n$ are ranging over real numbers only, or over all complex numbers too. The answer depends on the context.)

Question: How many functions are in the set $\text{Span}(\cos t, \sin t)$?

Answer: Infinitely many! Here are some functions in this set:

$$2 \cos t + 3 \sin t, \quad -5 \sin t, \quad 0, \quad \pi \cos t, \quad \dots$$

Problem 11.4. Let S be the set of polynomials $p(t)$ whose degree is ≤ 2 . Express S as a span.

Solution: The set S equals the set of polynomials of the form

$$at^2 + bt + c,$$

where a, b, c range over all numbers. Thus one possible answer is that $S = \text{Span}(t^2, t, 1)$. $\square$

Problem 11.5. Let T be the set of all solutions to $\dot{y} = 7y$. Express T as a span.

Solution: The set T is the set of functions of the form ce^{7t} , where c ranges over all numbers. Thus one possible answer is that $T = \text{Span}(e^{7t})$. (The linear combinations of a single function are just the scalar multiples of that function.) $\square$

Last week we showed that the set of all solutions to $\ddot{x} + x = 0$ equals the set of the linear combinations of two solutions. The same reasoning applies to *any* 2nd-order homogeneous linear ODE.

Conclusion: For *any* 2nd-order homogeneous linear ODE, the set of solutions can be expressed as the span of 2 solutions.

11.3. Linearly dependent functions. How do you know *which* two solutions to use? Will the span of *any* two solutions give the set of all solutions? No! It turns out that most pairs of solutions will work, but not every pair. To determine which pairs work, we need the notion of linear dependence.

Example 11.6 (Why the span of two solutions might not equal the set of all solutions). The functions $\cos t$ and $2 \cos t$ are two solutions to $\ddot{x} + x = 0$, but $\text{Span}(\cos t, 2 \cos t)$ consists only of the functions

$$a \cos t + b(2 \cos t) = (a + 2b) \cos t,$$

which vary only over the scalar multiples of $\cos t$. Thus

$$\text{Span}(\cos t, 2 \cos t) = \text{Span}(\cos t).$$

The solution $\sin t$ is missing from this set, so this is not the set of all solutions to $\ddot{x} + x = 0$.

Definition 11.7. Call two functions f and g **linearly dependent** when either f is a scalar multiple of g or g is a scalar multiple of f . Call f and g **linearly independent** otherwise.

Warning: The definition of linear dependence for three or more functions is more complicated. We'll discuss it later.

Question: Which of the following is a pair of linearly independent functions?

- $\cos t, 2 \cos t$
- $\cos t, \cos(t + \pi)$
- $\cos t, \cos(t - \pi/2)$
- $\cos t, 0$.

Answer: The third pair is linearly independent since $\cos(t - \pi/2) = \sin t$, and neither $\cos t$ nor $\sin t$ is a scalar multiple of the other function (they have zeros in different places, for instance). The second pair is linearly dependent since $\cos(t + \pi) = (-1)(\cos t)$. The fourth pair is linearly dependent since $0 = 0(\cos t)$. $\square$

It turns out that in solving a 2nd-order homogeneous linear ODE, any two solutions can be used to generate the others, *provided that they are linearly independent*:

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Strategy for solving a 2nd-order homogeneous linear ODE:

1. Find two solutions, say f and g , by any method (guessing is OK, as long as you plug f and g in to verify that they are solutions!)
2. Check that f and g are linearly independent (that is, check that f is not a scalar multiple of g and that g is not a scalar multiple of f).
3. If so, then the set of all solutions to the DE is $\text{Span}(f, g)$.

11.4. Example: a second-order homogeneous linear ODE with constant coefficients.

Problem 11.8. What are the solutions to

$$\ddot{y} + \dot{y} - 6y = 0 ? \quad (7)$$

Solution: Try $y = e^{rt}$, where r is a constant to be determined. Let's find out for which constants r this is really a solution. We get $\dot{y} = re^{rt}$ and $\ddot{y} = r^2e^{rt}$, so (7) becomes

$$\begin{aligned} r^2e^{rt} + re^{rt} - 6e^{rt} &= 0 \\ (r^2 + r - 6)e^{rt} &= 0. \end{aligned}$$

This holds as an equality of functions if and only if

$$\begin{aligned} r^2 + r - 6 &= 0 \\ (r - 2)(r + 3) &= 0 \\ r = 2 \quad \text{or} \quad r &= -3. \end{aligned}$$

So e^{2t} and e^{-3t} are solutions.

Luckily, neither is a constant times the other (the ratio is $e^{2t}/e^{-3t} = e^{5t}$, a nonconstant function); in other words, e^{2t} and e^{-3t} are linearly independent. Conclusion:

The set of all solutions to $\ddot{y} + \dot{y} - 6y = 0$ is $\text{Span}(e^{2t}, e^{-3t})$.

An equivalent way to express this answer:

The general solution to $\ddot{y} + \dot{y} - 6y = 0$ is $c_1e^{2t} + c_2e^{-3t}$, where c_1 and c_2 are parameters.

Remark 11.9. In the future, we'll jump directly from the ODE to solving $r^2 + r - 6 = 0$, now that we know how this works.

11.5. **Basis.** Here is a third way to express the answer to the previous problem:

The functions e^{2t} and e^{-3t} form a **basis** of the space of solutions to $\ddot{y} + \dot{y} - 6y = 0$.

This terminology means that

- $\text{Span}(e^{2t}, e^{-3t})$ is the space of solutions to $\ddot{y} + \dot{y} - 6y = 0$, and
- e^{2t} and e^{-3t} are linearly independent.

Think of the functions e^{2t} and e^{-3t} in the basis as the “basic building blocks”:

- the first condition says that every solution can be built from e^{2t} and e^{-3t} by taking linear combinations, and
- the second condition says that there is no redundancy in the list (neither building block could have been built from the other one).

The plural of basis is *bases*, pronounced BAY-sees.

11.6. How to solve any second-order homogeneous linear ODE with constant coefficients. To solve

$$a_2 \ddot{y} + a_1 \dot{y} + a_0 y = 0,$$

where a_2, a_1, a_0 are constants (with $a_2 \neq 0$), do the following:

1. Write down the **characteristic equation**

$$a_2 r^2 + a_1 r + a_0 = 0,$$

in which the coefficient of r^i is the coefficient of $y^{(i)}$ from the ODE. The polynomial $a_2 r^2 + a_1 r + a_0$ is called the **characteristic polynomial**. (For example, $\ddot{y} + 5y = 0$ has characteristic polynomial $r^2 + 5$.)

2. Solve the characteristic equation to list the complex roots with multiplicity. (To do this, factor the characteristic polynomial, or complete the square, or use the quadratic formula.)

- 3a. If the roots are *distinct* numbers $r_1 \neq r_2$ then the functions

$$\boxed{e^{r_1 t}, \quad e^{r_2 t}}$$

form a basis of the space of solutions to the ODE. In other words, the general solution is

$$c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

- 3b. If the roots are equal, r and r , then the functions

$$\boxed{e^{rt}, \quad te^{rt}}$$

form a basis of the space of solutions to the ODE. In other words, the general solution is

$$c_1 e^{rt} + c_2 te^{rt}.$$

(We'll explain later on why this works.) $\square$

In both cases, there is a basis consisting of 2 functions.

11.7. Complex roots. The general method in the previous section works even if some of the roots are not real.

Problem 11.10. What is a basis for the space of all solutions to $\ddot{x} + x = 0$?

Answer 1 (given last week): $\boxed{\cos t, \sin t}$.

(In other words, the general solution is $c_1 \cos t + c_2 \sin t$.) $\square$

Answer 2 (using the new general method): Characteristic polynomial: $r^2 + 1 = (r - i)(r + i)$. Roots: i and $-i$. Basis: $\boxed{e^{it}, e^{-it}}$. (In other words, the general solution is $c_1 e^{it} + c_2 e^{-it}$.) $\square$

The answers agree since any (complex) linear combination of e^{it} and e^{-it} is also a (complex) linear combination of $\cos t$ and $\sin t$, and vice versa. For example,

$$\begin{aligned} 5e^{it} + 3e^{-it} &= 5(\cos t + i \sin t) + 3(\cos t - i \sin t) \\ &= 8 \cos t + 2i \sin t. \end{aligned}$$

In other words, $\text{Span}(\cos t, \sin t)$ and $\text{Span}(e^{it}, e^{-it})$ (both defined using complex coefficients) are the same set of functions; each is the set of all solutions to $\ddot{x} + x = 0$.

Question 11.11. Which basis should be used, $\boxed{e^{it}, e^{-it}}$ or $\boxed{\cos t, \sin t}$?

Answer: It depends:

- The basis e^{it}, e^{-it} is easier to calculate with, but it's not immediately obvious which linear combinations of these functions are real-valued.
- The basis $\cos t, \sin t$ consisting of *real-valued* functions is useful for interpreting solutions in a physical system. The general real-valued solution is $c_1 \cos t + c_2 \sin t$ where c_1, c_2 are real numbers.

So we will be converting back and forth.

The same principles apply more generally:

Complex basis vs. real-valued basis. Let $y(t)$ be a complex-valued function of a real-valued variable t . If $\boxed{y, \bar{y}}$ is a basis for a space of functions, then $\boxed{\text{Re}(y), \text{Im}(y)}$ is another basis for the same space of functions, but having the advantage that it consists of real-valued functions.

Proof. (Not done in detail in lecture.) Any linear combination of y and $\bar{y}$ can be re-expressed as a linear combination of $\text{Re}(y)$ and $\text{Im}(y)$ by substituting

$$y = \text{Re}(y) + i \text{Im}(y), \quad \bar{y} = \text{Re}(y) - i \text{Im}(y).$$

Conversely, any linear combination of $\text{Re}(y)$ and $\text{Im}(y)$ can be re-expressed as a linear combination of y and $\bar{y}$ by substituting

$$\text{Re}(y) = \frac{y + \bar{y}}{2}, \quad \text{Im}(y) = \frac{y - \bar{y}}{2i}.$$

Thus $\text{Span}(\text{Re}(y), \text{Im}(y)) = \text{Span}(y, \bar{y})$.

To finish checking the $\text{Re}(y), \text{Im}(y)$ is a basis for the space, we need to check that they are linearly independent. If $\text{Im}(y)$ were a scalar multiple of $\text{Re}(y)$, say $\text{Re}(y) = f(t)$ and $\text{Im}(y) = af(t)$, then y and $\bar{y}$ would be multiples of $f(t)$ too, so one of y and $\bar{y}$ would be a multiple of the other, so y and $\bar{y}$ would be linearly dependent, which is nonsense since by assumption they form a basis. Thus $\text{Im}(y)$ cannot be a scalar multiple of $\text{Re}(y)$. Similarly, $\text{Re}(y)$ cannot be a scalar multiple of $\text{Im}(y)$. Thus $\text{Re}(y), \text{Im}(y)$ are linearly independent. $\square$

Question 11.12. Would it be OK to replace $\boxed{y, \bar{y}}$ instead by $\boxed{\text{Re}(y), i \text{Im}(y)}$?

Answer: Yes, but this would be less useful, because the whole point was to obtain a basis consisting of *real-valued* functions.

Question 11.13. Would it be OK to replace $\boxed{y, \bar{y}}$ instead by $\boxed{\operatorname{Re}(y), \operatorname{Re}(\bar{y})}$.

Answer: No, because if $y = f + ig$, then $\bar{y} = f - ig$, so $\operatorname{Re}(y)$ and $\operatorname{Re}(\bar{y})$ are both f ! They are linearly dependent, so they can't form a basis.

Here is an example of how this is used in practice:

Problem 11.14. Find a basis of solutions to

$$\ddot{y} + 4\dot{y} + 13y = 0$$

consisting of *real-valued* functions.

Solution:

Characteristic equation: $r^2 + 4r + 13 = 0$. The quadratic formula, or completing the square (rewriting the polynomial as $(r + 2)^2 + 9$), shows that the roots are $-2 + 3i$, $-2 - 3i$. Basis: $e^{(-2+3i)t}$, $e^{(-2-3i)t}$. But these functions are not real-valued!

So replace $y = e^{(-2+3i)t}$ and $\bar{y} = e^{(-2-3i)t}$ by $\operatorname{Re}(y)$ and $\operatorname{Im}(y)$, found by expanding

$$\begin{aligned} e^{(-2+3i)t} &= e^{-2t} e^{i(3t)} \\ &= e^{-2t} (\cos(3t) + i \sin(3t)) \\ &= e^{-2t} \cos(3t) + i e^{-2t} \sin(3t). \end{aligned}$$

Thus

$$\boxed{e^{-2t} \cos(3t), \quad e^{-2t} \sin(3t)}$$

is another basis, this time consisting of real-valued functions. $\square$

11.8. Harmonic oscillators and damped frequency. Let's apply all this to the spring-mass-dashpot system, assuming no external force.

11.8.1. Undamped case. If there is no damping, the DE is

$$m\ddot{x} + kx = 0.$$

Characteristic polynomial: $p(r) = mr^2 + k$.

Roots: $\pm\sqrt{-k/m} = \pm i\omega$, where $\boxed{\omega := \sqrt{k/m}}$.

Basis of solution space: $e^{i\omega t}$, $e^{-i\omega t}$.

Real-valued basis: $\cos \omega t$, $\sin \omega t$.

General real solution: $\boxed{c_1 \cos \omega t + c_2 \sin \omega t}$, where c_1, c_2 are real constants.

In other words, the real-valued solutions are all the sinusoid functions of angular frequency ω . They could also be written as $A \cos(\omega t - \phi)$, where A and ϕ are real constants.

This system, or any other system governed by the same DE, is also called a **simple harmonic oscillator**. The angular frequency ω is also called the **natural frequency** (or **resonant frequency**) of the oscillator.

11.8.2. *Damped case.* If there is damping, the DE is

$$m\ddot{x} + b\dot{x} + kx = 0.$$

Characteristic polynomial: $p(r) = mr^2 + br + k$.

Roots: $\frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$ (by the quadratic formula)

There are three cases, depending on the sign of $b^2 - 4mk$.

Case 1: $b^2 < 4mk$ (underdamped).

Then there are two complex roots $-s \pm i\omega_d$, where

$$s := \frac{b}{2m} \quad (\text{positive})$$

$$\text{damped frequency } \omega_d := \frac{\sqrt{4mk - b^2}}{2m} \quad (\text{positive})$$

Basis of solution space: $e^{(-s+i\omega_d)t}$, $e^{(-s-i\omega_d)t}$

Real-valued basis: $e^{-st} \cos(\omega_d t)$, $e^{-st} \sin(\omega_d t)$.

General real solution: $\boxed{e^{-st}(c_1 \cos(\omega_d t) + c_2 \sin(\omega_d t))}$, where c_1, c_2 are real constants.

This is a sinusoid multiplied by a decaying exponential. It can also be written as $e^{-st}(A \cos(\omega_d t - \phi))$ for some A and ϕ . Each nonzero solution tends to 0, but changes sign infinitely many times along the way. The system is called **underdamped**, because there was not enough damping to eliminate the oscillation completely.

The damping not only causes the solution to decay exponentially, but also *changes the frequency of the sinusoid*. The new angular frequency, ω_d , is called damped frequency. It is less than the undamped frequency we computed earlier (same formula, but with $b = 0$):

$$\omega_d = \frac{\sqrt{4mk - b^2}}{2m} < \frac{\sqrt{4mk}}{2m} = \sqrt{\frac{k}{m}} = \omega.$$

Warning: The damped solutions are not actually periodic (they don't repeat exactly, because of the decay). Sometimes $2\pi/\omega_d$ is called the **pseudo-period**.

Case 2: $b^2 = 4mk$ (critically damped).

There there is a repeated real root: $-\frac{b}{2m}$, $-\frac{b}{2m}$. Call it $-s$.

Basis of solution space: e^{-st} , te^{-st} .

General real solution: $\boxed{e^{-st}(c_1 + c_2 t)}$, where c_1, c_2 are real constants.

What happens to the solutions as $t \rightarrow +\infty$? The solution e^{-st} tends to 0. So does $te^{-st} = \frac{t}{e^{st}}$: even though the numerator t is tending to $+\infty$, the denominator e^{st} is tending to $+\infty$ faster (in a contest between exponentials and polynomials, exponentials always win). Thus all solutions eventually decay.

This case is when there is just enough damping to eliminate oscillation. The system is called **critically damped**.

Case 3: $b^2 > 4mk$ (overdamped).

In this case, the roots $\frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$ are real and distinct. Both roots are negative, since $\sqrt{b^2 - 4mk} < b$. Call them $-s_1$ and $-s_2$.

General real solution: $\boxed{c_1 e^{-s_1 t} + c_2 e^{-s_2 t}}$, where c_1, c_2 are real constants.

As in all the other damped cases, all solutions tend to 0 as $t \rightarrow +\infty$. The term corresponding to the *less negative* root eventually controls the rate of return to equilibrium. The system is called **overdamped**; there is so much damping that it is slowing the return to equilibrium.

Summary:

Case	Roots	Situation
$b = 0$	two complex roots $\pm i\omega$	undamped (simple harmonic oscillator)
$b^2 < 4mk$	two complex roots $-s \pm i\omega_d$	underdamped (damped oscillator)
$b^2 = 4mk$	repeated real root $-s, -s$	critically damped
$b^2 > 4mk$	distinct real roots $-s_1, -s_2$	overdamped

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Problem 11.15. Analyze the spring-mass-dashpot system with $m = 1$, $b = 2$, $k = 4$.

Solution: The ODE is

$$\ddot{x} + 2\dot{x} + 4x = 0.$$

Characteristic polynomial: $p(r) = r^2 + 2r + 4 = (r + 1)^2 + 3$.

Roots: $-1 \pm i\sqrt{3}$. These are complex, so the system is *underdamped*.

Basis of the solution space: $e^{(-1+i\sqrt{3})t}$, $e^{(-1-i\sqrt{3})t}$.

Real-valued basis: $e^{-t} \cos(\sqrt{3}t)$, $e^{-t} \sin(\sqrt{3}t)$.

General real solution: $\boxed{e^{-t}(a \cos(\sqrt{3}t) + b \sin(\sqrt{3}t))}$, where a, b are real constants.

The damped frequency is $\sqrt{3}$.

Question: Any nonzero solution $x = e^{-t}(a \cos(\sqrt{3}t) + b \sin(\sqrt{3}t))$ crosses the equilibrium position $x = 0$ infinitely many times.

How much time elapses between two consecutive crossings?

Possible answers:

1. $\pi\sqrt{3}$
2. $\pi/\sqrt{3}$
3. $2\pi\sqrt{3}$
4. $2\pi/\sqrt{3}$
5. $\sqrt{3}/\pi$
6. $\sqrt{3}/(2\pi)$
7. None of the above

Answer: $\pi/\sqrt{3}$. Why? The solution has the same zeros as the sinusoid $a\cos(\sqrt{3}t) + b\sin(\sqrt{3}t)$ of angular frequency $\sqrt{3}$, period $2\pi/\sqrt{3}$. But a sinusoid crosses 0 twice within each period, so the answer is half a period, $\pi/\sqrt{3}$.

Try the “Damped Vibrations” mathlet

<http://mathlets.org/mathlets/damped-vibrations/>

12. HIGHER-ORDER HOMOGENEOUS LINEAR ODES WITH CONSTANT COEFFICIENTS

Before talking about ODEs of order higher than 2, let’s continue the discussion of linear algebra topics such as span, linear dependence, and basis.

12.1. Vector spaces and span.

Question 12.1. Is $\text{Span}(f_1, \dots, f_n)$ always a vector space?

Answer: Yes, because

0. The zero function 0 is a linear combination of $f_1, \dots, f_n$, namely $0f_1 + \dots + 0f_n$.
1. Multiplying a linear combination of $f_1, \dots, f_n$ by a scalar gives another linear combination of $f_1, \dots, f_n$ (with different coefficients).
2. Adding two linear combinations of $f_1, \dots, f_n$ gives another linear combination of $f_1, \dots, f_n$. $\square$

12.2. Linearly dependent functions. Recall that *two* functions are called *linearly dependent* when one of them is a scalar multiple of the other one. To motivate the definition for *more than two* functions, first consider the following:

Question: True or false? The set of all solutions to $\ddot{x} + x = 0$ is

$$\text{Span}(\cos t, \sin t, 3\cos t + 4\sin t).$$

Answer: TRUE. Let’s explain why. The span of these three functions is the set of linear combinations

$$a\cos t + b\sin t + c(3\cos t + 4\sin t).$$

But each such linear combination is also just a linear combination of $\cos t$ and $\sin t$ alone: for example,

$$100 \cos t + 10 \sin t + 2(3 \cos t + 4 \sin t) = 106 \cos t + 18 \sin t.$$

Thus

$$\text{Span}(\cos t, \sin t, 3 \cos t + 4 \sin t) = \text{Span}(\cos t, \sin t),$$

which we already know is the set of all solutions to $\ddot{x} + x = 0$. $\square$

Even though the statement was true, including $3 \cos t + 4 \sin t$ in the list was redundant: it gave no new linear combinations. The general definition of linearly dependent functions captures this notion of redundancy:

Definition 12.2. Functions $f_1, \dots, f_n$ are **linearly dependent** (think redundant) if at least one of them is a linear combination of the others. Otherwise, call them **linearly independent**.

Example 12.3. The three functions $\cos t$, $\sin t$, and $3 \cos t + 4 \sin t$ are linearly dependent since the third function is a linear combination of the first two.

Remark 12.4. When there are only two functions, f_1 and f_2 , then to say that one of them is a linear combination of the others is the same as saying that one of them is a scalar multiple of the other one. So the new definition is compatible with the earlier definition for two functions.

Equivalent definition: Functions $f_1, \dots, f_n$ are **linearly dependent** if there exist numbers $c_1, \dots, c_n$ *not all zero* such that $c_1 f_1 + \dots + c_n f_n = 0$.

Why is this definition equivalent to the other one just given?

- If there exists such a nontrivial linear combination summing to 0, say

$$3f + 5g + 7h = 0,$$

then one can solve for one of the functions to express it as a linear combination of the others:

$$f = \left(-\frac{5}{3}\right)g + \left(-\frac{7}{3}\right)h.$$

- Conversely, if one of the functions is a linear combination of the others, say,

$$g = 6f + 8h,$$

then moving all the terms to the left side gives a nontrivial linear combination summing to 0:

$$(-6)f + g + (-8)h = 0.$$

Moral: When describing a vector space of functions (such as the set of solutions to a homogeneous linear ODE) as a span, it is most efficient to give it as the span of *linearly independent* functions.

12.3. Basis.

Definition 12.5. A **basis** of a vector space S is a list of functions $f_1, f_2, \dots$ such that

1. $\text{Span}(f_1, f_2, \dots) = S$, and
2. The functions $f_1, f_2, \dots$ are linearly **in**dependent.

Question: What is a basis for the space of solutions to $\dot{y} = 3y$?

Possible answers:

1. This is not a homogeneous linear ODE, so the solutions don't form a vector space. It's a trick question.
2. The function e^{3t} by itself is a basis.
3. The function $2e^{3t}$ by itself is a basis.
4. The basis is the set of all functions of the form ce^{3t} .

Answer: 2 or 3! First, answer 1 is wrong: each term *is* a (constant) function of t times either y or $\dot{y}$, so this *is* a homogeneous linear ODE, and the solutions *do* form a vector space. The basis is supposed to consist of linearly **in**dependent functions such that all the solutions can be built from them. Answer 4 is wrong since the functions in the basis are supposed to be linearly **in**dependent; if e^{3t} is in a basis, then $5e^{3t}$ should not be in the same basis since it is a linear combination of e^{3t} by itself. Answer 2 is correct since the solutions are exactly the functions ce^{3t} for all numbers c . Answer 3 is correct too since the functions $c(2e^{3t})$ also run through all solutions as c ranges over all numbers.

Key point: A vector space usually has infinitely many functions. To describe it compactly, give a *basis* of the vector space.

12.4. Dimension. Consider a vector space V . It turns out that, although V can have different bases, each basis has *the same number of functions* in it.

Definition 12.6. The **dimension** of a vector space is the number of functions in any basis.

Example 12.7. The space of solutions to $\ddot{x} + x = 0$ is 2-dimensional since the basis $\boxed{\cos t, \sin t}$ has 2 functions. (The basis $\boxed{e^{it}, e^{-it}}$ also has 2 functions.)

Example 12.8. The space of solutions to $\dot{y} = 3y$ is 1-dimensional.

In these two examples, the dimension equals the order of the homogeneous linear ODE. It turns out that this holds in general:

Dimension theorem for a homogeneous linear ODE. *The dimension of the space of solutions to an n^{th} order homogeneous linear ODE is n .*

In other words, the number of parameters needed in the general solution to an n^{th} order homogeneous linear ODE is n .

Remember when we proved that all solutions of $\ddot{x} + x = 0$ were linear combinations of $\cos t$ and $\sin t$, by showing that no matter what the values of $x(0)$ and $\dot{x}(0)$, we could find a linear combination of $\cos t$ and $\sin t$ that solved the DE with the same initial conditions? The same idea proves the dimension theorem for any homogeneous linear ODE.

Let's explain the idea for a 3rd-order ODE, in which we use 0 as starting time in the existence and uniqueness theorem. Define

$f :=$ the solution such that $y(0) = 1, \dot{y}(0) = 0, \ddot{y}(0) = 0$

$g :=$ the solution such that $y(0) = 0, \dot{y}(0) = 1, \ddot{y}(0) = 0$

$h :=$ the solution such that $y(0) = 0, \dot{y}(0) = 0, \ddot{y}(0) = 1$.

Then $af + bg + ch$ is the solution such that $y(0) = a, \dot{y}(0) = b, \ddot{y}(0) = c$,

so every solution $y(t)$, no matter what its values of $y(0), \dot{y}(0), \ddot{y}(0)$ are, is some linear combination $af + bg + ch$. Thus the set of all solutions to the DE is $\text{Span}(f, g, h)$.

If a, b, c are numbers such that $y := af + bg + ch$ is the zero function, then its values of $y(0), \dot{y}(0), \ddot{y}(0)$ must all be 0, which means that a, b, c are all 0. Thus f, g, h are linearly independent.

The previous two paragraphs imply that f, g, h is a basis for the space of solutions to the DE. There are 3 functions in this basis, so the dimension of the space of solutions is 3.

12.5. Solving a homogeneous linear ODE with constant coefficients. Earlier we gave a method to solve any *second-order* homogeneous linear ODEs with constant coefficients. Now we do the same for n^{th} order for *any* n .

Given

$$a_n y^{(n)} + \cdots + a_1 \dot{y} + a_0 y = 0, \quad (8)$$

where $a_n, \dots, a_0$ are constants, do the following:

1. Write down the **characteristic equation**

$$a_n r^n + \cdots + a_1 r + a_0 = 0,$$

in which the coefficient of r^i is the coefficient of $y^{(i)}$ from the ODE. The left hand side is called the **characteristic polynomial** $p(r)$. (For example, $\ddot{y} + 5y = 0$ has characteristic polynomial $r^2 + 5$.)

2. Factor $p(r)$ as

$$a_n(r - r_1)(r - r_2) \cdots (r - r_n)$$

where $r_1, \dots, r_n$ are (possibly complex) numbers.

3. If $r_1, \dots, r_n$ are *distinct*, then the functions $e^{r_1 t}, \dots, e^{r_n t}$ form a basis for the space of solutions to the ODE (8). In other words, the general solution is

$$c_1 e^{r_1 t} + \dots + c_n e^{r_n t}.$$

4. If $r_1, \dots, r_n$ are *not* distinct, then $e^{r_1 t}, \dots, e^{r_n t}$ cannot be a basis since some of these functions are redundant (definitely not linearly independent!) If a particular root r is repeated m times, then

$$\begin{array}{l} \text{replace } \overbrace{e^{rt}, \quad e^{rt}, \quad e^{rt}, \quad \dots, \quad e^{rt}}^{m \text{ copies}} \\ \text{by } e^{rt}, \quad te^{rt}, \quad t^2 e^{rt}, \quad \dots, \quad t^{m-1} e^{rt}. \end{array}$$

(We'll explain later on why this works.) $\square$

In all cases,

$$\boxed{\# \text{ functions in basis} = \# \text{ roots of } p(r) \text{ counted with multiplicity} = \text{order of ODE}},$$

as predicted by the dimension theorem.

Problem 12.9. Find the general solution to

$$y^{(6)} + 6y^{(5)} + 9y^{(4)} = 0.$$

Solution: The characteristic polynomial is

$$r^6 + 6r^5 + 9r^4 = r^4(r+3)^2,$$

whose roots listed with multiplicity are

$$0, 0, 0, 0, -3, -3.$$

Since the roots are not distinct, the basis is *not*

$$e^{0t}, \quad e^{0t}, \quad e^{0t}, \quad e^{0t}, \quad e^{-3t}, \quad e^{-3t}.$$

We need to replace the first block of four functions, and also the last block of two functions. So the correct basis is

$$\underbrace{e^{0t}, \quad te^{0t}, \quad t^2 e^{0t}, \quad t^3 e^{0t}}_{\text{first block}}, \quad \underbrace{e^{-3t}, \quad te^{-3t}}_{\text{last block}},$$

which simplifies to

$$1, \quad t, \quad t^2, \quad t^3, \quad e^{-3t}, \quad te^{-3t}.$$

Thus the general solution is

$$c_1 + c_2 t + c_3 t^2 + c_4 t^3 + c_5 e^{-3t} + c_6 t e^{-3t}.$$

(As expected, there is a 6-dimensional space of solutions to this 6th order ODE.) $\square$

Problem 12.10. Find the simplest constant-coefficient homogeneous linear ODE having $(5t + 7)e^{-t} - 9e^{2t}$ as one of its solutions.

Solution: The given function is a linear combination of

$$e^{-t}, \quad te^{-t}, \quad e^{2t}$$

so the roots of the characteristic polynomial (with multiplicity) should include $-1, -1, 2$. So the simplest characteristic polynomial is

$$(r + 1)(r + 1)(r - 2) = r^3 - 3r - 2$$

and the corresponding ODE is

$$y^{(3)} - 3\ddot{y} - 2y = 0. \quad \square$$

Remember when for a second-order ODE whose characteristic polynomial had complex roots we got a basis like

$$e^{(-2+3i)t}, \quad e^{(-2-3i)t},$$

consisting of a complex-valued function y and its complex conjugate $\bar{y}$? We explained that it was OK to replace $\boxed{y, \bar{y}}$ by a new basis $\boxed{\operatorname{Re} y, \operatorname{Im} y}$ consisting of real-valued functions.

We can do the same replacement even if $\boxed{y, \bar{y}}$ is just *part* of a basis.

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Here is an example of how this is used in practice:

Problem 12.11. Find a basis of solutions to

$$y^{(3)} + 3\ddot{y} + 9\dot{y} - 13y = 0$$

consisting of *real-valued* functions.

Solution: The characteristic polynomial is $p(r) := r^3 + 3r^2 + 9r - 13$. Checking the divisors of -13 (as instructed by the rational root test), we find that 1 is a root, so $r - 1$ is a factor. Long division (or solving for unknown coefficients) produces the other factor:

$$p(r) = (r - 1)(r^2 + 4r + 13).$$

Roots: $1, -2 + 3i, -2 - 3i$.

Basis: $e^t, e^{(-2+3i)t}, e^{(-2-3i)t}$.

Leave e^t as is but replace $y := e^{(-2+3i)t}$ and $\bar{y} = e^{(-2-3i)t}$ to get a new basis

$$\boxed{e^t, \quad e^{-2t} \cos(3t), \quad e^{-2t} \sin(3t)}$$

consisting of real-valued functions. $\square$

Midterm 1 covers everything up to here.

13. INHOMOGENEOUS LINEAR ODES WITH CONSTANT COEFFICIENTS

13.1. Operator notation.

13.1.1. The operator D .

- A function, like $f(t) = t^2$, takes an input number and returns another number.
- An **operator** takes an input *function* and returns another function.

For example, the differential operator $\frac{d}{dt}$ takes an input function $y(t)$ and returns $\frac{dy}{dt}$. This operator is also called D . So $De^{3t} = 3e^{3t}$, for instance (chain rule).

Any number can be viewed as the “multiply-by-the-number” operator: for instance, the operator 5 transforms the function $\sin t$ into the function $5 \sin t$. (Similarly, a function $f(t)$ can be viewed as the “multiply-by- $f(t)$ ” operator: the operator t^2 transforms $\cos t$ into $t^2 \cos t$.)

13.1.2. *Multiplying and adding operators.* To apply a product $L_1 L_2$ of operators to a function, first apply L_2 and then apply L_1 to the result: $(L_1 L_2)(f) = L_1(L_2(f))$. For instance, DDy means take the derivative of y , and then take the derivative of the result; therefore we write $D^2 y = \ddot{y}$.

To apply a sum of two operators, apply each operator to the function and add the results. For instance, $(D^2 + D)y = D^2 y + Dy = \ddot{y} + \dot{y}$.

Problem 13.1. Rewrite the ODE

$$2\ddot{y} + 3\dot{y} + 5y = 0,$$

in operator form.

Answer:

$$(2D^2 + 3D + 5)y = 0. \quad \square$$

The same argument shows that every constant-coefficient homogeneous linear ODE

$$a_n y^{(n)} + \cdots + a_0 y = 0$$

can be written simply as

$$p(D)y = 0,$$

where p is the characteristic polynomial.

13.2. **Linear operators.** An operator L is **linear** if

$$L(f + g) = Lf + Lg, \quad L(af) = aLf$$

for any functions f and g , and any number a . Any linear operator L respects linear combinations, meaning that

$$L(c_1 f_1 + \cdots + c_n f_n) = c_1 Lf_1 + \cdots + c_n Lf_n$$

for any numbers $c_1, \dots, c_n$ and functions $f_1, \dots, f_n$.

The operator D is linear. Also, any “multiply-by-a-function” operator is linear. Multiplying and adding these operators shows that

$$2D^2 + 3D + 5$$

is linear, as is any polynomial $p(D)$.

Remark 13.2. Any homogeneous linear ODE can be written as $Ly = 0$ for some linear operator L , even if the coefficients are *nonconstant* functions of t .

13.3. Time invariance.

Remark 13.3. Delaying an input signal $F(t)$ in time by a seconds gives a new input signal $F(t - a)$ (the minus sign is correct: the new input signal has the value at $t = a$ that the old input signal has at $t = 0$).

When p is a polynomial with constant coefficients, $p(D)$ is **time-invariant**, ‘which means that

$$\text{if } f(t) \text{ is a solution to } p(D)x = F(t)$$

and a is a number,

$$\text{then } f(t - a) \text{ is a solution to } p(D)x = F(t - a).$$

In words: if an input signal $F(t)$ is delayed in time by a seconds, then the output signal $f(t)$ is delayed by a seconds.

This can simplify the solution to some DEs:

Problem 13.4. Fill in the blank:

Given that $x(t) := \frac{1}{2} \cos t + \frac{1}{2} \sin t$ is a solution to $\dot{x} + x = \cos t$,

it follows that _____ is a solution to $\dot{x} + x = \sin t$.

(*Hint:* $\sin t = \cos(t - \pi/2)$.)

One answer is

$$\begin{aligned} x(t - \pi/2) &= \frac{1}{2} \cos(t - \pi/2) + \frac{1}{2} \sin(t - \pi/2) \\ &= \frac{1}{2} \sin t - \frac{1}{2} \cos t. \end{aligned}$$

A system that is defined by a linear time-invariant operator is called an **LTI system**.

13.4. Shortcut for applying an operator to an exponential function.

Warm-up problem: If r is a number, what is $(2D^2 + 3D + 5)e^{rt}$?

Solution: First, $De^{rt} = re^{rt}$ and $D^2e^{rt} = r^2e^{rt}$ (keep applying the chain rule). Thus

$$\begin{aligned}(2D^2 + 3D + 5)e^{rt} &= 2r^2e^{rt} + 3re^{rt} + 5e^{rt} \\ &= (2r^2 + 3r + 5)e^{rt}. \quad \square\end{aligned}$$

The same calculation, but with an arbitrary polynomial, proves the general rule:

Theorem 13.5. *For any polynomial p and any number r ,*

$$\boxed{p(D) e^{rt} = p(r)e^{rt}}.$$

13.5. Basis of solutions when there are repeated roots.

Problem 13.6. Find a basis of solutions to $\ddot{y} - 10\dot{y} + 25y = 0$.

Solution:

Characteristic polynomial: $p(r) = r^2 - 10r + 25 = (r - 5)^2$.

Roots: 5, 5.

Basis: $\boxed{e^{5t}, te^{5t}}$. $\square$

But why does this work? Using operators, we can now explain!

In operator form, the DE is

$$(D - 5)^2 y = 0.$$

The calculation

$$(D - 5)^2 e^{5t} = p(D) e^{5t} \stackrel{\text{shortcut}}{=} p(5)e^{5t} = 0e^{5t} = 0$$

shows that e^{5t} is one solution. But the DE is second-order, so the basis should have two functions. Taking the second function to be ce^{5t} for a *constant* c does not give a basis, since e^{5t}, ce^{5t} are linearly dependent.

Let's try variation of parameters! Plug in $y = ue^{5t}$, where u is a function to be determined. To calculate $(D - 5)^2 ue^{5t}$, let's apply $D - 5$ twice:

$$\begin{aligned}(D - 5) ue^{5t} &= (\dot{u}e^{5t} + u(5e^{5t})) - 5ue^{5t} \\ &= \dot{u}e^{5t}.\end{aligned}$$

Similarly,

$$(D - 5)^2 ue^{5t} = \ddot{u}e^{5t}.$$

Thus in order for ue^{5t} to be a solution to $(D - 5)^2 y = 0$ we must have

$$\begin{aligned}\ddot{u} &= 0 \\ \dot{u} &= c_1 \\ u &= c_1 t + c_2 \\ y &= (c_1 t + c_2)e^{5t} \\ y &= c_1 t e^{5t} + c_2 e^{5t}.\end{aligned}$$

In other words, the set of all solutions is $\text{Span}(te^{5t}, e^{5t})$. Neither te^{5t} nor e^{5t} is a *constant* multiple of the other, so they are linearly independent. Thus they form a basis.

A similar approach handles more complicated characteristic polynomials involving many repeated roots.

Remark 13.7. You don't have to go through the discussion of this section each time you want to solve $p(D)y = 0$; we are just explaining why the method given earlier actually works.

13.6. Exponential response. For any polynomial p and number r ,

$$p(D) e^{rt} = p(r) e^{rt},$$

so

$$\underset{\text{output signal}}{e^{rt}} \text{ is a particular solution to } p(D) y = \underset{\text{input signal}}{p(r) e^{rt}}.$$

New problem: What if the input signal is just e^{rt} ?

Answer (superposition): Multiply by the number $\frac{1}{p(r)}$ to get...

Exponential response formula (ERF).

For any polynomial p and any number r such that $p(r) \neq 0$,

$$\underset{\text{output signal}}{\frac{1}{p(r)} e^{rt}} \text{ is a particular solution to } p(D) y = \underset{\text{input signal}}{e^{rt}}.$$

In other words, multiply the input signal by the number $\frac{1}{p(r)}$ to get an output signal.

Problem 13.8. Find the general solution to $\ddot{y} + 7\dot{y} + 12y = -5e^{2t}$.

Solution:

Characteristic polynomial: $p(r) = r^2 + 7r + 12 = (r + 3)(r + 4)$.

Roots: $-3, -4$.

General solution to *homogeneous* equation: $y_h := c_1 e^{-3t} + c_2 e^{-4t}$.

ERF says:

$$\frac{1}{p(2)}e^{2t} \quad \text{is a particular solution to} \quad p(D)y = e^{2t};$$

i.e.,

$$\frac{1}{30}e^{2t} \quad \text{is a particular solution to} \quad \ddot{y} + 7\dot{y} + 12y = e^{2t},$$

so

$$-\frac{1}{6}e^{2t} \quad \text{is a particular solution to} \quad \ddot{y} + 7\dot{y} + 12y = -5e^{2t}.$$

call this y_p

General solution to inhomogeneous equation:

$$\begin{aligned} y &= y_p + y_h \\ &= -\frac{1}{6}e^{2t} + c_1e^{-3t} + c_2e^{-4t}. \quad \square \end{aligned}$$

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How to use ERF:

- ERF is a method for finding one particular solution to an **inhomogeneous** equation $p(D)y = e^{rt}$ such as $2\ddot{y} + 3\dot{y} + 5y = e^{10t}$.
- If instead the right hand side is a linear combination of different exponential functions, like $2e^{3t} + 5e^{7t}$, apply ERF with each of e^{3t} and e^{7t} by itself on the right, and then use superposition.
- Once one particular **inhomogeneous** solution is found, add it to the general **homogeneous** solution to get the *general inhomogeneous* solution.

The existence and uniqueness theorem says that

$$p(D)y = e^{rt}$$

should have a solution even if $p(r) = 0$ (when ERF does not apply). Here is how to find a particular solution in this bad case:

Generalized exponential response formula.

If p is a polynomial having r as a root with multiplicity m , then

$$z_p = \frac{1}{p^{(m)}(r)} t^m e^{rt} \quad \text{is a particular solution to} \quad p(D)z = e^{rt}.$$

output signal input signal

In other words, multiply the input signal by t^m , and then multiply by the number $\frac{1}{p^{(m)}(r)}$, where $p^{(m)}$ is the m^{th} derivative of p .

(The proof that this works involves a generalized shortcut, obtained by applying $\frac{\partial}{\partial R}$ to the shortcut equation

$$p(D) e^{Rt} = p(R) e^{Rt}$$

m times and then setting the variable R equal to the number r . We'll skip it.)

Generalized ERF comes up less often than regular ERF, since in most applications, $p(r) \neq 0$.

13.7. Sinusoidal response (complex replacement). Suppose that $p(t)$ is a real polynomial, and ω is a real number. Let z_p denote a complex-valued function.

Problem 13.9. Fill in the blank:

$$\begin{aligned} &\text{If } p(D) z_p = e^{i\omega t} \\ &\text{then } p(D) \underline{\hspace{1cm}} = \cos \omega t. \end{aligned}$$

Answer: Taking real parts of both sides shows that $\text{Re}(z_p)$ works.

The observation above leads to the **complex replacement method**, which we now explain.

Given: The inhomogeneous linear ODE

$$p(D) x = \cos \omega t,$$

where p is a real polynomial, and ω is a real number.

Goal: To find *one particular solution* x_p .

Method:

1. Replace $\cos \omega t$ by $e^{i\omega t}$ (whose real part is $\cos \omega t$) and use a different letter z for the unknown function in this new “complex replacement DE”:

$$p(D) z = \underset{\text{complex replacement}}{e^{i\omega t}}.$$

2. Find a particular solution z_p to the complex replacement DE. (Use ERF for this, provided that $p(i\omega) \neq 0$.)
3. Take the real part: $x_p := \text{Re}(z_p)$. Then x_p is a particular solution to the *original* DE.

Problem 13.10. Find a particular solution x_p to

$$\ddot{x} + \dot{x} + 2x = \cos 2t.$$

Solution: The characteristic polynomial is $p(r) := r^2 + r + 2$.

Step 1. Since $\cos 2t$ is the real part of e^{2it} , replace $\cos 2t$ by e^{2it} :

$$\ddot{z} + \dot{z} + 2z = e^{2it}.$$

Step 2. ERF says that one particular solution to this new ODE is

$$z_p := \frac{1}{p(2i)} e^{2it} = \frac{1}{-2 + 2i} e^{2it}.$$

Step 3. A particular solution to the original ODE is

$$x_p := \operatorname{Re}(z_p) = \operatorname{Re} \left(\frac{1}{-2 + 2i} e^{2it} \right).$$

This is a sinusoid expressed in complex form.

It might be more useful to have the answer in amplitude-phase form or as a linear combination of $\cos$ and $\sin$, but we are given $x_p = \operatorname{Re}(ce^{2it})$ with $c := \frac{1}{-2 + 2i}$. To convert, we need to rewrite $\bar{c}$ as $Ae^{i\phi}$ or $a + bi$, using

$$\begin{array}{ccc} & Ae^{i\phi} & \\ // & & // \\ \bar{c} & \text{---} & a + bi. \end{array}$$

Converting to amplitude-phase form. The number $-2 + 2i$ has absolute value $2\sqrt{2}$ and angle $3\pi/4$, so

$$\begin{aligned} -2 + 2i &= 2\sqrt{2}e^{i(3\pi/4)} \\ c &= \frac{1}{-2 + 2i} = \frac{1}{2\sqrt{2}}e^{i(-3\pi/4)} \\ \bar{c} &= \frac{1}{2\sqrt{2}}e^{i(3\pi/4)}, \end{aligned}$$

which is supposed to be $Ae^{i\phi}$. Thus the amplitude is $A = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$ and the phase lag is $\phi = 3\pi/4$. **Conclusion:** In amplitude-phase form,

$$x_p = \frac{\sqrt{2}}{4} \cos(2t - 3\pi/4).$$

Converting to a linear combination of $\cos$ and $\sin$. We have

$$\begin{aligned} c &= \frac{1}{-2 + 2i} = \frac{1}{-2 + 2i} \left(\frac{-2 - 2i}{-2 - 2i} \right) = \frac{-2 - 2i}{8} = -\frac{1}{4} - \frac{1}{4}i \\ \bar{c} &= -\frac{1}{4} + \frac{1}{4}i, \end{aligned}$$

which is supposed to be $a + bi$, so $a = -1/4$ and $b = 1/4$. **Conclusion:**

$$x_p = -\frac{1}{4} \cos 2t + \frac{1}{4} \sin 2t .$$

Try the “Amplitude and Phase: Second Order IV” mathlet

<http://mathlets.org/mathlets/amplitude-and-phase-second-order-iv/>

with $m = 1$, $b = 1$, $k = 2$, $\omega = 2$ to see the **input signal** $\cos 2t$, and output signal (in yellow). Can you see the amplitude and phase lag of the output signal? The **red segment** indicates the time lag $t_0 = \phi/\omega = (3\pi/4)/2 = 3\pi/8 \approx 1.18$. $\square$

13.8. Complex gain, gain, and phase lag for an ODE. Earlier we talked about complex gain for any LTI system. Now we’ll calculate it in the special case where the LTI system is defined by an operator $p(D)$.

When solving

$$p(D) x = \cos \omega t,$$

the **input signal** is $\cos \omega t$ and the **output signal** is the steady-state solution x_p . (We’ll say later what steady-state means.) Here is what happens in general, assuming $p(i\omega) \neq 0$:

- The complex replacement ODE

$$p(D) z = e^{i\omega t}$$

has complex input signal $e^{i\omega t}$ and complex output signal $\frac{1}{p(i\omega)} e^{i\omega t}$ by ERF, so

$$\text{complex gain } G = \frac{1}{p(i\omega)}$$

(a complex number). We can write $z_p = Ge^{i\omega t}$.

- The original ODE

$$p(D) x = \cos \omega t$$

has sinusoid output signal $x_p := \text{Re}(Ge^{i\omega t})$.

- The angular frequency of the output signal is the same as the angular frequency of the input signal: ω .
- For this system,

$$\text{gain} = |G| = \frac{1}{|p(i\omega)|}$$

and

$$\text{phase lag} = -\arg G = \arg p(i\omega) .$$

Remark 13.11. What happens if we change the input signal $\cos \omega t$ to a different sinusoidal function of angular frequency ω ? As mentioned when we introduced complex gain, this multiplies the complex input signal and the complex output signal by the same complex number, so the complex gain is the same. Thus the complex gain, gain, and phase lag are given by the same formulas as above: they depend only on the system and on ω .

Complex replacement is helpful also with other real input signals, with any real-valued function that can be written as the real part of a reasonably simple complex input signal. Here are some examples:

Real input signal	Complex replacement
$\cos \omega t$	$e^{i\omega t}$
$A \cos(\omega t - \phi)$	$Ae^{-i\phi}e^{i\omega t}$
$a \cos \omega t + b \sin \omega t$	$(a - bi)e^{i\omega t}$
$e^{at} \cos \omega t$	$e^{(a+i\omega)t}$

Each function in the first column is the real part of the corresponding function in the second column. The nice thing about these examples is that the complex replacement is a constant times a complex exponential, so ERF (or generalized ERF) applies.

13.9. Stability.

13.9.1. Steady-state solution, transient.

Problem 13.12. What is the general solution to $\ddot{x} + 7\dot{x} + 12x = \cos 2t$?

Solution: The characteristic polynomial is

$$p(r) = r^2 + 7r + 12 = (r + 3)(r + 4).$$

The complex gain is

$$G = \frac{1}{p(2i)} = \frac{1}{(2i)^2 + 7(2i) + 12} = \frac{1}{8 + 14i}.$$

Complex replacement and ERF show that

$$x_p = \operatorname{Re} \left(\frac{1}{8 + 14i} e^{2it} \right)$$

is a particular solution.

On the other hand, the general solution to the associated homogeneous ODE is

$$x_h = c_1 e^{-3t} + c_2 e^{-4t}.$$

Therefore the general solution to the original inhomogeneous ODE is

$$\begin{aligned}
 x &= x_p + x_h \\
 &= \underbrace{\operatorname{Re} \left(\frac{1}{8 + 14i} e^{2it} \right)}_{\text{steady-state solution}} + \underbrace{c_1 e^{-3t} + c_2 e^{-4t}}_{\text{transient}}. \quad \square
 \end{aligned}$$

In general, for a forced damped oscillator, complex replacement and ERF will produce a periodic output signal, and that particular solution is called the **steady-state solution**. Every other solution is the steady-state solution plus a **transient**, where the transient is a function that decays to 0 as $t \rightarrow +\infty$. As time progresses, the solution $x(t)$ will approximate the steady-state solution more and more closely.

Changing the initial conditions gives a new solution $x_{\text{new}}(t)$. But this changes only the c_1, c_2 above, so $x_{\text{new}}(t)$ will approximate the same steady-state solution that $x(t)$ approximates, and $x_{\text{new}}(t) - x(t)$ will tend to 0 as $t \rightarrow +\infty$. A system like this, in which changes in the initial conditions have vanishing effect on the long-term behavior of the solution (i.e., $x_{\text{new}}(t) - x(t)$ tends to 0 as $t \rightarrow +\infty$), is called **stable**.

Try the “Forced Damped Vibration” mathlet

<http://mathlets.org/mathlets/forced-damped-vibration/>

Notice that changing the initial conditions (dragging the yellow square on the left) does not change the long-term behavior of the output signal (yellow curve) much. So this is a stable system.

13.9.2. *Testing a second-order system for stability in terms of roots.* Stability depends on the shape of the solution to the associated homogeneous solution. In the problem above, this was $c_1 e^{-3t} + c_2 e^{-4t}$, which decays as $t \rightarrow +\infty$ no matter what c_1 and c_2 are, because -3 and -4 are negative. For a general 2nd-order constant-coefficient linear ODE, stability depends on the roots of the characteristic polynomial, as shown in the following table:

Roots	General solution x_h	Condition for stability	Characteristic poly.
complex $a \pm bi$	$e^{at}(c_1 \cos(bt) + c_2 \sin(bt))$	$a < 0$	$r^2 - 2ar + (a^2 + b^2)$
repeated real s, s	$e^{st}(c_1 + c_2 t)$	$s < 0$	$r^2 - 2sr + s^2$
distinct real r_1, r_2	$c_1 e^{r_1 t} + c_2 e^{r_2 t}$	$r_1, r_2 < 0$	$r^2 - (r_1 + r_2)r + r_1 r_2$

More generally:

Theorem 13.13 (Stability test in terms of roots). *A constant-coefficient linear ODE of any order is stable if and only if every root of the characteristic polynomial has negative real part.*

13.9.3. *Testing a second-order system for stability in terms of coefficients.* In the 2nd-order case, there is a shortcut for testing whether every root has negative real part:

Lemma 13.14. *For a real polynomial $X^2 + a_1X + a_0$, the following conditions are equivalent:*

- (i) *Every root has negative real part.*
- (ii) *All the coefficients are positive: $a_1 > 0$ and $a_0 > 0$.*

Proof. *Case 1: complex roots $a \pm bi$ with $b \neq 0$, so the polynomial is $X^2 - 2a + (a^2 + b^2)$.*

Then (i) says $a < 0$. And (ii) says $-2a > 0$ and $a^2 + b^2$. These are the same, since $a^2 + b^2$ is automatically positive.

Case 2: real roots r_1, r_2 , so the polynomial is $X^2 - (r_1 + r_2)X + r_1r_2$. Then (i) says $r_1, r_2 < 0$. And (ii) says $-(r_1 + r_2) > 0$ and $r_1r_2 > 0$. These are the same: if (i) holds, then (ii) holds; also, if (ii) holds, then at least one of r_1 and r_2 is negative, but then the inequality $r_1r_2 > 0$ forces the other one to be negative too. $\square$

The same test works when the leading coefficient is any positive number, not necessarily 1.

Thus, to check for stability, instead of checking whether every root has negative real part, we can simply check whether the coefficients are positive:

Theorem 13.15 (Stability test in terms of coefficients, 2nd-order case). *Assume that a_0, a_1, a_2 are real numbers with $a_2 > 0$. The ODE*

$$(a_2D^2 + a_1D + a_0)x = F(t)$$

is stable if and only if $a_1 > 0$ and $a_0 > 0$.

There is a generalization of the coefficient test to higher-order ODEs, called the [Routh–Hurwitz conditions for stability](#), but the conditions are much more complicated.

13.10. **Resonance.** Recall that a harmonic oscillator has a natural frequency. [Resonance](#) is a phenomenon that occurs when a harmonic oscillator is driven with an input sinusoid whose frequency is close to or equal to the natural frequency.

- “Near resonance” (frequency is *close to* the natural frequency): The oscillations of the output signal will be much larger than the oscillations of the input signal — the gain will be large. In fact, the closer that the input frequency gets to the natural frequency, the larger the gain becomes.
- “Pure resonance” (frequency is *equal to* the natural frequency): The oscillations grow with time, and there is no steady-state solution. But in a realistic physical situation, there is at least a tiny amount of damping, which prevents the runaway growth, so that the oscillations are bounded (but still large).

We now explain all of this by solving ODEs explicitly.

13.10.1. *Warm-up: harmonic oscillator with no input signal.* A typical ODE modeling a harmonic oscillator is

$$\ddot{x} + 9x = 0.$$

Characteristic polynomial: $r^2 + 9$.

Roots: $\pm 3i$.

Basis of solutions: e^{3it} , e^{-3it} .

Real-valued basis: $\cos 3t$, $\sin 3t$.

General real-valued solution: $a \cos 3t + b \sin 3t$, for real numbers a, b . These are all the sinusoids with angular frequency 3. The natural frequency is 3.

13.10.2. *Near resonance.* Now let's drive the harmonic oscillator with an input sinusoid. A typical ODE modeling this situation is

$$\ddot{x} + 9x = \underset{\text{input signal}}{\cos \omega t}.$$

The complex replacement ODE is

$$\ddot{z} + 9z = e^{i\omega t}.$$

Characteristic polynomial: $p(r) = r^2 + 9$.

Assume that $\omega \neq 3$, so that $i\omega$ is not a root of $p(r)$. Then ERF gives the particular solution

$$z_p := \frac{1}{p(i\omega)} e^{i\omega t} = \frac{1}{9 - \omega^2} e^{i\omega t}.$$

Then a particular solution to the original ODE is

$$x_p := \frac{1}{9 - \omega^2} \cos \omega t.$$

$$\text{Complex gain: } G = \frac{1}{p(i\omega)} = \frac{1}{9 - \omega^2}.$$

Gain: $|G| = \frac{1}{|9 - \omega^2|}$. This becomes very large as ω approaches 3.

Phase lag: $-\arg G$, which is 0 or π depending on whether $\omega < 3$ or $\omega > 3$.

Try the “Harmonic Frequency Response: Variable Input Frequency” mathlet

<http://mathlets.org/mathlets/harmonic-frequency-response-i/>

to see this. (In this mathlet, the natural frequency is 1, and the frequency of the input signal is adjustable.)

In engineering, the graph of gain as a function of ω is called a **Bode plot** (Bode is pronounced BOH-dee). (Actually, engineers usually instead use a log-log plot: they plot $\log(\text{gain})$ as a function of $\log \omega$). On the other hand, a **Nyquist plot** shows the *trajectory* of the complex gain G as ω varies.

Also try the “Harmonic Frequency Response: Variable Natural Frequency” mathlet

http://mathlets.org/mathlets/
harmonic-frequency-response-variable-natural-frequency/

(In this one, the input signal is fixed to be $\sin t$, and the natural frequency is adjustable. RMS stands for [root mean square](#), which for a sinusoid is amplitude/ $\sqrt{2}$.)

13.10.3. *Pure resonance.*

Question 13.16. What happens if $\omega = 3$ exactly?

This time, the complex replacement ODE

$$\ddot{z} + 9z = e^{3it}$$

cannot be solved by ERF, since $3i$ is a root of $p(r) = r^2 + 9$. This one requires generalized ERF. First, $p(r)$ has distinct roots $3i$ and $-3i$, so $m = 1$, and $p^{(m)}(r) = p'(r) = 2r = 6i$ at $r = 3i$. Generalized ERF gives

$$\begin{aligned} z_p &:= \frac{1}{6i} t e^{3it} \\ &= -\frac{i}{6} t (\cos(3t) + i \sin(3t)) \\ &= \frac{1}{6} t (-i \cos(3t) + \sin(3t)), \end{aligned}$$

so

$$x_p := \frac{1}{6} t \sin(3t)$$

is a particular solution to the original ODE. This is not a sinusoid, but an oscillating function whose oscillations grow without bound as time progresses.

13.10.4. *Resonance with damping.* In a realistic physical situation, there is at least a tiny amount of damping, and this prevents the runaway growth of the previous section.

Question 13.17. What happens if $\omega = 3$ exactly, but there is a tiny amount of damping, so that the ODE is

$$\ddot{x} + \underset{\text{damping term}}{b\dot{x}} + 9x = \underset{\text{input signal}}{\cos \omega t}$$

for some small positive constant b ?

New characteristic polynomial: $p(r) = r^2 + br + 9$. Since $3i$ is no longer a root, ERF applies.

$$\text{Complex gain: } G = \frac{1}{p(3i)} = \frac{1}{3bi}.$$

Gain: $|G| = \frac{1}{3b}$. This is large, but the oscillations are bounded; there is a steady-state solution.

Try the “Amplitude and Phase: First Order” mathlet

<http://mathlets.org/mathlets/amplitude-and-phase-1st-order/>

Try the “Amplitude and Phase: Second Order I” mathlet

<http://mathlets.org/mathlets/amplitude-and-phase-2nd-order/>

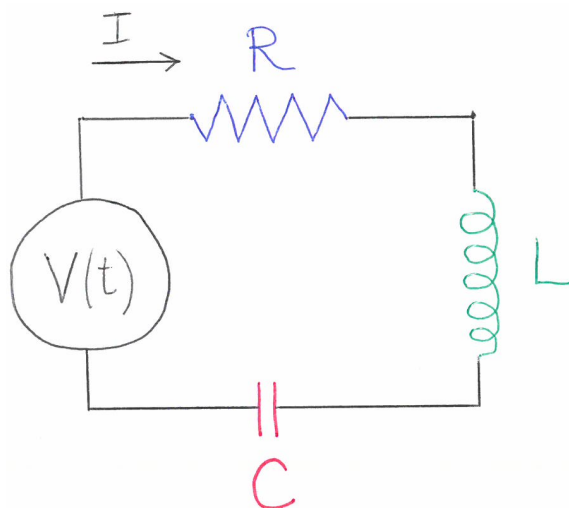
Try the “Amplitude and Phase: Second Order II” mathlet

<http://mathlets.org/mathlets/amplitude-and-phase-2nd-order-ii/>

Try the “Amplitude and Phase: Second Order III” mathlet

<http://mathlets.org/mathlets/amplitude-and-phase-2nd-order-iii/>

13.11. **RLC circuits.** Let’s model a circuit with a voltage source, resistor, inductor, and capacitor attached in series: an **RLC circuit**.



Variables and functions (with SI units):

t : time (s)

R : resistance of the resistor (ohms)

L : inductance of the inductor (henries)

C : capacitance of the capacitor (farads)

Q : charge on the capacitor (coulombs)

I : current (amperes)

V : voltage source (volts)

V_R : voltage drop across the resistor (volts)

V_L : voltage drop across the inductor (volts)

V_C : voltage drop across the capacitor (volts).

The independent variable is t . The quantities R , L , C are constants. Everything else is a function of t .

Equations: Physics says

$$I = \dot{Q}$$

$$V_R = IR \quad \text{Ohm's law}$$

$$V_L = L\dot{I} \quad \text{Faraday's law}$$

$$V_C = \frac{1}{C}Q$$

$$V = V_R + V_L + V_C \quad \text{Kirchhoff's voltage law.}$$

The last equation can be rearranged into

$$V_L + V_R + V_C = V,$$

which becomes

$$L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = V(t),$$

a second-order inhomogeneous linear ODE with unknown function $Q(t)$. Mathematically, this is equivalent to the spring-mass-dashpot ODE

$$m\ddot{x} + b\dot{x} + kx = F_{\text{external}}(t),$$

with the following table of analogies:

Spring-mass-dashpot system		RLC circuit	
displacement	x	Q	charge
velocity	$\dot{x}$	I	current
mass	m	L	inductance
damping constant	b	R	resistance
spring constant	k	$1/C$	1/capacitance
external force	$F_{\text{external}}(t)$	$V(t)$	voltage source

Similarly, a harmonic oscillator (undamped) is analogous to an LC circuit (no resistor).

Remark 13.18. Differentiating the DE involving Q gives a DE involving I :

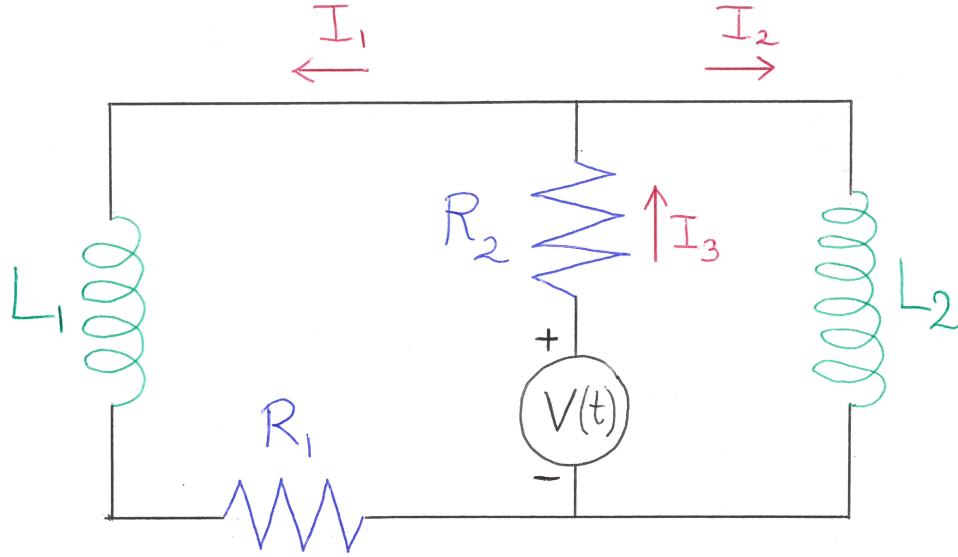
$$L\ddot{I} + R\dot{I} + \frac{1}{C}I = \dot{V}.$$

Try the “Series RLC Circuit” mathlet

<http://mathlets.org/mathlets/series-rlc-circuit/>

14. INTRODUCTION TO LINEAR SYSTEMS OF ODEs

14.1. **Motivation: modeling a two-loop circuit.** Consider a two-loop circuit



in which $R_1 = 8$ ohms, $R_2 = 4$ ohms, $L_1 = 2$ henries, $L_2 = 1$ henry.

- **Kirchhoff's current law** says that at each junction, the current flowing in equals the current flowing out; applying this at the junction at the top gives

$$I_3 = I_1 + I_2$$

(and the bottom junction gives the same).

- **Kirchhoff's voltage law** says that around each loop in the circuit, the sum of the electric potential differences (voltages) is 0:

$$V - R_2 I_3 - L_1 \dot{I}_1 - R_1 I_1 = 0$$

$$V - R_2 I_3 - L_2 \dot{I}_2 = 0.$$

(As one goes around the left loop counterclockwise, the electric potential increases by V as one crosses the voltage source, and then *decreases* as one crosses the resistor R_2 since the current I_3 flows from high potential to low potential, and so on.)

Substitute $I_3 = I_1 + I_2$, and substitute the given values:

$$V - 4(I_1 + I_2) - 2\dot{I}_1 - 8I_1 = 0$$

$$V - 4(I_1 + I_2) - \dot{I}_2 = 0.$$

Isolate $\dot{I}_1$ and $\dot{I}_2$:

$$\dot{I}_1 = -6I_1 - 2I_2 + V/2$$

$$\dot{I}_2 = -4I_1 - 4I_2 + V,$$

where V is a function of t alone.

If the function $V(t)$ is given, this is a system in *two* unknown functions $I_1(t)$ and $I_2(t)$. We will develop methods for solving for $I_1(t)$ and $I_2(t)$.

14.2. Definitions.

Question: Consider the system

$$\dot{x} = 2t^2x + 3y$$

$$\dot{y} = 5x - 7e^ty$$

involving *two* unknown functions, $x(t)$ and $y(t)$. Which of the following describes this system?

Possible answers:

- first-order homogeneous linear system of ODEs
- second-order homogeneous linear system of ODEs
- first-order inhomogeneous linear system of ODEs
- second-order inhomogeneous linear system of ODEs
- first-order homogeneous linear system of PDEs
- second-order homogeneous linear system of PDEs
- first-order inhomogeneous linear system of PDEs
- second-order inhomogeneous linear system of PDEs

Answer: It's a first-order homogeneous linear system of ODEs. The system is **first-order** since it involves only the first derivatives of the unknown functions. This is a **homogeneous linear** system since every summand is a function of t times one of $x, \dot{x}, \dots, y, \dot{y}, \dots$. (If there were also terms that were functions of t , then it would be an **inhomogeneous linear** system.) The equations are **ODEs** since the functions are still functions of only one variable, t . $\square$

14.3. Rewriting a linear system of ODEs in matrix form.

- The homogeneous system in the question can be written in matrix form,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2t^2 & 3 \\ 5 & -7e^t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$
$$\dot{\mathbf{x}} = A(t) \mathbf{x},$$

by defining

$$\begin{array}{c} \mathbf{x} \\ \text{vector-valued function} \end{array} := \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad \begin{array}{c} A(t) \\ \text{matrix-valued function} \end{array} := \begin{pmatrix} 2t^2 & 3 \\ 5 & -7e^t \end{pmatrix}.$$

- Similarly, in the two-loop circuit example, if we define

$$\begin{array}{c} \mathbf{I} \\ \text{vector-valued function} \end{array} := \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}, \quad \text{and} \quad \begin{array}{c} \mathbf{q}(t) \\ \text{vector-valued function} \end{array} := \begin{pmatrix} V(t)/2 \\ V(t) \end{pmatrix}$$

then the inhomogeneous system can be written compactly as

$$\dot{\mathbf{I}} = \begin{pmatrix} -6 & -2 \\ -4 & -4 \end{pmatrix} \mathbf{I} + \mathbf{q}(t).$$

In this example, the matrix-valued function is constant.

14.4. Theory. Before trying to solve such systems, we might want to have some assurance that solutions exist. Fortunately, as in the case of a single linear ODE, they do:

Existence and uniqueness theorem for a linear system of ODEs. *Let $A(t)$ be a matrix-valued function and let $\mathbf{q}(t)$ be a vector-valued function, both continuous on an open interval I . Let $a \in I$, and let $\mathbf{b}$ be a vector. Then there exists a unique solution $\mathbf{x}(t)$ to the system*

$$\dot{\mathbf{x}} = A(t) \mathbf{x} + \mathbf{q}(t)$$

satisfying the initial condition $\mathbf{x}(a) = \mathbf{b}$.

(Of course, the sizes of these matrices and vectors should match in order for this to make sense.)

Remark 14.1. Write $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$. Then the vector initial condition $\mathbf{x}(a) = \mathbf{b}$ amounts to n scalar initial conditions: $x_1(a) = b_1, \dots, x_n(a) = b_n$.

Once the system $\dot{\mathbf{x}} = A(t) \mathbf{x}$ and the starting time a are fixed, there is a bijection (one-to-one correspondence)

$$\{\text{solutions to } \dot{\mathbf{x}} = A(t) \mathbf{x}\} \longleftrightarrow \{\text{possibilities for } \mathbf{b}\}$$

under which each solution $\mathbf{x}(t)$ corresponds to its initial condition vector $\mathbf{b} := \mathbf{x}(a)$ (the existence and uniqueness theorem says that for each $\mathbf{b}$, there is one solution $\mathbf{x}(t)$). Adding solutions corresponds to adding their $\mathbf{b}$ vectors, and scalar multiplication of solutions corresponds to scalar multiplication of their $\mathbf{b}$ vectors too. Therefore the concepts of span, linear

independence, basis, and dimension on the left side correspond to the same concepts on the right side. In particular,

$$\text{dimension of } \{\text{solutions to } \dot{\mathbf{x}} = A(t) \mathbf{x}\} = \text{dimension of } \{\text{possibilities for } \mathbf{b}\}.$$

The latter dimension is n , since $\mathbf{b}$ ranges over all vectors in $\mathbb{R}^n$. Conclusion:

Dimension theorem for a homogeneous linear system of ODEs. *For any first-order homogeneous linear system of n ODEs in n unknown functions*

$$\dot{\mathbf{x}} = A(t) \mathbf{x},$$

the set of solutions is an n -dimensional vector space.

14.5. Converting a second-order ODE to a system of two first-order ODEs.

Problem 14.2. Convert $\ddot{x} + 5\dot{x} + 6x = 0$ to a first-order system of ODEs.

Solution: Introduce a new function variable $y := \dot{x}$. Now try to express the derivatives $\dot{x}$ and $\dot{y}$ in terms of x and y :

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= \ddot{x} = -5\dot{x} - 6x = -6x - 5y.\end{aligned}$$

In matrix form, this is $\dot{\mathbf{x}} = A\mathbf{x}$ with $A = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}$. $\square$

(The matrix $\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}$ arising this way is called the **companion matrix** of the polynomial $r^2 + 5r + 6$.)

Remark 14.3. For constant-coefficient ODEs, the characteristic polynomial of the second-order ODE (scaled, if necessary, to have leading coefficient 1) equals the characteristic polynomial (to be defined soon) of the matrix of the first-order system.

Remark 14.4. Given a 3^{rd} -order ODE with unknown function x , we can convert it to a system of first-order ODEs by introducing $y := \dot{x}$ and $z := \ddot{x}$. In general, we can convert an n^{th} -order ODE to a system of n first-order ODEs.

Remark 14.5. One can also convert *systems* of higher-order ODEs to systems of first-order ODEs. For example, a system of 4 fifth-order ODEs can be converted to a system of 20 first-order ODEs. That's why it's enough to study *first-order* systems.

14.6. Converting a system of two first-order ODEs to a second-order ODE. Conversely, given a system of two first-order ODEs, one can eliminate function variables to find a second-order ODE satisfied by one of the functions.

Problem 14.6. Given that

$$\begin{aligned}\dot{x} &= 2x - y \\ \dot{y} &= 5x + 7y,\end{aligned}$$

find a second-order ODE involving only x .

Solution: Solve for y in the first equation ($y = 2x - \dot{x}$) and substitute into the second:

$$2\dot{x} - \ddot{x} = 5x + 7(2x - \dot{x}).$$

This simplifies to

$$\ddot{x} - 9\dot{x} + 19x = 0. \quad \square$$

Remark 14.7. First-order systems with more than two equations can be converted too, but the conversion is not so easy.

Remark 14.8. In principle, we could solve a first-order linear system of ODEs by first converting it in this way. But usually it is better just to leave it as a system.

15. HOMOGENEOUS LINEAR SYSTEMS OF ODEs

15.1. Guessing solutions. Consider a first-order 2×2 homogeneous linear system of ODEs with *constant* coefficients:

$$\dot{\mathbf{x}} = A\mathbf{x},$$

where A is an 2×2 matrix with constant entries.

The dimension theorem predicts that the space of solutions is 2-dimensional, so if we are clever enough to guess 2 solutions and they turn out to be linearly independent, then we know the general solution!

The solutions to the ODE $\dot{x} = ax$ were the functions ce^{at} . So let's try

$$\mathbf{x} = e^{\lambda t}\mathbf{v},$$

where λ is a number and $\mathbf{v}$ is a nonzero constant vector. (Note: In contrast with ce^{at} , we put the $e^{\lambda t}$ first when writing $e^{\lambda t}\mathbf{v}$ in order to follow the convention of writing the scalar first in a scalar-vector multiplication. Some people nevertheless write $\mathbf{v}e^{\lambda t}$ instead of $e^{\lambda t}\mathbf{v}$; it means the same thing.)

Question 15.1. For which pairs $(\lambda, \mathbf{v})$ consisting of a scalar and a nonzero vector is the vector-valued function $\mathbf{x} = e^{\lambda t}\mathbf{v}$ a solution to the system $\dot{\mathbf{x}} = A\mathbf{x}$?

Solution: Plug it in, to see what has to happen in order for it to be a solution:

$$\lambda e^{\lambda t} \mathbf{v} = A e^{\lambda t} \mathbf{v} \quad (\text{for all } t).$$

Interchanging sides and dividing by $e^{\lambda t}$ (also a reversible operation) shows that this is equivalent to

$$\boxed{A\mathbf{v} = \lambda\mathbf{v}}.$$

15.2. Eigenvalues and eigenvectors. Given A , we face the problem of

- finding all possible λ , and
- for each λ , finding all vectors $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$.

This is called the **eigenvalue-eigenvector problem**.

(Eigen is the German word for “own”, as in “the matrix’s own vectors” — the eigenvectors belong to the matrix.)

Definition 15.2. Suppose that A is an $n \times n$ matrix.

- An **eigenvalue** of A is a scalar λ such that $A\mathbf{v} = \lambda\mathbf{v}$ for some nonzero vector $\mathbf{v}$.
 - An **eigenvector** of A associated to a given λ is a nonzero vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$.
- (**Warning:** Other authors consider $\mathbf{0}$ to be an eigenvector too.)

Try the “Matrix Vector” mathlet

<http://mathlets.org/mathlets/matrix-vector/>

Problem 15.3. Let $A = \begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix}$ and let $\mathbf{v} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$. Is $\mathbf{v}$ an eigenvector of A ?

Solution: The calculation

$$A\mathbf{v} = \begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} = 2\mathbf{v},$$

shows that $\mathbf{v}$ is an eigenvector, and that the associated eigenvalue is 2. $\square$

In order to find eigenvalues and eigenvectors of a matrix, we need a few concepts from linear algebra.

15.3. Identity matrix. Review of 18.02: not covered in lecture.

The **diagonal** of a square matrix consists of the entries along the straight line from the upper left to the lower right:

$$\begin{pmatrix} 4 & 6 & 9 \\ 1 & 7 & 8 \\ 2 & 3 & 5 \end{pmatrix}.$$

The $n \times n$ **identity matrix** is the matrix with ones along the diagonal and zeros elsewhere. For example, the 2×2 identity matrix is

$$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Key property: $I\mathbf{v} = \mathbf{v}$ for any vector $\mathbf{v}$. (Check this yourself in the 2×2 case!)

15.4. Trace. Review of 18.02: not covered in lecture.

Definition 15.4. The **trace** of a square matrix A is the sum of the entries along the diagonal. It is denoted $\text{tr } A$.

Example 15.5. If $A = \begin{pmatrix} 4 & 6 & 9 \\ 1 & 7 & 8 \\ 2 & 3 & 5 \end{pmatrix}$, then $\text{tr } A = 4 + 7 + 5 = 16$.

15.5. Determinant. Review of 18.02: not covered in lecture.

To each square matrix A is associated a number $\det A$ called the **determinant**.

Key property: $A\mathbf{v} = \mathbf{0}$ has a nonzero solution $\mathbf{v}$ if and only if $\det A = 0$.

(Thus the determinant “determines” whether a system of linear equations has a nonzero solution.) In the 2×2 case, the determinant is given by the formula

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Warning: Trace and determinant make sense only for *square* matrices.

15.6. Characteristic polynomial of a square matrix. Review of 18.02: not covered in lecture.

Use λ to denote a scalar-valued variable.

Definition 15.6. The **characteristic polynomial** of an $n \times n$ matrix A is $\det(\lambda I - A)$. This is a degree n polynomial in the variable λ and its leading coefficient is 1, so the polynomial looks like $\lambda^n + \dots$.

The reason for this definition will be clear in the next section when we show how to compute eigenvalues.

(Warning: This is not the same concept as the characteristic polynomial of a constant-coefficient linear ODE, but there is a connection, arising when such a DE is converted to a first-order system of linear ODEs.)

Remark 15.7. We often calculate the characteristic polynomial using $\det(A - \lambda I)$ instead. This turns out to be the same as $\det(\lambda I - A)$, except negated when n is odd. (The reason is that changing the signs of all n rows of the matrix $A - \lambda I$ flips the sign of the determinant n times.) Usually we care only about the *roots* of the polynomial, so negating the whole polynomial doesn’t make a difference. In any case, $\det(A - \lambda I) = \det(\lambda I - A)$ for 2×2 matrices (since 2 is even).

Problem 15.8. What is the characteristic polynomial of $A := \begin{pmatrix} 7 & 2 \\ 3 & 5 \end{pmatrix}$?

Solution: We have

$$\begin{aligned} A - \lambda I &= \begin{pmatrix} 7 & 2 \\ 3 & 5 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 7 - \lambda & 2 \\ 3 & 5 - \lambda \end{pmatrix} \\ \det(A - \lambda I) &= (7 - \lambda)(5 - \lambda) - 2(3) = \boxed{\lambda^2 - 12\lambda + 29}. \end{aligned}$$

Here is a shortcut for 2×2 matrices:

Theorem 15.9. *If A is a 2×2 matrix, then the characteristic polynomial of A is*

$$\lambda^2 - (\operatorname{tr} A)\lambda + (\det A).$$

Proof. Write $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then the characteristic polynomial is

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} \\ &= (a - \lambda)(d - \lambda) - bc \\ &= \lambda^2 - (a + d)\lambda + (ad - bc) \\ &= \lambda^2 - (\operatorname{tr} A)\lambda + (\det A). \end{aligned} \quad \square$$

We can solve Problem 15.8 again, using this shortcut: the matrix $A := \begin{pmatrix} 7 & 2 \\ 3 & 5 \end{pmatrix}$ has $\operatorname{tr} A = 12$ and $\det A = 29$, so the characteristic polynomial of A is $\lambda^2 - 12\lambda + 29$.

Remark 15.10. Suppose that $n > 2$. Then, for an $n \times n$ matrix A , the characteristic polynomial has the form

$$\lambda^n - (\operatorname{tr} A)\lambda^{n-1} + \cdots \pm \det A$$

where the $\pm$ is $+$ if n is even, and $-$ if n is odd. So knowing $\operatorname{tr} A$ and $\det A$ determines *some* of the coefficients of the characteristic polynomial, but not *all* of them.

15.7. Computing all the eigenvalues.

Warm-up problem: Given a square matrix A , how can we test if 5 is an eigenvalue?

Solution: The following are equivalent:

- 5 is an eigenvalue.

- There exists a nonzero solution to

$$A\mathbf{v} = 5\mathbf{v}$$

$$5\mathbf{v} - A\mathbf{v} = \mathbf{0}$$

$$5I\mathbf{v} - A\mathbf{v} = \mathbf{0}$$

$$(5I - A)\mathbf{v} = \mathbf{0}.$$

- $\det(5I - A) = 0$.
- Evaluating the characteristic polynomial $\det(\lambda I - A)$ at 5 gives 0.
- 5 is a root of the characteristic polynomial. $\square$

The same test works for any number in place of 5. (Now that we know how this works, we never again have to go through the argument above.) Conclusion:

eigenvalues = roots of the characteristic polynomial.

Steps to find all the eigenvalues of a square matrix A :

1. Calculate the characteristic polynomial $\det(\lambda I - A)$ or $\det(A - \lambda I)$.
2. The roots of this polynomial are all the eigenvalues of A .

Problem 15.11. Find all the eigenvalues of $A := \begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix}$.

Solution: We have $\text{tr } A = 1 + 0 = 1$ and $\det A = 0 - 2 = -2$, so the characteristic polynomial is

$$p(\lambda) = \lambda^2 - (\text{tr } A)\lambda + (\det A) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1).$$

Its roots are 2 and -1 ; these are the eigenvalues. $\square$

The **multiplicity** of an eigenvalue is just its multiplicity as a root of the characteristic polynomial.

15.8. Computing eigenvectors.

Problem 15.12. Find all the eigenvectors of $A := \begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix}$ associated with the eigenvalue 2.

Solution: By definition, an eigenvector associated to the eigenvalue 2 is a nonzero vector $\mathbf{v} = \begin{pmatrix} v \\ w \end{pmatrix}$ satisfying

$$\begin{aligned} A\mathbf{v} &= 2\mathbf{v} \\ (A - 2I)\mathbf{v} &= \mathbf{0} \\ \left(\begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right) \mathbf{v} &= \mathbf{0} \\ \begin{pmatrix} -1 & -2 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \end{aligned}$$

which is equivalent to

$$-v - 2w = 0.$$

We set w to be any number c , and solve for v to get the general solution $\begin{pmatrix} -2c \\ c \end{pmatrix} = c \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. In other words, the eigenvectors with eigenvalue 2 are all the nonzero scalar multiples of $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$. $\square$

Remark 15.13. In this example, the matrix equation became two copies of the same equation $-v - 2w = 0$. More generally, for any 2×2 matrix A and eigenvalue λ , one of the two equations will be a scalar multiple of the other, so again we need to consider only one of them. In particular, the system of two equations will always have a nonzero solution (as there must be, by definition of eigenvalue).

A similar calculation shows that the eigenvectors of A associated with the eigenvalue -1 are the nonzero scalar multiples of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

To summarize:

Steps to find all the eigenvectors associated to a given eigenvalue λ of a 2×2 matrix A :

1. Calculate $A - \lambda I$.
2. Expand $(A - \lambda I)\mathbf{v} = \mathbf{0}$ using $\mathbf{v} = \begin{pmatrix} v \\ w \end{pmatrix}$; this gives a system of two equations in x and y .
3. Solve the system; one of the equations will be redundant, so nonzero solutions will exist.
4. The nonzero solution vectors $\begin{pmatrix} v \\ w \end{pmatrix}$ are the eigenvectors associated to λ .

Remark 15.14. Let A be a 2×2 matrix.

- If A is $aI = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ for some number a , then the only eigenvalue is a (with multiplicity 2), and every nonzero vector is an eigenvector with eigenvalue a .
- Otherwise, for each eigenvalue λ , the system $(A - \lambda I)\mathbf{v} = 0$ amounts to one nontrivial equation (the other is redundant), so the eigenvectors associated to λ will be the nonzero scalar multiples of a single nonzero vector. In this case, if λ is real, then the set of all real eigenvectors (together with $\mathbf{0}$) is a line through the origin, called the **eigenline** of λ .

March 5

15.9. Solving a 2×2 homogeneous linear system of ODEs with constant coefficients.

Steps to find a basis of solutions to $\dot{\mathbf{x}} = A\mathbf{x}$, given a 2×2 constant matrix A with distinct eigenvalues:

1. Compute the characteristic polynomial $\det(\lambda I - A)$ or $\det(A - \lambda I)$ or $\lambda^2 - (\text{tr } A)\lambda + (\det A)$.
2. Find the roots λ_1 and λ_2 of the characteristic polynomial; these are the eigenvalues.
3. Solve $(A - \lambda_1 I)\mathbf{v} = 0$ to find an eigenvector $\mathbf{v}_1$ associated with λ_1 . (Assuming that λ_1 is not a repeated root, the eigenvectors associated to λ_1 will be just the nonzero scalar multiples of $\mathbf{v}_1$.)
4. Solve $(A - \lambda_2 I)\mathbf{v} = 0$ to find an eigenvector $\mathbf{v}_2$ associated to λ_2 .
5. Then $e^{\lambda_1 t}\mathbf{v}_1$ and $e^{\lambda_2 t}\mathbf{v}_2$ form a basis for the space of solutions. (Under our assumption $\lambda_1 \neq \lambda_2$, these two vector-valued functions are linearly independent.)

The “simple” solutions forming a basis, here of the shape $e^{\lambda t}\mathbf{v}$, are sometimes called **normal modes**. There is not a precise mathematical definition of normal mode, however, since what counts as simple is subjective.

Problem 15.15. Find the solution to

$$\begin{aligned}\dot{x} &= x - 2y \\ \dot{y} &= -x \\ x(0) &= -1 \\ y(0) &= 8.\end{aligned}$$

Solution: This is $\dot{\mathbf{x}} = A\mathbf{x}$ with $A := \begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix}$.

We already found the eigenvalues and eigenvectors of A :

- Eigenvalues: $2, -1$.

- Eigenvector associated to the eigenvalue 2 : $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

- Eigenvector associated to the eigenvalue -1 : $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Basis of the space of solutions: $e^{2t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}, e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

General solution: $\mathbf{x}(t) = c_1 e^{2t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Finally, set $t = 0$ and plug in the initial conditions:

$$\begin{pmatrix} -1 \\ 8 \end{pmatrix} = c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This is a system of linear equations

$$-2c_1 + c_2 = -1$$

$$c_1 + c_2 = 8.$$

Solving it gives $c_1 = 3$ and $c_2 = 5$. Putting these values back into the general solution gives

$$\boxed{\mathbf{x}(t) = 3e^{2t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + 5e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}.$$

In other words,

$$x(t) = -6e^{2t} + 5e^{-t},$$

$$y(t) = 3e^{2t} + 5e^{-t}.$$

Since there were many opportunities to make errors, it would be wise to check the answer by verifying that these functions satisfy the original DEs and initial condition:

$$\dot{x} = -12e^{2t} - 5e^{-t} = x - 2y$$

$$\dot{y} = 6e^{2t} - 5e^{-t} = -x$$

$$x(0) = -6 + 5 = -1$$

$$y(0) = 3 + 5 = 8. \quad \text{😊}$$

Remark 15.16. The system of linear equations involving c_1 and c_2 could have been written in matrix form:

$$\begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 8 \end{pmatrix}.$$

This point of view will be helpful for more complicated systems.

15.10. Complex eigenvalues.

Suppose that A is a real 2×2 matrix whose eigenvalues are not real.

- If λ is one of the eigenvalues, the other is $\bar{\lambda}$.
- If $\mathbf{v}$ is an eigenvector associated to λ , then $\bar{\mathbf{v}}$ is an eigenvector associated to $\bar{\lambda}$.
- $e^{\lambda t} \mathbf{v}, e^{\bar{\lambda} t} \bar{\mathbf{v}}$ is a basis for the space of solutions.
- $\operatorname{Re}(e^{\lambda t} \mathbf{v}), \operatorname{Im}(e^{\lambda t} \mathbf{v})$ is a basis consisting of *real* vector-valued functions.

15.11. Phase plane.

Question 15.17. Let A be a constant 2×2 real matrix. How can you visualize a real-valued solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ to $\dot{\mathbf{x}} = A\mathbf{x}$?

Answer: Make two plots, the first showing $x(t)$ as a function of t , and the second showing $y(t)$ as a function of t .

Better answer: Draw the solution as a *parametrized curve* $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ in the **phase plane** with axes x and y . In other words, plot the point $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ for every real number t (including negative t). The ODE specifies, in terms of the current position, which direction the phase plane point will move next (and how fast).

Question 15.18. Suppose that λ is a real eigenvalue of A , and $\mathbf{v}$ is a real eigenvector associated to λ . Then $e^{\lambda t} \mathbf{v}$ is a solution to $\dot{\mathbf{x}} = A\mathbf{x}$. (It is *the* solution satisfying $\mathbf{x}(0) = \mathbf{v}$.) Evaluating $e^{\lambda t} \mathbf{v}$ at any time t gives a positive scalar multiple of $\mathbf{v}$, so the trajectory is contained in the ray through $\mathbf{v}$. What is the direction of the trajectory?

Two approaches:

1. Consider the length and direction of $e^{\lambda t} \mathbf{v}$ as t changes.
2. Use the ODE itself to get the velocity vector at each point.

Answers:

- If $\lambda > 0$, the phase point tends to infinity (repelled from $(0, 0)$).
- If $\lambda < 0$, the phase point tends to $(0, 0)$ (attracted to $(0, 0)$).
- If $\lambda = 0$, the phase point is stationary at $\mathbf{v}$! The point $\mathbf{v}$ is called an **critical point** since $\dot{\mathbf{x}} = \mathbf{0}$ there. $\square$

Question: One of the solutions to $\dot{\mathbf{x}} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \mathbf{x}$ is

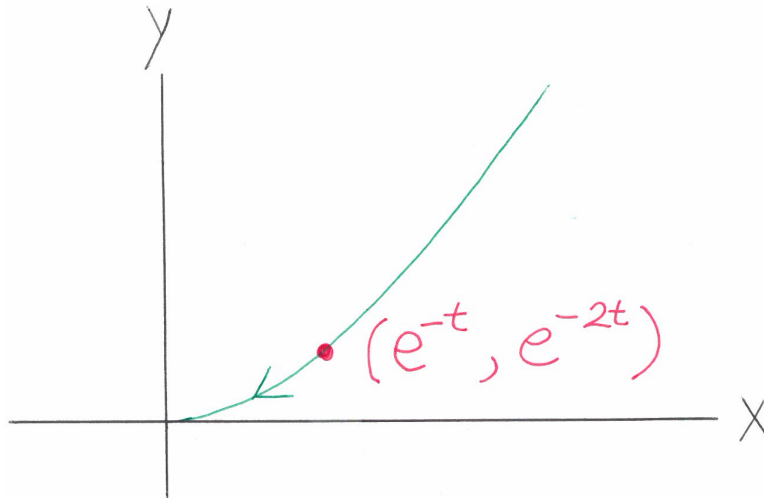
$$\begin{pmatrix} x \\ y \end{pmatrix} = e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{-2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{-t} \\ e^{-2t} \end{pmatrix}.$$

Which of the following describes the motion in the phase plane with axes x and y as $t \rightarrow +\infty$?

Possible answers:

- approaching infinity, along a curve asymptotic to the x -axis
- approaching infinity, along a curve asymptotic to the y -axis
- approaching infinity, along a straight line
- approaching the origin, along a curve tangent to the x -axis
- approaching the origin, along a curve tangent to the y -axis
- approaching the origin, along a straight line
- spiraling
- none of the above

Answer: Approaching the origin, along a curve tangent to the x -axis. As $t \rightarrow +\infty$, both $x = e^{-t}$ and $y = e^{-2t}$ tend to 0, but the y -coordinate tends to 0 faster than the x -coordinate, so the trajectory is tangent to the x -axis. (In fact, the y -coordinate is always the square of the x -coordinate, so the trajectory is part of the parabola $y = x^2$.) $\square$



The phase plane trajectory by itself does not describe a solution fully, since it does not show at what time each point is reached. The trajectory contains no information about speed, though one can specify the direction by drawing an arrow on the trajectory.

The **phase portrait** (or **phase diagram**) is the diagram showing *all* the trajectories in the phase plane. Every point in the plane belongs to exactly one trajectory.

We are now ready to classify all possibilities for the phase portrait in terms of the eigenvalue behavior. The most common cases are introduced with **green text**; the others are degenerate cases.

Try the “Linear Phase Portraits: Matrix Entry” mathlet

<http://mathlets.org/mathlets/linear-phase-portraits-matrix-entry/>

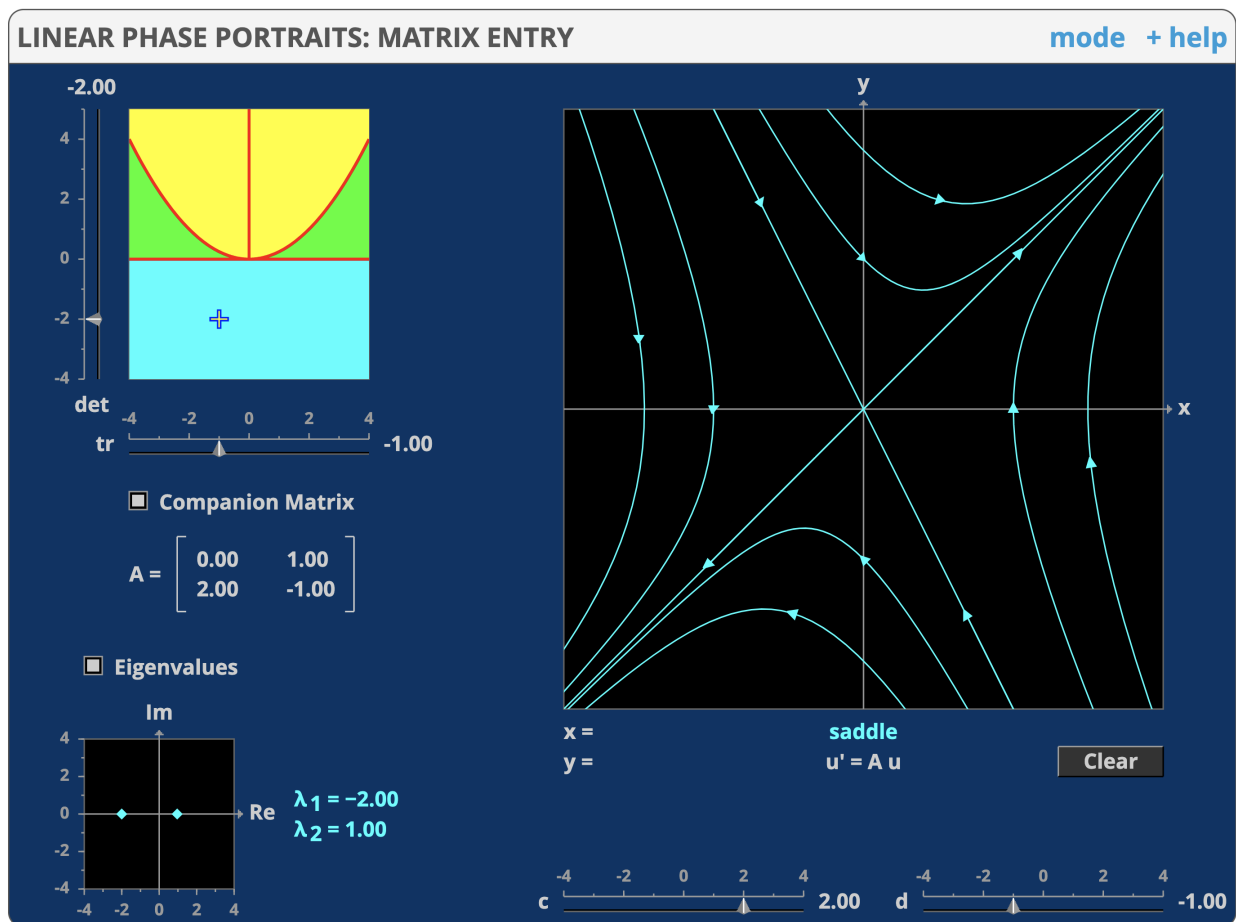
See if you can get every possibility listed below.

15.11.1. *Distinct real eigenvalues.* Suppose that the eigenvalues λ_1, λ_2 are real and distinct.

Let $\mathbf{v}_1, \mathbf{v}_2$ be corresponding eigenvectors. The set of all eigenvectors associated to λ_1 consists of all scalar multiples of $\mathbf{v}_1$; these form the eigenline of λ_1 . General solution:

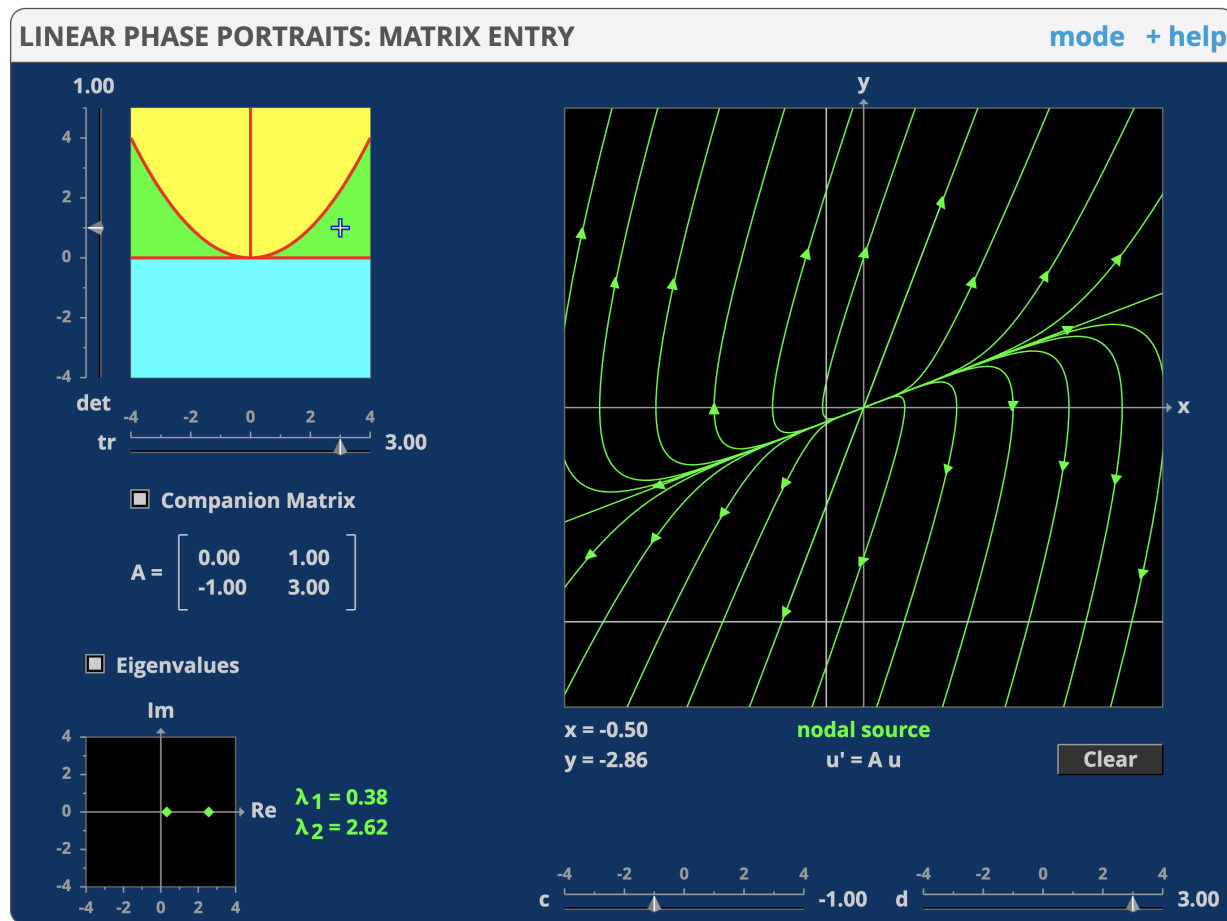
$$c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2.$$

Opposite sign: $\lambda_1 > 0, \lambda_2 < 0$. This is called a **saddle**. Trajectories flow outward along the positive eigenline (the eigenline of λ_1) and inward along the negative eigenline (the eigenline of λ_2). Other trajectories are asymptotic to both eigenlines, tending to infinity towards the positive eigenline. (Typical solution: $\mathbf{x} = e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-3t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. When $t = +1000$, this is approximately a large positive multiple of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. When $t = -1000$, this is approximately a large positive multiple of $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.)

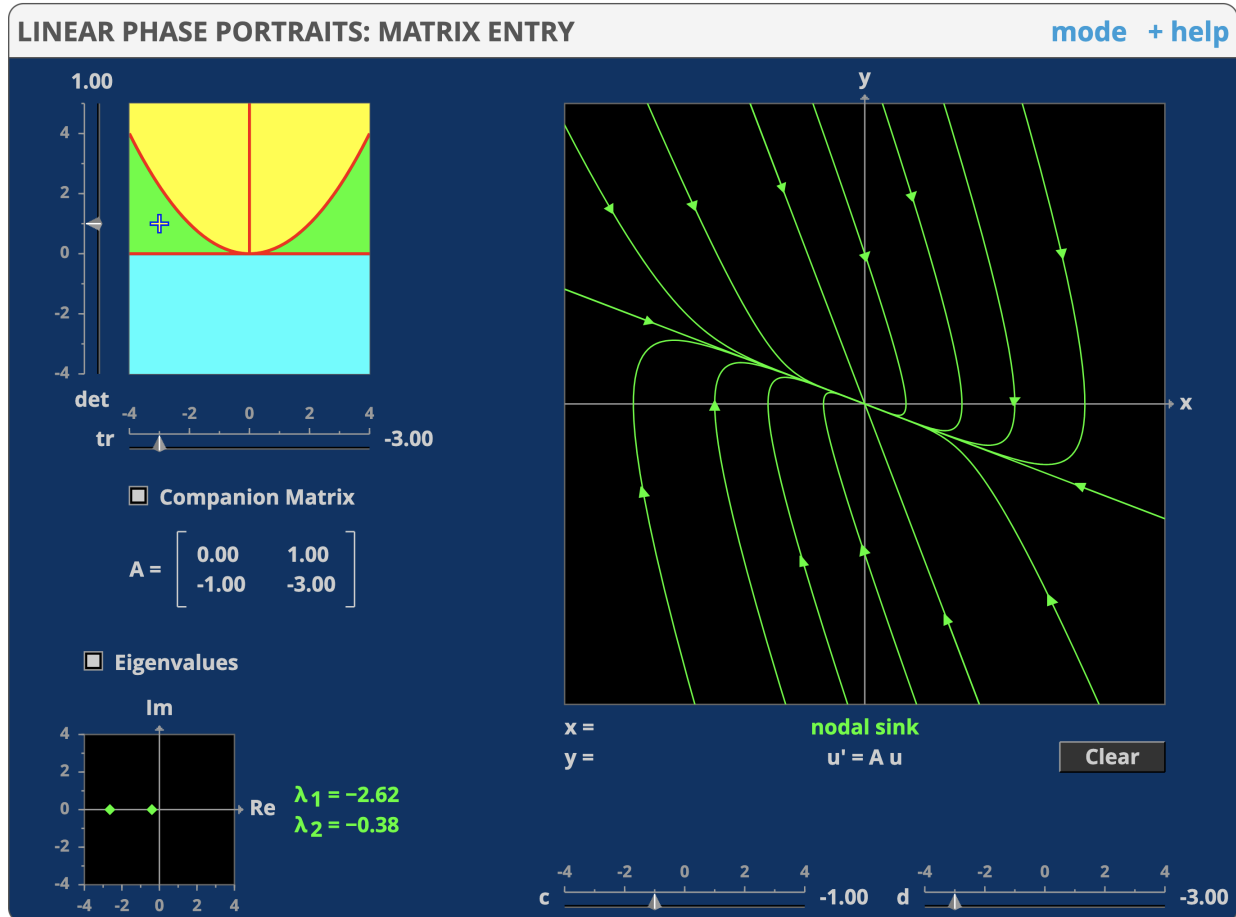


In the next two cases, in which the eigenvalues have the same sign, we'll want to know which is bigger. If $|\lambda_1| > |\lambda_2|$, call λ_1 the **fast** eigenvalue and λ_2 the **slow** eigenvalue; use the same adjectives for the eigenlines.

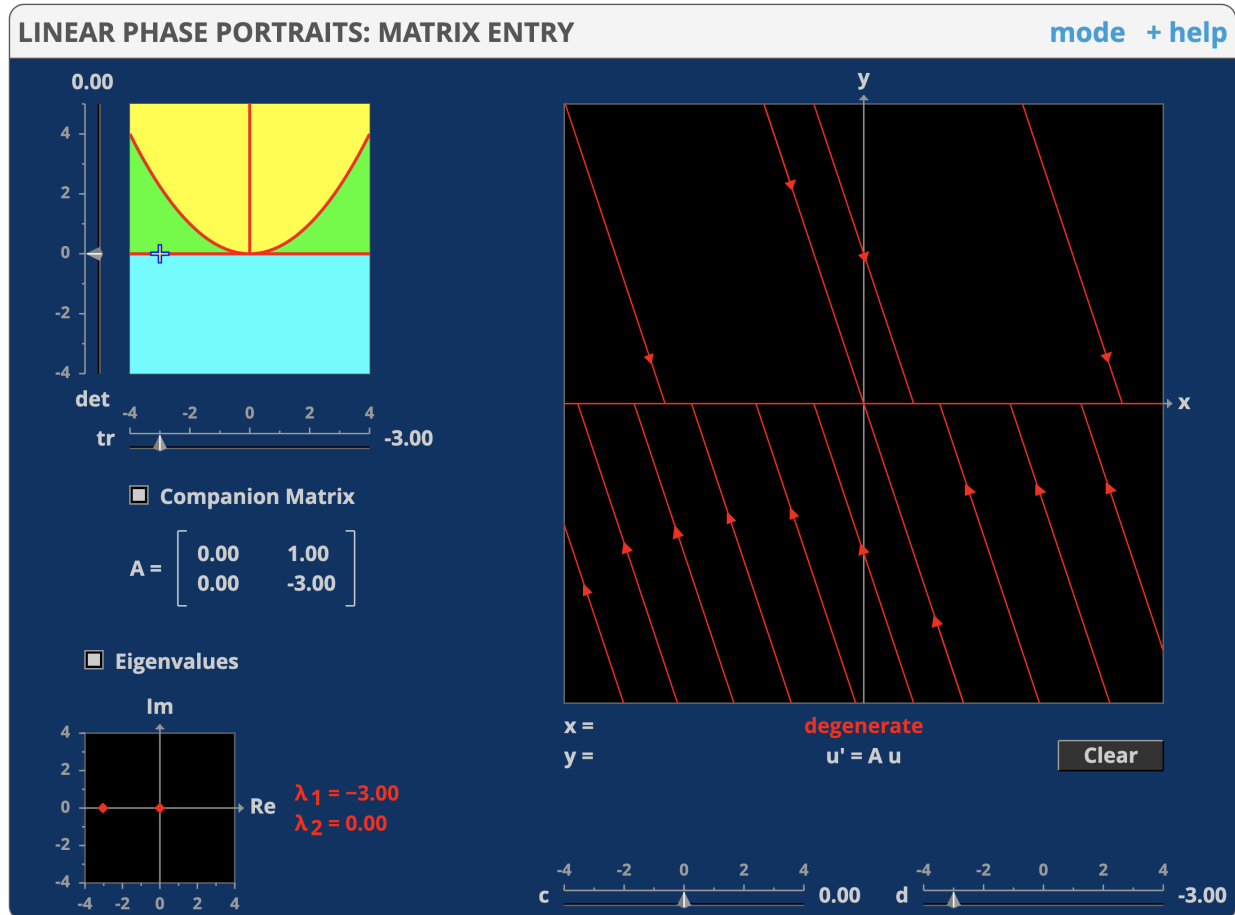
Both positive: $\lambda_1, \lambda_2 > 0$. This is called a **repelling node** (or **node source**). All nonzero trajectories flow from $(0,0)$ towards infinity. Trajectories not contained in the eigenlines are tangent to the slow eigenline at $(0,0)$, and far from $(0,0)$ have direction approximately parallel to the fast eigenline.



Both negative: $\lambda_1, \lambda_2 < 0$. This is called an **attracting node** (or **node sink**). All nonzero trajectories flow from infinity towards $(0,0)$. Trajectories not contained in the eigenlines are tangent to the slow eigenline at $(0,0)$, and far from $(0,0)$ have direction approximately parallel to the fast eigenline.



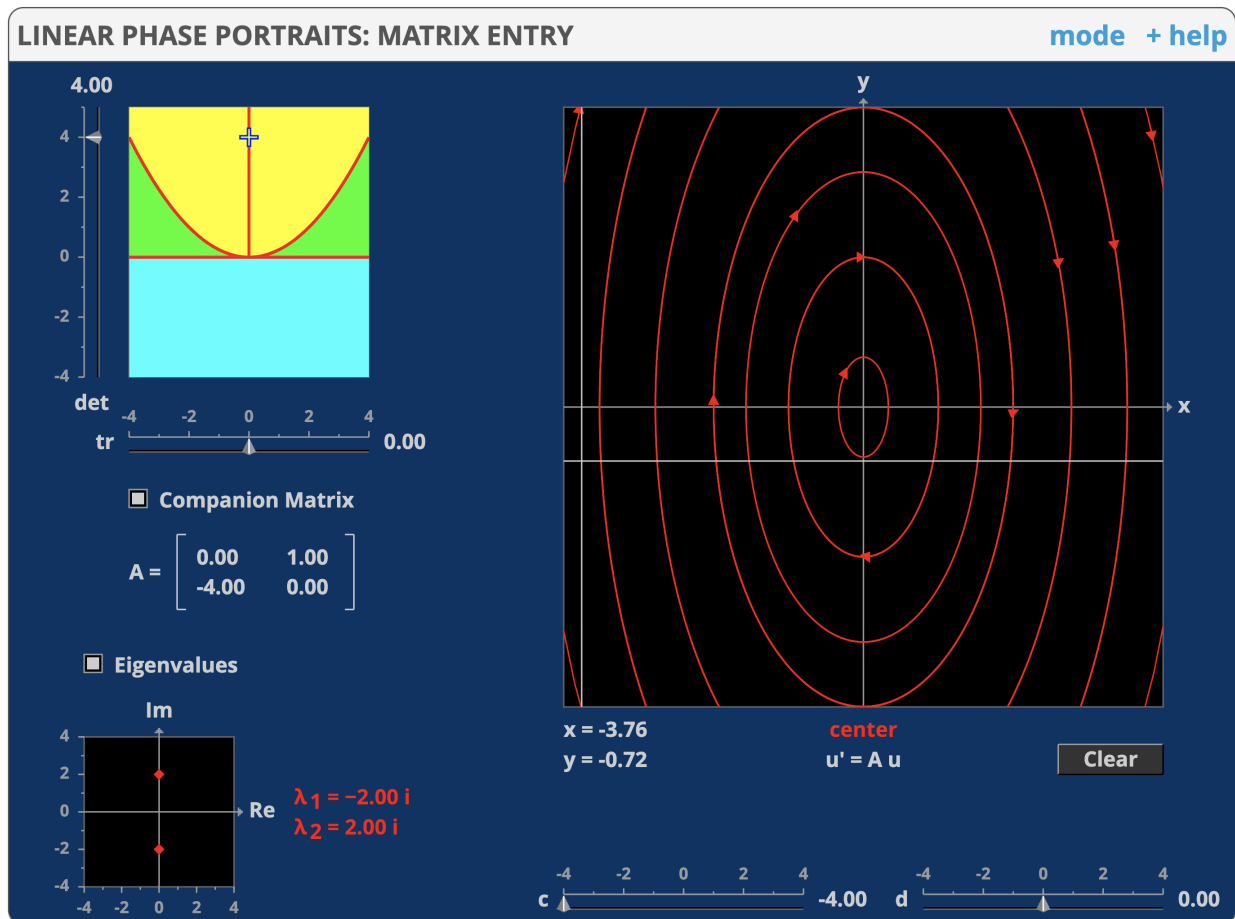
One eigenvalue is zero: $\lambda_1 \neq 0, \lambda_2 = 0$. General solution: $c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 \mathbf{v}_2$. This is a degenerate case called a **comb**. It could also be described by the words **nonisolated critical points**, since every point on the 0 eigenline is stationary. Other trajectories are along lines parallel to the other eigenline, tending to infinity if $\lambda_1 > 0$, and approaching the 0 eigenline if $\lambda_1 < 0$ (the case shown below).



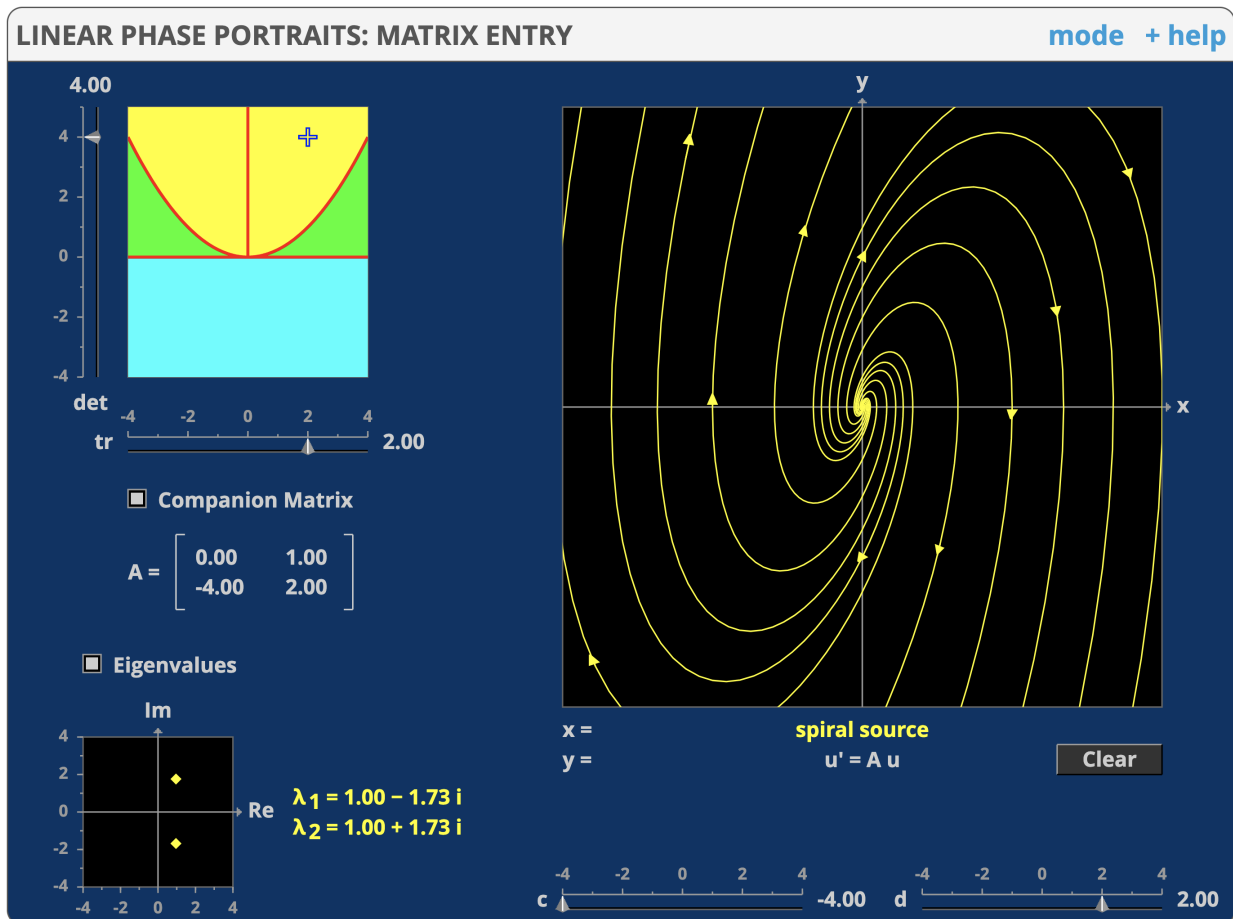
15.11.2. *Complex eigenvalues.* Suppose that the eigenvalues λ_1, λ_2 are not real. Then they are $a \pm bi$ for some real numbers a, b . In $e^{(a+bi)t}$, the number a controls repulsion/attraction, while b controls rotation (angular frequency).

Zero real part: $a = 0$. This is called a **center**. The nonzero trajectories are concentric ellipses. Solutions are periodic with period $2\pi/b$.

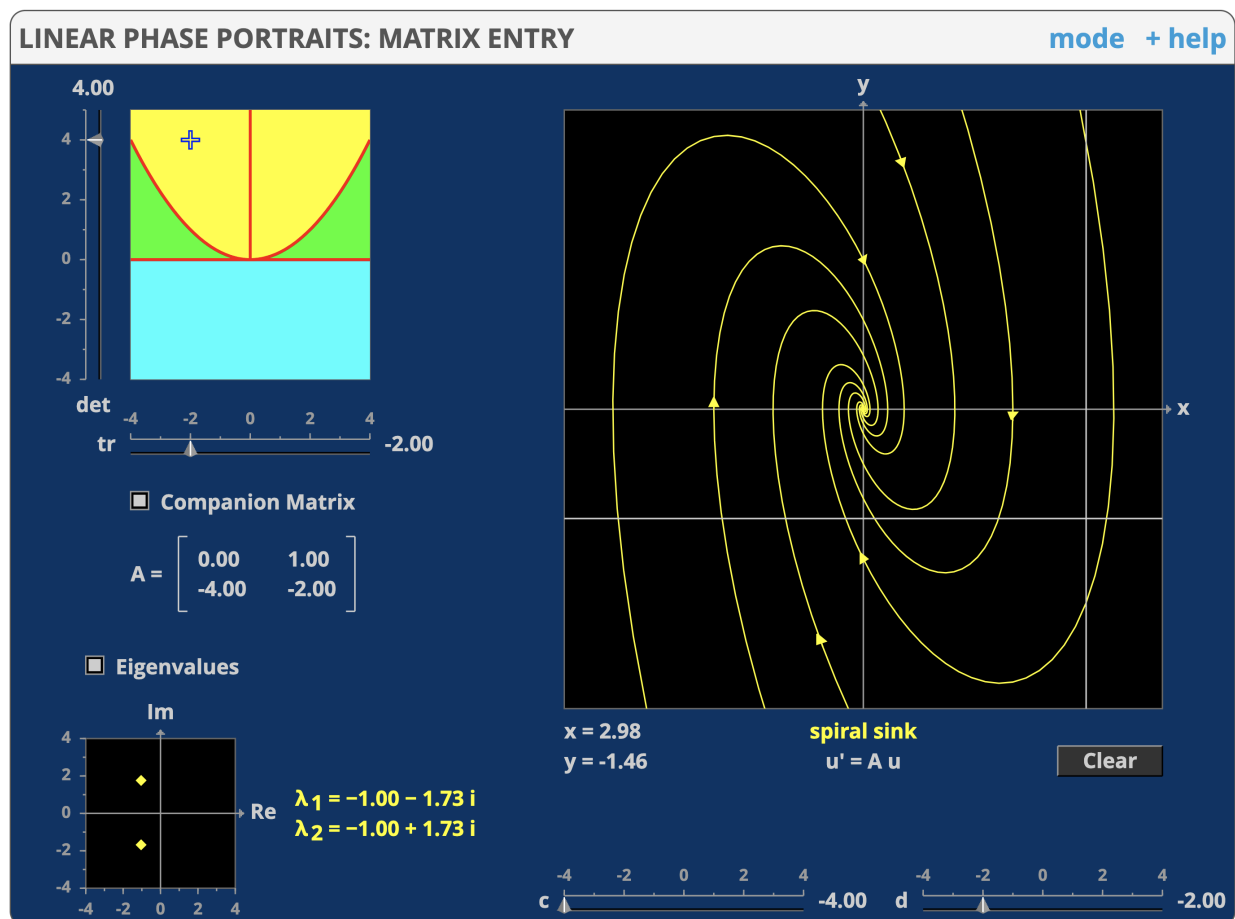
(Typical solution: $\mathbf{x} = \operatorname{Re} \left[e^{it} \begin{pmatrix} 2 \\ -i \end{pmatrix} \right] = \begin{pmatrix} 2 \cos t \\ \sin t \end{pmatrix}$, a parametrization of a wide ellipse.)



Positive real part: $a > 0$. This is called a **repelling spiral** (or **spiral source**). All nonzero trajectories spiral outward.



Negative real part: $a < 0$. This is called an **attracting spiral** (or **spiral sink**). All nonzero trajectories spiral inward.



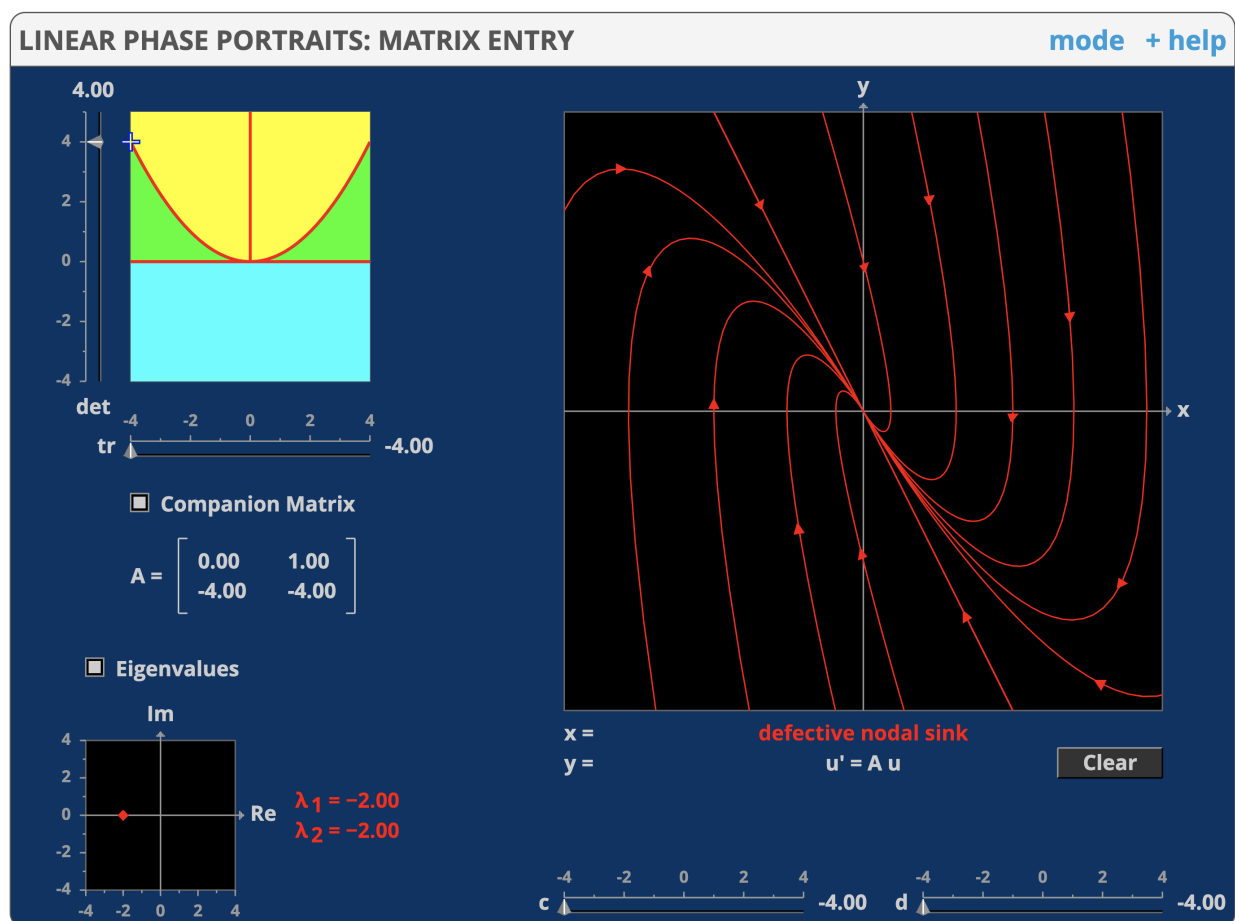
In these spiraling or rotating cases, how can one determine whether trajectories go clockwise or counterclockwise? It's complicated to see this in terms of eigenvalues and eigenvectors, but easy to see by testing a single velocity vector.

Problem 15.19. The phase portrait for $\dot{\mathbf{x}} = A\mathbf{x}$ with $A := \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$ is a center. Do the trajectories go clockwise or counterclockwise?

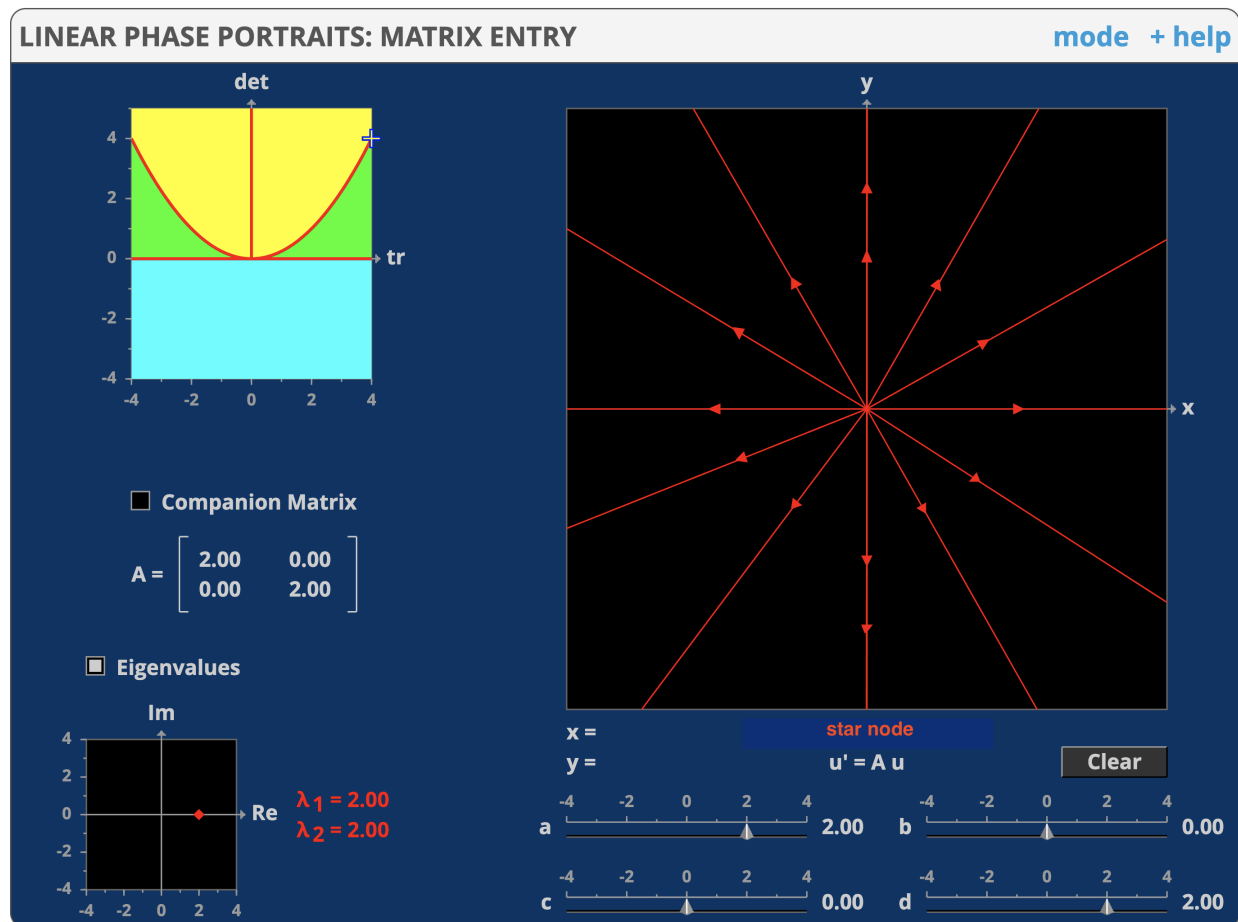
Solution: The velocity vector at $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is $\dot{\mathbf{x}} = A\mathbf{x} = A \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, which is the first column of A , namely $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Since the velocity vector at $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ has negative y -coordinate, the trajectory is going clockwise.

15.11.3. *Repeated real eigenvalue.* Suppose that there is a repeated real eigenvalue, say λ . The eigenspace of λ (the set of all eigenvectors associated to λ) could be either 1-dimensional or 2-dimensional.

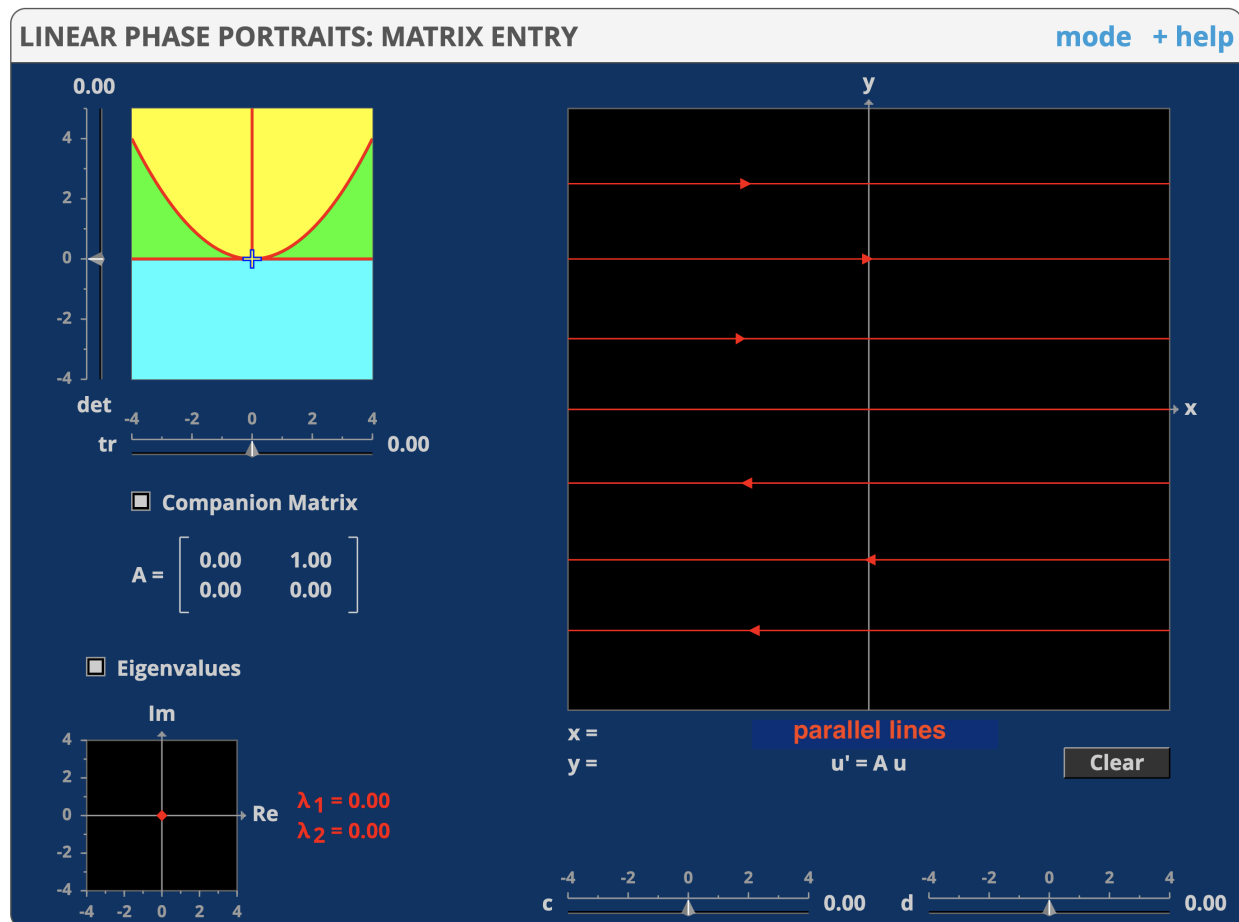
$\lambda \neq 0$ and $A \neq \lambda I$ (1-dimensional eigenspace): This is called a **degenerate node** (or **improper node** or **defective node**). There is just one eigenline. It serves as both the slow eigenline and the fast eigenline: every trajectory not contained in it is tangent to it at $(0,0)$, and approximately parallel to it when far from $(0,0)$. Such trajectories are repelled from $(0,0)$ if $\lambda > 0$, and attracted to $(0,0)$ if $\lambda < 0$. (This is a borderline case between a node and a spiral.)



$\lambda \neq 0$ and $A = \lambda I$ (2-dimensional eigenspace): This is called a **star node**. Every vector is an eigenvector. Nonzero trajectories are along rays, repelled from $(0,0)$ if $\lambda > 0$ (the case shown below), and attracted to $(0,0)$ if $\lambda < 0$.



$\lambda = 0$ and $A \neq 0$ (1-dimensional eigenspace): This could be called **parallel lines**. Points on the eigenline are stationary. All other trajectories are lines parallel to the eigenline.



$\lambda = 0$ and $A = 0$ (2-dimensional eigenspace): This could be called **stationary**. Every point is stationary. (No diagram included, since there isn't much to show!)

15.11.4. *Summary.* Although all of the above may be needed for homework problems, for exams you are expected to know only the main cases listed in green above and also the case of a center, not the other “borderline” cases.

Steps to sketch a phase portrait of $\dot{\mathbf{x}} = A\mathbf{x}$ (when A has distinct nonzero eigenvalues):

1. Find the eigenvalues of A .
2. If the eigenvalues are distinct real numbers ($(\text{tr } A)^2 - 4\det A > 0$) and are nonzero, find and draw the two eigenlines, and indicate the direction of motion along each (repelling/attracting according to eigenvalue being $+/ -$).
 - If opposite sign, **saddle**. Other trajectories are asymptotic to both eigenlines, in the direction matching that of the nearby eigenline.
 - If same sign, then **repelling/attracting node**. Other trajectories are tangent to the slow eigenline at $(0, 0)$.
3. If the eigenvalues are complex, say $a \pm bi$, check the sign of a :
 - If $+$, **repelling spiral**.
 - If $-$, **attracting spiral**.
 - If 0 , center.

To determine whether it is clockwise or counterclockwise, choose a starting vector $\mathbf{x}(0)$ and compute the velocity vector $\dot{\mathbf{x}}(0)$ there as $A\mathbf{x}(0)$ to see which direction the particle will move next.

Problem 15.20. Sketch the phase portrait of the system $\dot{\mathbf{x}} = \begin{pmatrix} -5 & -2 \\ -1 & -4 \end{pmatrix} \mathbf{x}$.

Solution: Call the matrix A . Then $\text{tr } A = -9$ and $\det A = 20 - 2 = 18$, so the characteristic polynomial is $\lambda^2 + 9\lambda + 18 = (\lambda + 6)(\lambda + 3)$. The eigenvalues are the roots, which are -6 and -3 .

Eigenvectors of -6 : These are the solutions to

$$(A - (-6)I)\mathbf{v} = \mathbf{0}$$

$$\begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which is the linear system

$$\begin{aligned} v - 2w &= 0 \\ -v + 2w &= 0. \end{aligned}$$

Here w can be any number c , and then $v = 2c$, so

$$\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 2c \\ c \end{pmatrix} = c \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Thus the eigenline of -6 is the line through the origin in the direction of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Eigenvectors of -3 : These are the solutions to

$$(A - (-3)I)\mathbf{v} = \mathbf{0}$$

$$\begin{pmatrix} -2 & -2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which is the linear system

$$\begin{aligned} -2v - 2w &= 0 \\ -v - w &= 0. \end{aligned}$$

Here w can be any number c , and then $v = -c$, so

$$\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} -c \\ c \end{pmatrix} = c \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Thus the eigenline of -3 is the line through the origin in the direction of $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. This is the slow eigenline, since $|-3| < |-6|$.

The eigenvalues are distinct real numbers, and they are negative, so the phase portrait is an attracting node. The trajectories along the eigenlines tend to $(0,0)$ as $t \rightarrow +\infty$ because the eigenvalues are negative. All other trajectories tend to $(0,0)$ too, and are tangent at $(0,0)$ to the slow eigenline (the line in the direction of $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$).

Problem 15.21. Sketch the phase portrait of the system $\dot{\mathbf{x}} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x}$.

Solution: Call the matrix A , so $A = 2I$. Every vector $\mathbf{v}$ satisfies $A\mathbf{v} = 2I\mathbf{v} = 2\mathbf{v}$, so every nonzero vector is an eigenvector associated with the eigenvalue 2. At every position in the phase plane, the system $\dot{\mathbf{x}} = A\mathbf{x}$ says that the velocity vector is 2 times the position vector, so every trajectory moves out radially along a ray. This phase portrait is called a star node (a degenerate case).

(If instead A were $-2I$, then every trajectory would tend to $(0,0)$ along a ray.)

Problem 15.22. Bonus problem: not done in lecture. Sketch the phase portrait of the system $\dot{\mathbf{x}} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \mathbf{x}$.

Solution: Call the matrix A . Then $\text{tr } A = 4$ and $\det A = 4$. Characteristic polynomial: $\lambda^2 - 4\lambda + 4$. Eigenvalues: 2, 2.

Eigenvectors of 2: These are the solutions to

$$(A - 2I)\mathbf{v} = \mathbf{0}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which is the linear system

$$w = 0$$

$$0 = 0.$$

Here v can be any number c , but $w = 0$, so

$$\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} c \\ 0 \end{pmatrix} = c \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Thus there is only one eigenline, and it is horizontal. Trajectories inside this line (other than the one that sits at $\mathbf{0}$) tend to infinity along the line, since the eigenvalue is positive. All other trajectories are tangent to the eigenline at $(0, 0)$, and tend to infinity as $t \rightarrow +\infty$ while becoming approximately parallel to the eigenline. This phase portrait is called a degenerate node.

15.12. Trace-determinant plane. The type of phase portrait is determined by the eigenvalues λ_1, λ_2 (except in the case of a repeated eigenvalue, when one needs to know whether A is a scalar times I). And the eigenvalues are determined by the characteristic polynomial

$$\det(\lambda I - A) = \lambda^2 - (\text{tr } A)\lambda + (\det A) = (\lambda - \lambda_1)(\lambda - \lambda_2).$$

(Comparing coefficients shows that $\text{tr } A = \lambda_1 + \lambda_2$ and $\det A = \lambda_1 \lambda_2$.)

Therefore the classification of phase portraits can be re-expressed in terms of $\text{tr } A$ and $\det A$. First, by the quadratic formula, the number of real eigenvalues is determined by the sign of the discriminant $(\text{tr } A)^2 - 4 \det A$.

15.12.1. Distinct real eigenvalues. Suppose that $(\text{tr } A)^2 - 4 \det A > 0$.
Then the eigenvalues are real and distinct.

- If $\det A < 0$, then $\lambda_1 \lambda_2 < 0$, so the eigenvalues have opposite sign: saddle.
- If $\det A > 0$, then the eigenvalues have the same sign.
 - If $\text{tr } A > 0$, repelling node.
 - If $\text{tr } A < 0$, attracting node.
- If $\det A = 0$, then one eigenvalue is 0; comb.

15.12.2. *Complex eigenvalues.* Suppose that $(\operatorname{tr} A)^2 - 4 \det A < 0$.

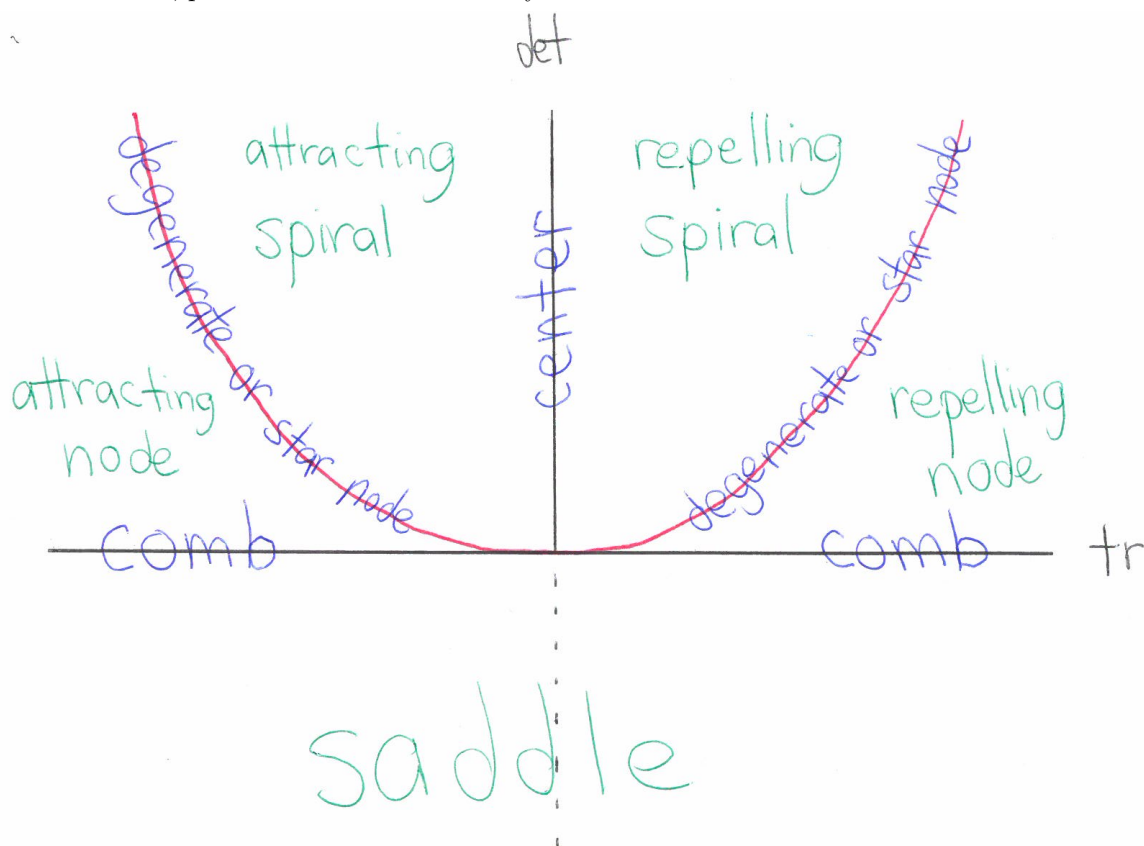
Then the eigenvalues are $a \pm bi$, and their sum is $\operatorname{tr} A = 2a$.

- If $\operatorname{tr} A = 0$, center.
- If $\operatorname{tr} A > 0$, repelling spiral.
- If $\operatorname{tr} A < 0$, attracting spiral.

15.12.3. *Repeated real eigenvalues.* Suppose that $(\operatorname{tr} A)^2 - 4 \det A = 0$.

Then we get a repeated real eigenvalue λ, λ , and $\operatorname{tr} A = 2\lambda$.

- If $\operatorname{tr} A \neq 0$, degenerate node or star node.
- If $\operatorname{tr} A = 0$, parallel lines or the stationary case.



The **trace-determinant plane** is the plane with axes tr and $\det$. This is completely different from the phase plane (because the axes are different).

Whereas the phase portrait shows all possible trajectories for a system $\dot{\mathbf{x}} = A\mathbf{x}$, the trace-determinant plane has just one point for the system. The position of that point contains information about the kind of phase portrait.

Above the parabola $\det = \frac{1}{4} \operatorname{tr}^2$, the eigenvalues are complex. Below the parabola, the eigenvalues are real and distinct.

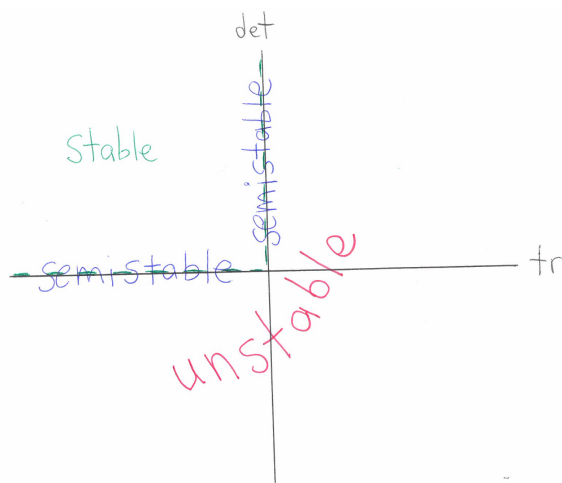
15.13. **Stability.** Consider a system $\dot{\mathbf{x}} = A\mathbf{x}$.

- If all trajectories tend to $\mathbf{0}$ as $t \rightarrow +\infty$, the system is called **stable**.
- If some trajectories are unbounded as $t \rightarrow +\infty$, then the system is called **unstable**.
- In the borderline case in which all solutions are bounded, but do not all tend to $\mathbf{0}$, the system is called **semistable** or **neutrally stable**. Example: a center.

The tests for stability are the same as for a single higher-order ODE, in terms of the roots or coefficients of the characteristic polynomial:

$$\begin{aligned} \text{stable} &\iff \text{each } e^{\lambda t} \text{ appearing in the general solution tends to } 0 \\ &\iff \text{all eigenvalues have negative real part} \\ &\iff \text{the characteristic polynomial has positive coefficients} \\ &\iff \text{tr } A < 0 \text{ and } \det A > 0. \end{aligned}$$

(The green tests are for the 2×2 case only.)



15.14. **Structural stability.** Stability is a question of what happens to solutions of a fixed system of ODEs. What happens if the system of ODEs itself is changed, by changing the matrix A ? There is a new definition to describe this:

Definition 15.23. If the phase portrait type is robust in the sense that small perturbations in the entries of A cannot change the type of the phase portrait, then the system is called **structurally stable**.

Warning: A system $\dot{\mathbf{x}} = A\mathbf{x}$ can be structurally stable without being stable, and can be stable without being structurally stable. It is unfortunate that the two concepts have similar names, since they are independent of each other.

The structurally stable cases are those corresponding to the large regions in the trace-determinant plane, not the borderline cases. For a 2×2 matrix A , the system $\dot{\mathbf{x}} = A\mathbf{x}$ is structurally stable if and only if A has either

- distinct nonzero real eigenvalues (saddle, repelling node, or attracting node), or
- complex eigenvalues with nonzero real part (spiral).

15.15. Energy conservation and energy loss.

15.15.1. *Conservation of energy in the harmonic oscillator.* Consider the harmonic oscillator described by $m\ddot{x} + kx = 0$. Let's check conservation of energy.

Kinetic energy: $\text{KE} = \frac{m\dot{x}^2}{2}$.

Potential energy PE is a function of x , and

$$\underbrace{\text{PE}(x) - \text{PE}(0)}_{\text{change in PE}} = - \underbrace{\int_0^x F_{\text{spring}}(X) dX}_{\text{work done by } F_{\text{spring}}} = - \int_0^x -kX dX = \frac{kx^2}{2}.$$

If we declare $\text{PE} = 0$ at position 0, then $\text{PE}(x) = \frac{kx^2}{2}$.

Total energy:

$$E = \text{KE} + \text{PE} = \frac{m\dot{x}^2}{2} + \frac{kx^2}{2}.$$

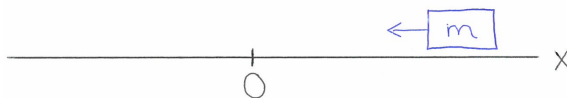
How does total energy change with time?

$$\dot{E} = m\dot{x}\ddot{x} + kx\dot{x} = \dot{x}(m\ddot{x} + kx) = 0.$$

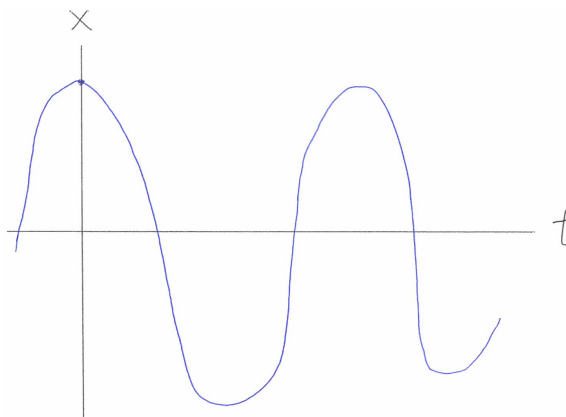
So energy is conserved.

15.15.2. *Phase plane depiction of the harmonic oscillator.* Three ways to depict the harmonic oscillator:

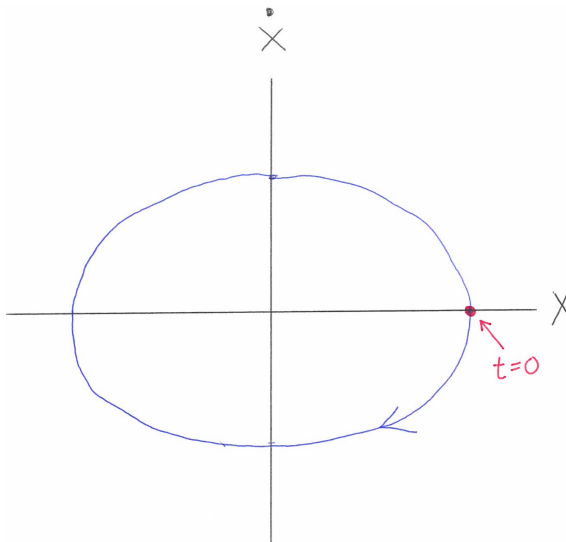
- Movie showing the motion of the mass directly:



- Graph of x as a function of t :



- Trajectory of $\begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix}$ (a parametrized curve) in the phase plane whose horizontal axis shows x and whose vertical axis shows $\dot{x}$ (the phase plane after converting the second-order ODE to a 2×2 system):



Warning: Do not confuse this trajectory with motion in physical space! The cart is not moving along this trajectory. Instead, each point on the trajectory records the position and velocity at a given time.

Let's start with the mass to the right of equilibrium, and then let go. At $t = 0$, we have $x > 0$ and $\dot{x} = 0$. At the first time the mass crosses equilibrium, $x = 0$ and $\dot{x} < 0$. When the mass reaches its leftmost point, $x < 0$ and $\dot{x} = 0$ again. These give three points on the phase plane trajectory.

Here are two ways to see that the whole trajectory is an ellipse:

1. We have $x = A \cos \omega t$ for some A and ω . Thus $\dot{x} = -A\omega \sin \omega t$. The parametrized curve

$$\begin{pmatrix} A \cos \omega t \\ -A\omega \sin \omega t \end{pmatrix}$$

is an ellipse. (This is like the parametrization of a circle, but with axes stretched by different amounts.)

2. Rearrange $E = \frac{m\dot{x}^2}{2} + \frac{kx^2}{2}$ as

$$\frac{x^2}{2E/k} + \frac{\dot{x}^2}{2E/m} = 1.$$

This is an ellipse with semi-axes $\sqrt{2E/k}$ and $\sqrt{2Em}$.

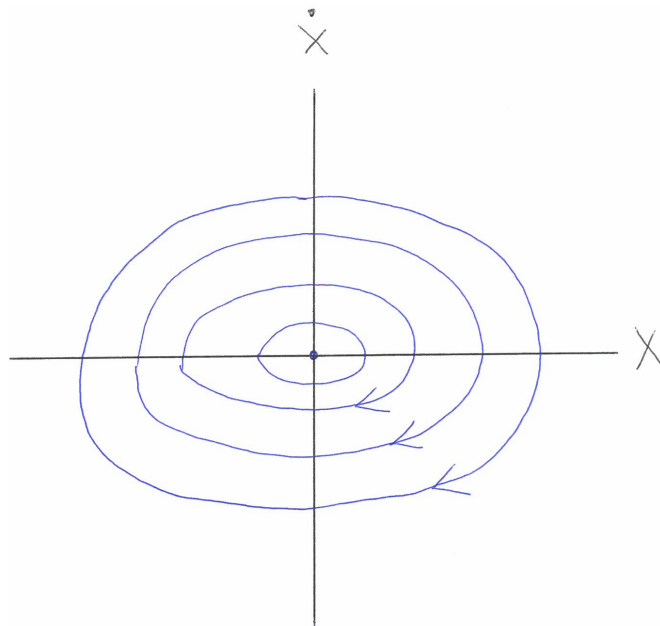
Question: In which direction is the ellipse traversed?

Possible answers:

1. Clockwise.
2. Counterclockwise.
3. It depends on the initial conditions.

Answer: Clockwise. Above the horizontal axis, $\dot{x} > 0$, which means that x is increasing.

Changing the initial conditions changes E , which changes the ellipse. The family of all such trajectories is a nested family of ellipses, the phase portrait of the system.

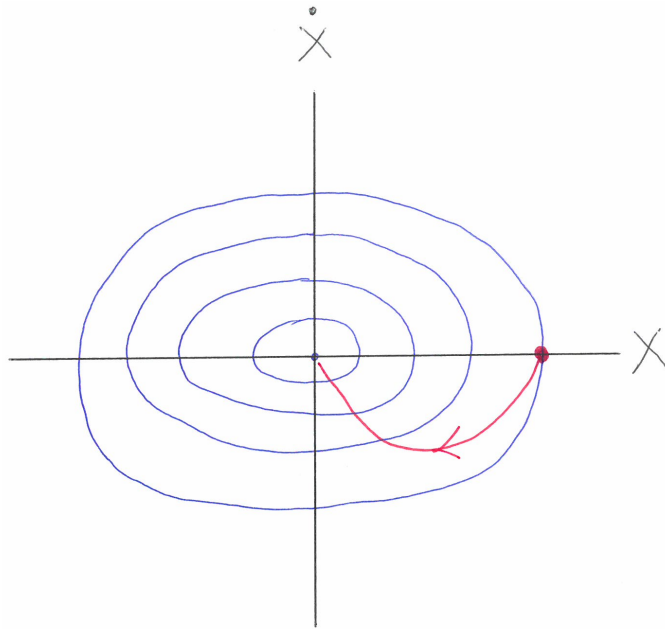


15.15.3. *Energy loss in the damped oscillator.* Now consider a damped oscillator described by $m\ddot{x} + b\dot{x} + kx = 0$. Now

$$\dot{E} = \dot{x}(m\ddot{x} + kx) = -b\dot{x}^2.$$

Energy is lost to friction. The dashpot heats up. The phase plane trajectory crosses through the equal-energy ellipses, inwards towards the origin.

It may spiral in (underdamped case), or approach the origin more directly (critically damped and overdamped cases).



Midterm 2 covers everything up to here.

16. MATRICES AS LINEAR TRANSFORMATIONS

This whole section is review from 18.02 — not covered in 18.03 lecture.

16.1. Functions with vector input and/or vector output.

Scalar input, scalar output: Alternative notation for the function $f(x) := \sin x$:

$$\begin{aligned} f: \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto \sin x. \end{aligned}$$

The first line specifies that

- the set of allowable inputs is $\mathbb{R}$ (the **domain**), and
- every output is an element of $\mathbb{R}$ (the **codomain**).

The second line shows a typical input x in the domain, and shows which element of the codomain it is mapped to.

(Note: The codomain specifies only the *type* of output; every output is an element of the codomain, but there might be other elements of the codomain that are not outputs. In contrast, the **range** of f (also called the **image** of f) is the set of all actual outputs of f , which in this example is the interval $[-1, 1]$. In general, the range is a subset of the codomain.)

Scalar input, vector output (vector-valued function):

$$\mathbf{r}: \mathbb{R} \longrightarrow \mathbb{R}^2$$

$$t \longmapsto \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}.$$

(We use a bold letter $\mathbf{r}$ instead of r since its values are vectors.) For example, $\mathbf{r}(\pi/3) = \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix}$.

Vector input, scalar output (multivariable function):

$$f: \mathbb{R}^3 \longrightarrow \mathbb{R}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto x^2 + y^2 - z^2.$$

It could also be written $f(x, y, z) := x^2 + y^2 - z^2$. For example, $f(1, 2, 3) = 1 + 4 - 9 = -4$.

Vector input, vector output: Can we have functions with vector input *and* vector output? Yes! See Example 16.1 below.

16.2. Going from a matrix to a linear transformation.

Warm-up: Given a number, say 3, we get a function

$$f: \mathbb{R} \longrightarrow \mathbb{R}$$

$$x \longmapsto 3x$$

that multiplies each input by 3. What is the higher-dimensional analogue?

Example 16.1. Given the 2×3 matrix $\begin{pmatrix} 6 & 7 & 8 \\ 2 & 3 & 5 \end{pmatrix}$, we get a function

$$\mathbf{f}: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

$$\mathbf{x} \longmapsto \begin{pmatrix} 6 & 7 & 8 \\ 2 & 3 & 5 \end{pmatrix} \mathbf{x}$$

with vector input and vector output! Explicitly,

$$\mathbf{f} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6x + 7y + 8z \\ 2x + 3y + 5z \end{pmatrix}.$$

Each coordinate of the output is a linear combination of the input variables.

What does this function do to $\mathbf{e}_2$? Answer: $\mathbf{f}(\mathbf{e}_2) = \mathbf{f} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$.

In general, an $m \times n$ matrix A gives rise to a function

$$\mathbf{f}: \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

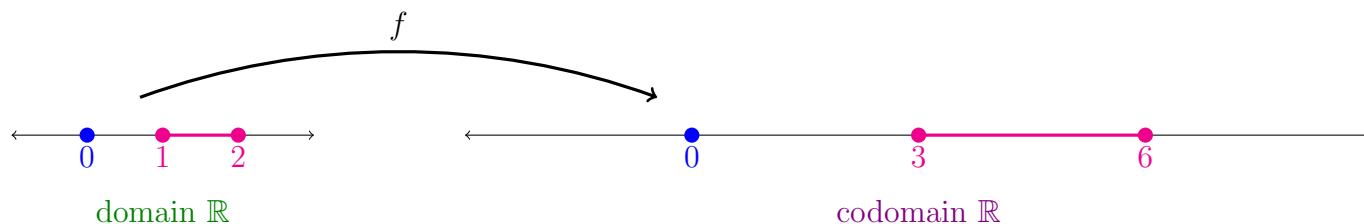
$$\mathbf{x} \longmapsto A\mathbf{x}$$

(note that m and n get reversed). Functions arising in this way are called **linear transformations**. Sometimes we use the matrix A itself instead of $\mathbf{f}$ to denote the linear transformation.

16.3. Depicting a linear transformation with a domain-codomain diagram.

16.3.1. *A single-variable function.* To visualize a function like $f(x) := 3x$, we would usually draw its graph in $\mathbb{R}^2$, the line $y = 3x$.

But there is another way: Draw the domain and the codomain (two copies of the real line), and show what certain features in the domain get transformed to:



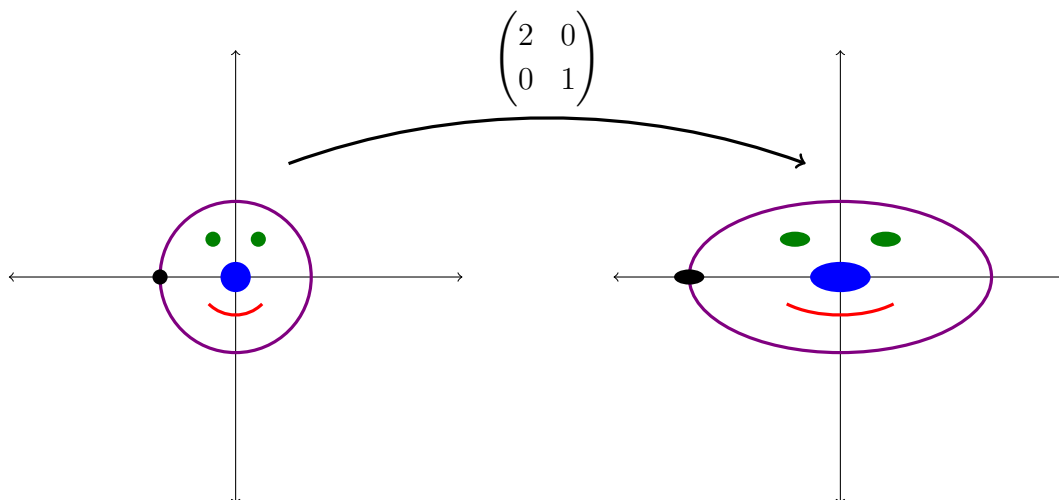
For example, f maps the point 2 to the point 6. The diagram shows how f expands everything by a factor of 3.

16.3.2. *Higher-dimensional analogue.* Consider the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and the associated linear transformation

$$\mathbf{f}: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ y \end{pmatrix}.$$

Drawing a graph of $\mathbf{f}$ would require 4 dimensions (2 for the input, 2 for the output), so instead let's draw a domain-codomain diagram. How does $\mathbf{f}$ transform the van Gogh unit smiley?



For example, the ear at $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ is mapped to $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$. The linear transformation $\mathbf{f}$ stretches in the horizontal direction only.

16.4. Going from a linear transformation to a matrix. Given a linear transformation $\mathbf{f}$, how do we reconstruct the matrix A ?

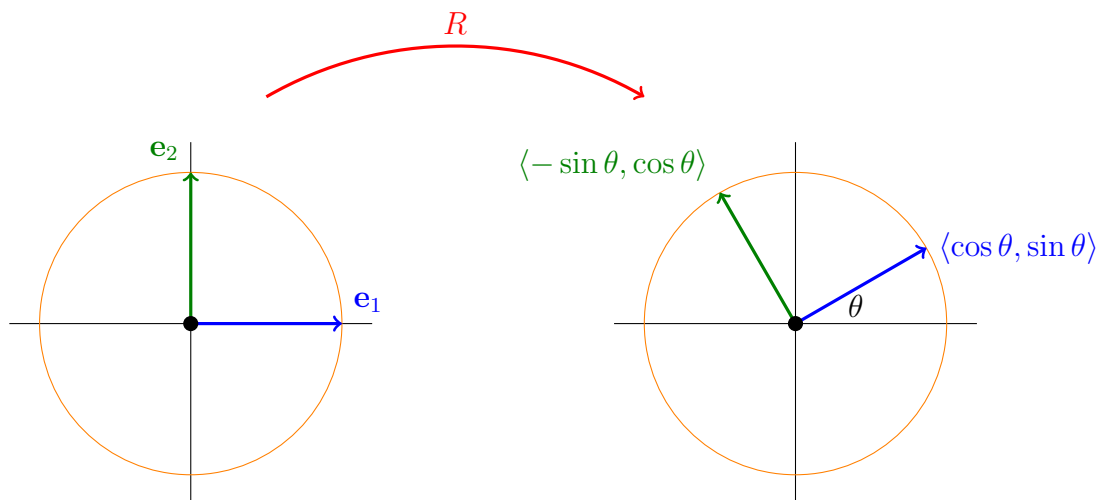
Answer 1: If you know a *formula* for $\mathbf{f}$, just read off the entries of the matrix. For example, if $\mathbf{f} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6x + 7y \\ 2x + 5z \end{pmatrix}$, then $A = \begin{pmatrix} 6 & 7 & 0 \\ 2 & 0 & 5 \end{pmatrix}$.

Answer 2: If you know only a *geometric description* of the linear transformation $\mathbf{f}$, then get A as the matrix whose columns are $\mathbf{f}(\mathbf{e}_1)$, $\mathbf{f}(\mathbf{e}_2)$, etc.

Here is an example illustrating Answer 2:

Question 16.2. Given θ , there is a 2×2 matrix R that rotates each vector in $\mathbb{R}^2$ counter-clockwise by the angle θ . What is it?

Solution.



The diagram above shows that

$$(\text{first column of } R) = R\mathbf{e}_1 = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$$

$$(\text{second column of } R) = R\mathbf{e}_2 = \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix},$$

so

$$R = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

□

16.5. Composition. Matrix multiplication corresponds to composition of the linear transformations. More explicitly, if

- $\mathbf{f}: \mathbb{R}^n \rightarrow \mathbb{R}^m$ corresponds to a matrix A , and
- $\mathbf{g}: \mathbb{R}^p \rightarrow \mathbb{R}^n$ corresponds to a matrix B ,

then the **composition**

$$\mathbb{R}^p \xrightarrow{\mathbf{g}} \mathbb{R}^n \xrightarrow{\mathbf{f}} \mathbb{R}^m$$

$$\quad \quad \quad \text{f} \circ \mathbf{g}$$

$$\mathbf{x} \longmapsto \mathbf{f}(\mathbf{g}(\mathbf{x}))$$

corresponds to the matrix product AB . In other words, if you multiply $\mathbf{x}$ by B (apply the inner function $\mathbf{g}$ first!) and then multiply the result by A , you get the same output vector as if you multiplied $\mathbf{x}$ by AB :

$$A(B\mathbf{x}) = (AB)\mathbf{x}.$$

Matrix multiplication is defined as it is to make this true.

Problem 16.3. Consider the following two functions from $\mathbb{R}^2$ to $\mathbb{R}^2$:

- 90° counterclockwise rotation about the origin, represented by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.
- reflection across the x -axis, represented by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

We can compose these two functions, first rotating and then reflecting, to get a third function. What matrix represents this third function?

Solution 1: The composition maps

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{\text{rotate}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{\text{reflect}} \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{\text{rotate}} \begin{pmatrix} -1 \\ 0 \end{pmatrix} \xrightarrow{\text{reflect}} \begin{pmatrix} -1 \\ 0 \end{pmatrix};$$

the answer is the matrix having these outputs as the first and second columns:

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \quad \square$$

Solution 2: Multiply the matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} := \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \quad \square$$

reflection rotation

Question 16.4. Why did we write the rotation second?

Answer: Applying this matrix product to a vector $\mathbf{v}$ means that you apply rotation first (just as when computing $f(g(x))$, you apply g first).

Example 16.5. Let α and β be any two angles. Let R_α be the 2×2 rotation matrix for α , and so on. Then

$$R_\alpha R_\beta = R_{\alpha+\beta}$$

because rotating by β and then rotating by α has the same effect as rotating by $\alpha + \beta$. (In this example the order did not matter, but in other examples it does.) What famous formulas do you get if you compare entries in this identity?

16.6. **Identity matrix.** The [identity transformation](#)

$$\mathbf{f}: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1x + 0y + 0z \\ 0x + 1y + 0z \\ 0x + 0y + 1z \end{pmatrix}.$$

is associated to the 3×3 [identity matrix](#)

$$I := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(You can guess what the 4×4 identity matrix looks like. The positions of the 1s form what is called the [diagonal](#) of the matrix.)

The matrix I acts like the number 1:

$$IA = A \quad \text{and} \quad AI = A$$

whenever the multiplication makes sense.

March 12

The March 12 lecture can be found in [these notes](#) after the March 14 review.

March 14

17. REVIEW

How can you check your answers? To check...

- that a function is a solution to a DE: plug it in (and check initial conditions too).
- that a list of functions is a basis of solutions to a (system of) linear ODE:
 - check that each function is a solution,
 - check that they are linearly independent (no linear combination of them is 0, except for the combination in which all coefficients are 0), and
 - check that the number of functions is as predicted by the dimension theorem.
- that λ is an eigenvalue of A :
 - plug it into the characteristic polynomial to check that it gives 0, or
 - check that $\det(A - \lambda I) = 0$, or
 - try to find an eigenvector (the resulting system should have a redundant equation).
- that $\mathbf{v}$ is an eigenvector of A : evaluate $A\mathbf{v}$ and check that it is a scalar multiple of $\mathbf{v}$.

- a phase portrait for $\dot{\mathbf{x}} = A\mathbf{x}$: compute the velocity vector at a point or two by evaluating $A\mathbf{x}$ at specific points $\mathbf{x}$ (see problem below for an example). Also, check that the point $(\text{tr } A, \det A)$ in the trace-determinant plane is in the region for the type you expected.

Problem 17.1. A pie is launched straight up from the Earth's surface, accelerating almost immediately to more than escape velocity(!). It then floats freely, subject to the Earth's gravitational field. Let $x(t)$ be its distance from the center of the Earth. Plot a possible trajectory in the $(x, \dot{x})$ -phase plane.

Solution: Let R be the radius of the Earth. Then $x(0) = R$, and $\dot{x}$ almost immediately rises from 0 to some large v_0 . After that, x increases forever, tending to ∞ (the pie is escaping), while $\dot{x}$ decreases forever (due to gravity) but stays positive.

Thus the phase point goes from $(R, 0)$ quickly to approximately (R, v_0) , and then goes down and right, with horizontal coordinate x tending towards infinity, always staying in the first quadrant ($x > 0$ and $\dot{x} > 0$ always).

To say more, we need to model the situation with equations. Let M be the mass of the Earth. Let m be the mass of the pie. Let G be the gravitational constant. Physics says that the gravitational force is $-GMm/x^2$, which must equal $m\ddot{x}$. Potential energy measures the negative of the work done by gravity (as the pie goes out, the work done on it is negative, but the pie's potential energy is increasing); this leads to

$$\text{PE} = - \int -\frac{GMm}{x^2} dx = -\frac{GMm}{x}.$$

(There is an arbitrary constant that could be added.) On the other hand, the kinetic energy of the pie is

$$\text{KE} = \frac{1}{2}m\dot{x}^2.$$

By conservation of energy,

$$\frac{1}{2}m\dot{x}^2 - \frac{GMm}{x} = E \tag{9}$$

for some constant E . As $x \rightarrow \infty$, $-GMm/x \rightarrow 0$, so E represents the limiting kinetic energy, which is positive ($E < 0$ would mean that there was not enough energy to escape the Earth, and $E = 0$ would mean that the pie started with exactly escape velocity).

Equation (9) in x and $\dot{x}$ (with everything else being constant) is the implicit equation of the trajectory. For example, if units are redefined so that $m/2$ and GMm and E are all 1, and we set $y = \dot{x}$, then the equation says that $y^2 - 1/x = 1$, so $y = \sqrt{1 + 1/x}$, so the trajectory is asymptotic to the line $y = 1$, approaching it from above.

Problem 17.2. A voltage source providing $V = 10 \cos(3t)$ volts is attached in series with a 6 ohm resistor and a 2 henry inductor in a circuit. Find the steady-state current, and the phase lag of the current relative to V .

Solution: Kirchhoff's voltage law says that $V - 6I - 2\dot{I} = 0$, so

$$2\dot{I} + 6I = 10 \cos(3t). \quad (10)$$

This is a first-order linear ODE, so we could use an integrating factor or variation of parameters, but it will be easier to use complex replacement. That is, solve

$$2\dot{\tilde{I}} + 6\tilde{I} = 10e^{3it}$$

for the *complex* current $\tilde{I}(t)$ instead. The characteristic polynomial is $p(r) = 2r + 6$, so ERF says that one particular solution to the complex replacement ODE is

$$\tilde{I}(t) = \frac{1}{p(3i)} 10e^{3it} = \frac{1}{6 + 6i} 10e^{3it},$$

and one particular solution to the original inhomogeneous ODE (10) is

$$\operatorname{Re} \left(\frac{1}{6 + 6i} 10e^{3it} \right).$$

On the other hand, the only root of the characteristic polynomial is -3 , so the general solution to the associated homogeneous ODE

$$2\dot{I} + 6I = 0$$

is ce^{-3t} . Thus the general solution to the original inhomogeneous ODE (10) is

$$I(t) = \operatorname{Re} \left(\frac{1}{6 + 6i} 10e^{3it} \right) + ce^{-3t}.$$

The first term is the steady-state solution (because it is a solution and is a sinusoidal function), and the second term is the transient (since it tends to 0 as $t \rightarrow \infty$).

To find the relative phase lag, first find the complex gain. The complex gain is $G = \frac{1}{p(3i)} = \frac{1}{6+6i}$. Then the phase lag is $-\arg G = \arg(6 + 6i) = \pi/4$.

Remark 17.3. If, before solving, we divided both sides by 2 to rewrite (10) as

$$\dot{I} + 3I = 5 \cos(3t).$$

then the new complex replacement ODE would have $5e^{3it}$ on the right hand side, but and the new characteristic polynomial would be $p(r) = r + 3$, so we would get

$$\tilde{I}(t) = \frac{1}{3 + 3i} 5e^{3it},$$

which is the same complex solution as before (just written differently), so upon taking real parts we would get the same steady-state solution as before. Complex gain depends on what you are calling the input and output (it is defined as complex output divided by complex input), so if you are now viewing $5 \cos(3t)$ as input instead of the actual input voltage $10 \cos(3t)$, then the new complex gain is $\frac{1}{3+3i}$ instead of $\frac{1}{6+6i}$.

Problem 17.4. Given $A := \begin{pmatrix} 1 & -4 \\ 1 & 1 \end{pmatrix}$, sketch the phase portrait for $\dot{\mathbf{x}} = A\mathbf{x}$.

Solution: We have $\text{tr } A = 1 + 1 = 2$ and $\det A = 1 - (-4) = 5$, so the characteristic polynomial is $\lambda^2 - 2\lambda + 5 = (\lambda - 1)^2 + 4$, and the eigenvalues are $1 \pm 2i$. Since the eigenvalues are not real, and have positive real part, the phase portrait is a repelling spiral (spiral source).

Check: The point $(\text{tr } A, \det A) = (2, 5)$ lies above the parabola $\det = \frac{1}{4} \text{tr}^2$ and to the right of the vertical axis.

Do the trajectories go clockwise or counterclockwise? The velocity vector at $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is

$$\dot{\mathbf{x}} = A\mathbf{x} = \begin{pmatrix} 1 & -4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

so the trajectories are counterclockwise.

Further question: Is the system stable?

Solution 1: No, because the real parts of the eigenvalues are not negative.

Solution 2: No, because the coefficients of the characteristic polynomial are not all positive.

Solution 3: No, because the point $(\text{tr } A, \det A) = (2, 5)$ is in the first quadrant, not in the interior of the second quadrant.

Further question: Is the system structurally stable?

Solution 1: Yes, it is not a boundary case, since the eigenvalues are distinct and nonzero.

Solution 2: Yes, because the point $(\text{tr } A, \det A) = (2, 5)$ is not on the horizontal axis, the positive vertical axis, or the curve $\det = \frac{1}{4} \text{tr}^2$.

Solution 3: Yes, repelling spiral is one of the structurally stable types.

There was not time in lecture on March 14 to discuss the remaining parts below of Problem 17.4, but we did discuss Problem 17.6 instead.

Further question: What kind of function is the second coordinate $y(t)$ of $\mathbf{x}(t)$?

Solution: If $\mathbf{v}$ is an eigenvector associated to $1 + 2i$, then $\overline{\mathbf{v}}$ is an eigenvector associated to $1 - 2i$, so the vector-valued functions

$$e^{(1+2i)t}\mathbf{v}, \quad e^{(1-2i)t}\overline{\mathbf{v}}$$

form a basis of the space of solutions. Therefore $\mathbf{x}(t)$ is a linear combination of these. Taking second coordinates shows that $y(t)$ is a linear combination of $e^{(1+2i)t}$ and $e^{(1-2i)t}$. We can replace the latter two functions by the real and imaginary parts of the first of them, so $y(t)$

is also a linear combination of $e^t \cos(2t)$ and $e^t \sin(2t)$. In other words, $y(t)$ is e^t times a sinusoidal function with $\omega = 2$:

$$y(t) = e^t A \cos(2t - \phi)$$

for some A and ϕ .

Further question: If $\mathbf{x}(0)$ is on the positive x -axis, what is the next time t that $\mathbf{x}(t)$ lies on the x -axis?

Solution 1: We have

$$y(t) = c_1 e^t \cos(2t) + c_2 e^t \sin(2t)$$

for some real numbers c_1 and c_2 . Since $\mathbf{x}(0)$ is on the positive x -axis, $y(0) = 0$. Plugging $t = 0$ into the formula for $y(t)$ gives $0 = c_1$, so

$$y(t) = c_2 e^t \sin(2t).$$

The first positive time at which $y(t) = 0$ is when the angle $2t$ equals π , so $t = \pi/2$.

Solution 2: To determine when $y(t) = 0$, we can ignore the e^t factor and focus on the sinusoidal factor. The sinusoidal factor of $y(t)$ crosses 0 twice within each period. The period is $P = 2\pi/\omega = \pi$, so the time interval between crossings is $P/2 = \pi/2$.

Further question: At that first time when it crosses the x -axis again, what is its distance to the origin compared to its initial distance to the origin?

Solution: The y -coordinate is 0 at both $t = 0$ and $t = \pi/2$, so we need only study $x(t)$. The same argument as for $y(t)$ shows that $x(t) = e^t A' \cos(2t - \phi')$ for some $A' > 0$ and some ϕ' . When t goes from 0 to $\pi/2$, the angle $2t - \phi'$ increases by π , so the cosine changes sign, while e^t increases from 1 to $e^{\pi/2}$. Thus the distance is multiplied by $e^{\pi/2}$.

Question 17.5. Can two different trajectories in the (x, y) -phase plane for a system $\dot{\mathbf{x}} = A\mathbf{x}$ ever intersect?

Answer: No. If a trajectory passes through a point $\mathbf{v}$, then its behavior before and after are uniquely determined, by the existence and uniqueness theorem. (They can approach the same point as $t \rightarrow \infty$, however.)

Problem 17.6. Estimate the angular frequency ω for which the steady-state solution to

$$(D^3 + D^2 + 4D + 3.9)x = \cos(\omega t)$$

has largest amplitude.

Solution: Let $p(r) = r^3 + r^2 + 4r + 3.9$. The complex replacement ODE

$$p(D)z = e^{i\omega t}$$

has a solution $\frac{1}{p(i\omega)}e^{i\omega t}$, by ERF, as long as $p(i\omega) \neq 0$. Thus the complex gain is $\frac{1}{p(i\omega)}$ and the gain is $\frac{1}{|p(i\omega)|}$, which is largest when $i\omega$ is close to a root of $p(r)$.

Now

$$p(r) \approx q(r) := r^3 + r^2 + 4r + 4 = (r + 1)(r^2 + 4).$$

The roots of $q(r)$ are -1 and $\pm 2i$ and these are close to the roots of $p(r)$. In particular, the positive numbers ω such that $i\omega$ is close to a root of $p(r)$ are the numbers ω close to 2. Thus the amplitude of the solution is maximized for a value of ω close to 2.

March 17

Midterm 2

March 12

18. INTRODUCTION TO LINEAR SYSTEMS

18.1. Why? Recall that complicated real-world problems often give rise to a system of linear ODEs. We package the unknown functions into a vector-valued function $\mathbf{x}(t)$ and write the system in matrix form, such as $\dot{\mathbf{x}} = A\mathbf{x}$. To find a basis of solutions (when the matrix A is constant), we compute the eigenvalues and eigenvectors of A . Finding the eigenvectors involves solving a system of linear equations

$$(A - \lambda I)\mathbf{v} = \mathbf{0},$$

in which A and λ are known and $\mathbf{v}$ is unknown. In the 2×2 case, this is easy, but to handle more complicated situations, we'll need better methods for solving linear systems.

Solving a linear system comes up also when we have a general solution to an ODE (or system of ODEs) and want to use given initial conditions to find the coefficients in the particular solution.

And there are many other applications of solving linear systems in all branches of science and engineering (e.g., balancing a chemical equation).

18.2. Intersecting lines in $\mathbb{R}^2$. Given a system of two equations in two unknowns, each equation describes a line (assuming that the equation is not just constant=constant). The solution to the system is the intersection of the two lines.

From geometry, you know that there are three possibilities:

- The lines intersect at one point: one solution. (This is what happens most of the time.) Example:

$$x + y = 1$$

$$x - 2y = 2.$$

- The lines are the same, so their intersection is a line: infinitely many solutions. Example:

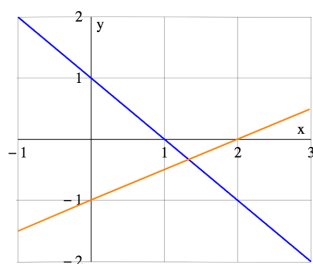
$$x + y = 1$$

$$2x + 2y = 2.$$

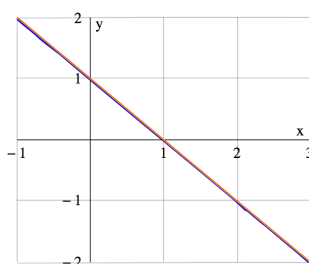
- The lines are parallel, so their intersection is empty: no solutions. Example:

$$x + y = 1$$

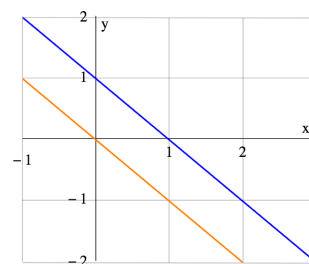
$$x + y = 0.$$



point



line



empty

With more equations and more unknowns, there are more possibilities, and we want to describe them all. For this, we need to develop more linear algebra.

18.3. Augmented matrix. A linear system

$$2x + 5y + 7z = 15$$

$$x + z = 1$$

can be written in matrix form $A\mathbf{x} = \mathbf{b}$:

$$\begin{pmatrix} 2 & 5 & 7 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 15 \\ 1 \end{pmatrix}$$

A
 $\mathbf{x}$
 $\mathbf{b}$

and can be represented by the augmented matrix

$$\left(\begin{array}{ccc|c} 2 & 5 & 7 & 15 \\ 1 & 0 & 1 & 1 \end{array} \right)$$

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(augmented with an extra column containing the right hand sides). Each row corresponds to an equation. Each column except the last one corresponds to a variable.

A linear system is **homogeneous** if the right hand sides (the constants) are all zero, and **inhomogeneous** otherwise. So a linear system is homogeneous if and only if the zero vector is a solution.

19. SOLVING LINEAR SYSTEMS

19.1. Equation operations. A good way to solve a linear system is to perform the following operations repeatedly, in some order:

- Multiply an equation by a nonzero number.
- Interchange two equations.
- Add a multiple of one equation to another equation.

The solution set is unchanged at each step.

19.2. Row operations. The equation operations correspond to operations on the augmented matrix, called **elementary row operations**:

- Multiply a row by a nonzero number.
- Interchange two rows.
- Add a multiple of one row to another row (while leaving the first row as it was).

19.3. Overview of method.

Overview of how to solve a linear system $A\mathbf{x} = \mathbf{b}$:

1. Use row operations (*Gaussian elimination*) to convert the augmented matrix to a particularly simple form, called *row-echelon form*.
2. Solve the new system by *back-substitution*.

19.4. Gaussian elimination. **Gaussian elimination** is an algorithm for converting any matrix into row-echelon form by performing row operations. Here are the steps:

1. Find the leftmost nonzero column, and the first nonzero entry in that column (read from the top down).
2. If that entry is not already in the first row, interchange its row with the first row.
3. Make all other entries of the column zero by adding suitable multiples of the first row to the others.
4. At this point, the first row is done, so ignore it, and repeat the steps above for the remaining submatrix (with one fewer row). In each iteration, ignore the rows already taken care of.
5. Stop when all the remaining rows consist entirely of zeros. Then the whole matrix will be in what is called *row-echelon form*.

Problem 19.1. This will be done in recitation on 3/18/25. Apply Gaussian elimination to convert the 4×7 matrix

$$\begin{pmatrix} 0 & 0 & 6 & 2 & -4 & -8 & 8 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 6 & -9 & 0 & 11 & -19 & 3 & 0 \end{pmatrix}$$

to row-echelon form. (This example is taken from Hill, *Elementary linear algebra with applications*, p. 17.)

Solution:

Step 1. The leftmost nonzero column is the first one, and its first nonzero entry is the 2:

$$\begin{pmatrix} 0 & 0 & 6 & 2 & -4 & -8 & 8 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 6 & -9 & 0 & 11 & -19 & 3 & 0 \end{pmatrix}.$$

Step 2. The 2 is not in the first row, so interchange its row with the first row:

$$\begin{pmatrix} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 6 & 2 & -4 & -8 & 8 \\ 6 & -9 & 0 & 11 & -19 & 3 & 0 \end{pmatrix}.$$

Step 3. To make all other entries of the column zero, we need to add -3 times the first row to the last row (the other rows are OK already):

$$\begin{pmatrix} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 6 & 2 & -4 & -8 & 8 \\ 0 & 0 & -3 & -1 & 2 & 0 & -6 \end{pmatrix}.$$

Step 4. Now the first row is done. Start over with the 3×7 submatrix that remains beneath it:

$$\begin{pmatrix} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 6 & 2 & -4 & -8 & 8 \\ 0 & 0 & -3 & -1 & 2 & 0 & -6 \end{pmatrix}.$$

Step 1. The leftmost nonzero column is now the third column, and its first nonzero entry is the 3:

$$\begin{pmatrix} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 6 & 2 & -4 & -8 & 8 \\ 0 & 0 & -3 & -1 & 2 & 0 & -6 \end{pmatrix}.$$

Step 2. The **3** is already in the first row of the submatrix (we are ignoring the first row of the whole matrix), so no interchange is necessary.

Step 3. To make all other entries of the column zero, add -2 times the (new) first row to the (new) second row, and 1 times the (new) first row to the (new) third row:

$$\begin{pmatrix} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & \mathbf{3} & 1 & -2 & -4 & 4 \\ 0 & 0 & \mathbf{0} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{0} & 0 & 0 & -4 & -2 \end{pmatrix}.$$

Step 4. Now the first and second row of the original matrix are done. Start over with the 2×7 submatrix beneath them:

$$\begin{pmatrix} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & -2 \end{pmatrix}.$$

Step 1. The leftmost nonzero column is now the penultimate column, and its first nonzero entry is the **-4** at the bottom:

$$\begin{pmatrix} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{0} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{-4} & -2 \end{pmatrix}.$$

Step 2. The **-4** is not in the first row of the submatrix, so interchange its row with the first row of the submatrix:

$$\begin{pmatrix} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{-4} & -2 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{0} & 0 \end{pmatrix}.$$

Step 3. The other entry in this column of the submatrix is already 0, so this step is not necessary.

Now the first three rows are done. What remains below them is all zeros, so stop! The matrix is now in row-echelon form:

$$\begin{pmatrix} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 0 & 0 & 0 & -4 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad \square$$

19.5. **Row-echelon form.** What does row-echelon form mean? Before explaining this, we need a few preliminary definitions.

- A **zero row** of a matrix is a row consisting entirely of zeros.
- A **nonzero row** of a matrix is a row with at least one nonzero entry. In each nonzero row, the first nonzero entry is called the **pivot**.

Example 19.2. The following 4×5 matrix has one zero row, and three pivots (shown in red):

$$\begin{pmatrix} 0 & -5 & 4 & 4 & 3 \\ 2 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}.$$

Definition 19.3. A matrix is in **row-echelon form** if it satisfies both of the following conditions:

1. All the zero rows (if any) are grouped at the bottom of the matrix.
2. Each pivot lies farther to the right than the pivots of higher rows.

Warning: Some books require also that each pivot be a 1. We are not going to require this for row-echelon form, but we will require it for *reduced* row-echelon form later on.

19.6. Back-substitution.

Key point of row-echelon form: Matrices in row-echelon form correspond to systems that are ready to be solved immediately by **back-substitution**: solve for each variable in reverse order, while introducing a parameter for each variable not directly expressed in terms of later variables, and substitute values into earlier equations once they are known.

Problem 19.4. Suppose that we are solving a linear system with unknowns x, y, z, v, w . Suppose that we already wrote down the augmented matrix and used Gaussian elimination to convert it to row-echelon form, resulting in

$$\left(\begin{array}{ccccc|c} 1 & 2 & 0 & 2 & 3 & 4 \\ 0 & -1 & 2 & 3 & 1 & 5 \\ 0 & 0 & 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right).$$

This is slightly more complicated than the example actually done in lecture.

Find the general solution to the system.

Solution: The new augmented matrix (the one shown, in row-echelon form) represents the inhomogeneous linear system

$$\begin{aligned}x + 2y + 2v + 3w &= 4 \\-y + 2z + 3v + w &= 5 \\2w &= 6 \\0 &= 0.\end{aligned}$$

We solve for the variables in reverse order, using the equations from the bottom up. Start by solving for the last variable, w :

$$w = 3.$$

There is no equation for v in terms of the later variable w , so v can be any number; set

$$v = c_1 \quad \text{for a parameter } c_1.$$

There is no equation for z in terms of the later variables v and w , so set

$$z = c_2 \quad \text{for a parameter } c_2.$$

Substitute the values of w , v , z into the previous equation, and solve for y :

$$\begin{aligned}-y + 2c_2 + 3c_1 + 3 &= 5 \\y &= 3c_1 + 2c_2 - 2.\end{aligned}$$

Similarly, solve for x :

$$\begin{aligned}x + 2(3c_1 + 2c_2 - 2) + 2c_1 + 3(3) &= 4 \\x &= -8c_1 - 4c_2 - 1.\end{aligned}$$

Conclusion: The general solution is

$$\begin{aligned}
 \begin{pmatrix} x \\ y \\ z \\ v \\ w \end{pmatrix} &= \begin{pmatrix} -8c_1 - 4c_2 - 1 \\ 3c_1 + 2c_2 - 2 \\ c_2 \\ c_1 \\ 3 \end{pmatrix} \\
 &= \begin{pmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 3 \end{pmatrix} + \begin{pmatrix} -8c_1 \\ 3c_1 \\ 0 \\ c_1 \\ 0 \end{pmatrix} + \begin{pmatrix} -4c_2 \\ 2c_2 \\ c_2 \\ 0 \\ 0 \end{pmatrix} \\
 &= \begin{pmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 3 \end{pmatrix} + c_1 \begin{pmatrix} -8 \\ 3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -4 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix},
 \end{aligned}$$

where c_1, c_2 are parameters. $\square$

Suppose that a matrix is in row-echelon form.

Some columns contain a pivot, and some do not; a column that contains a pivot is called a **pivot column**. Correspondingly, there are two kinds of variables:

- A variable corresponding to a pivot column is called a **dependent variable** (or **pivot variable**).
- The other variables are called **free variables**.

(The augmented column does not correspond to any variable.)

In Problem 19.4, the first, second, and fifth columns were the pivot columns, so x, y, w were dependent variables; v, z were free variables.

Warning: If your matrix is not in row-echelon form yet, don't talk about pivot columns and pivot variables!

19.7. Reduced row-echelon form. With even more row operations, one can simplify a matrix in row-echelon form to an even more special form:

Definition 19.5. A matrix is in **reduced row-echelon form (RREF)** if it satisfies all of the following conditions:

1. It is in row-echelon form.
2. Each pivot is a 1.

3. In each pivot column, all the entries are 0 except for the pivot itself.

19.8. Gauss–Jordan elimination. The presentation of the algorithm and Problem 19.6 below will be done in recitation on 3/18/25.

Gauss–Jordan elimination is an algorithm for converting any matrix into *reduced* row-echelon form by performing row operations. Here are the steps:

1. Use Gaussian elimination to convert the matrix to row-echelon form.
2. Divide the last nonzero row by its pivot, to make the pivot 1.
3. Make all entries in that pivot's column 0 by adding suitable multiples of the pivot's row to the rows above.
4. At this point, the row in question (and all rows below it) are done. Ignore them, and go back to Step 2, but now with the remaining submatrix, above the row just completed.

Eventually the whole matrix will be in reduced row-echelon form.

Problem 19.6. Convert the 4×7 matrix

$$\begin{pmatrix} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 0 & 0 & 0 & -4 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

to reduced row-echelon form.

Solution:

Step 1. The matrix is already in row-echelon form.

Step 2. The last nonzero row is the third row, and its pivot is the -4 , so divide the third row by -4 :

$$\begin{pmatrix} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Step 3. To make all other entries of that pivot's column 0, add -1 times the third row to the first row, and add 4 times the third row to the second row:

$$\begin{pmatrix} 2 & -3 & 1 & 4 & -7 & 0 & 3/2 \\ 0 & 0 & 3 & 1 & -2 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Step 4. Now the last two rows are done:

$$\begin{pmatrix} 2 & -3 & 1 & 4 & -7 & 0 & 3/2 \\ 0 & 0 & 3 & 1 & -2 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Go back to Step 2, but with the 2×7 submatrix above them.

Step 2. The last nonzero row of the new matrix (ignoring the bottom two rows of the original matrix) is the second row, and its pivot is the **3**, so we divide the second row by **3**:

$$\begin{pmatrix} 2 & -3 & 1 & 4 & -7 & 0 & 3/2 \\ 0 & 0 & 1 & 1/3 & -2/3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Step 3. To make the other entries of that pivot's column 0, add -1 times the second row to the first row:

$$\begin{pmatrix} 2 & -3 & 0 & 11/3 & -19/3 & 0 & -1/2 \\ 0 & 0 & 1 & 1/3 & -2/3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Step 4. Now the last three rows are done:

$$\begin{pmatrix} 2 & -3 & 0 & 11/3 & -19/3 & 0 & -1/2 \\ 0 & 0 & 1 & 1/3 & -2/3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Go back to Step 2, but with the 1×7 submatrix above them.

Step 2. The last nonzero row of the new matrix is the only remaining row (the first row), and its pivot is the initial **2**, so we divide the first row by **2**:

$$\begin{pmatrix} 1 & -3/2 & 0 & 11/6 & -19/6 & 0 & -1/4 \\ 0 & 0 & 1 & 1/3 & -2/3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The matrix is now in reduced row-echelon form. $\square$

Problem 19.7. Suppose that we are solving a linear system for unknowns x, y, z, v, w . Suppose that we have used Gauss–Jordan elimination to put the augmented matrix in reduced

row-echelon form, and the result is

$$\left(\begin{array}{ccccc|c} \textcolor{red}{1} & 0 & -2 & 0 & 7 & \textcolor{violet}{3} \\ 0 & \textcolor{red}{1} & 6 & 0 & 8 & \textcolor{violet}{4} \\ 0 & 0 & 0 & \textcolor{red}{1} & 9 & \textcolor{violet}{5} \end{array} \right).$$

(Check: This really is in reduced row-echelon form!) Find the general solution to the system.

Solution: The system to be solved is

$$\begin{aligned} \textcolor{red}{x} - 2z + 7w &= 3 \\ \textcolor{violet}{y} + 6z + 8w &= 4 \\ \textcolor{red}{v} + 9w &= 5. \end{aligned}$$

Back-substitution:

$$\begin{aligned} w &= c_1 \quad (\text{free variable}) \\ v &= -9w + 5 = -9c_1 + 5 \\ z &= c_2 \quad (\text{free variable}) \\ y &= -6z - 8w + 4 = -6c_2 - 8c_1 + 4 \\ x &= 2z - 7w + 3 = 2c_2 - 7c_1 + 3. \end{aligned}$$

(Notice that no substitution was required: we could solve for each variable directly!) Answer:

$$\begin{pmatrix} x \\ y \\ z \\ v \\ w \end{pmatrix} = \begin{pmatrix} 2c_2 - 7c_1 + 3 \\ -6c_2 - 8c_1 + 4 \\ c_2 \\ -9c_1 + 5 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 0 \\ 5 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} -7 \\ -8 \\ 0 \\ -9 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ -6 \\ 1 \\ 0 \\ 0 \end{pmatrix}. \quad \square$$

Moral: After using Gaussian elimination to put an augmented matrix into row-echelon form, there are two ways to finish solving the linear system:

- Do back-substitution.
- Do the extra row operations need to get the matrix into *reduced* row-echelon form (Gauss–Jordan elimination), and then do (a much easier) back-substitution.

You can experiment to find out which is faster for you.

Remark 19.8. Performing row operations on A in a different order than specified by Gaussian elimination and Gauss–Jordan elimination can lead to different row-echelon forms. But it turns out that row operations leading to *reduced row-echelon form* always give the same result, a matrix that we will write as $\text{RREF}(A)$.

19.9. **Comparing inhomogeneous and homogeneous linear systems.** Recall that the general solution to the inhomogeneous system

$$\begin{aligned}x + 2y + 2v + 3w &= 4 \\ -y + 2z + 3v + w &= 5 \\ 2w &= 6 \\ 0 &= 0\end{aligned}$$

was

$$\begin{pmatrix} x \\ y \\ z \\ v \\ w \end{pmatrix} = \underbrace{\begin{pmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 3 \end{pmatrix}}_{\text{particular solution}} + \underbrace{c_1 \begin{pmatrix} -8 \\ 3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -4 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{\text{general homogeneous solution}},$$

where c_1, c_2 are parameters. (The labels under the braces haven't been explained yet.)

Doing Gaussian elimination and back-substitution again would show that the general solution to the associated *homogeneous* system

$$\begin{aligned}x + 2y + 2v + 3w &= 0 \\ -y + 2z + 3v + w &= 0 \\ 2w &= 0 \\ 0 &= 0\end{aligned}$$

is

$$\begin{pmatrix} x \\ y \\ z \\ v \\ w \end{pmatrix} = c_1 \begin{pmatrix} -8 \\ 3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -4 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

where c_1, c_2 are parameters. Thus the set of solutions is

$$\text{Span} \left(\begin{pmatrix} -8 \\ 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right).$$

20. HOMOGENEOUS LINEAR SYSTEMS AND LINEAR ALGEBRA CONCEPTS

For a while, we are going to assume that vectors have real numbers as coordinates, and that all scalars are real numbers. This is so we can describe things geometrically in $\mathbb{R}^n$ more easily. But eventually, we will work with vectors in $\mathbb{C}^n$ whose coordinates can be complex numbers, and will allow scalar multiplication by complex numbers.

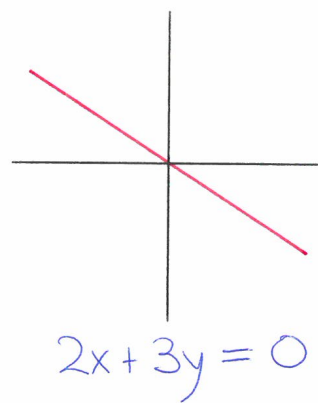
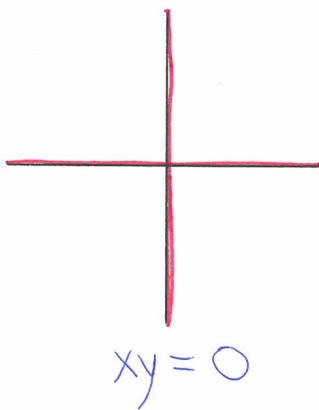
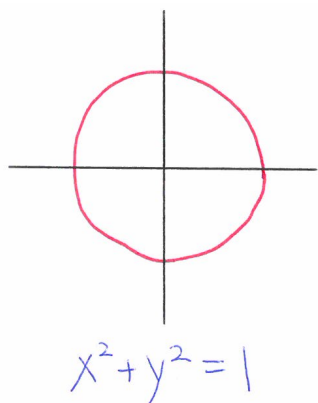
20.1. Vector space.

Definition 20.1. Suppose that S is a set consisting of some of the vectors in $\mathbb{R}^n$ (for some fixed value of n). Call S a **vector space** if all of the following are true:

0. The zero vector $\mathbf{0}$ is in S .
1. Multiplying any one vector in S by any scalar gives another vector in S .
2. Adding any two vectors in S gives another vector in S .

Question: Which of the following subsets of $\mathbb{R}^2$ are vector spaces?

- (a) The set of all vectors $\begin{pmatrix} x \\ y \end{pmatrix}$ satisfying $x^2 + y^2 = 1$.
- (b) The set of all vectors $\begin{pmatrix} x \\ y \end{pmatrix}$ satisfying $xy = 0$.
- (c) The set of all vectors $\begin{pmatrix} x \\ y \end{pmatrix}$ satisfying $2x + 3y = 0$.



Answer: Only (c) is a vector space.

Explanation: Let S be the set. For S to be a vector space, it must satisfy all three conditions.

Example (a) doesn't even satisfy condition 0, because the zero vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is not in S . So (a) is not a vector space.

Example (b) satisfies condition 0: the zero vector is in S . It satisfies condition 1 too: If $\begin{pmatrix} x \\ y \end{pmatrix}$ is one vector in S (so $xy = 0$) and c is any scalar, then the vector $c \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} cx \\ cy \end{pmatrix}$ satisfies $(cx)(cy) = c^2xy = c^2(0) = 0$. But it does not satisfy condition 2 for *every* pair of vectors in S : for example, $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$ are in S , but their sum $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is not in S . So (b) is not a vector space.

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Example (c) is a vector space, as we will now check.

- Condition 0: The zero vector is in S .
- Condition 1: If $\begin{pmatrix} x \\ y \end{pmatrix}$ is any element of S (so $2x + 3y = 0$) and c is any scalar, then multiplying the equation by c gives $2(cx) + 3(cy) = 0$, which shows that the vector $c \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} cx \\ cy \end{pmatrix}$ is in S .
- Condition 2: If $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ are in S (so $2x_1 + 3y_1 = 0$ and $2x_2 + 3y_2 = 0$), then adding the equations shows that $2(x_1 + x_2) + 3(y_1 + y_2) = 0$, which says that the vector $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$ is in S .

Thus S is a vector space. $\square$

Definition 20.2. A vector space contained in another vector space V is called a **subspace** of V .

For example, the subset described in (c) above is a subspace of $\mathbb{R}^2$.

Subspaces of $\mathbb{R}^2$ (it turns out that this is the complete list):

- $\{\mathbf{0}\}$ (the set containing only the origin)
- a line through the origin
- the whole plane $\mathbb{R}^2$.

Subspaces of $\mathbb{R}^3$ (again, the complete list):

- $\{\mathbf{0}\}$
- a line through the origin
- a plane through the origin
- the whole space $\mathbb{R}^3$.

20.2. Span. This is review from 18.02.

Definition 20.3. A **linear combination** of vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ is a vector of the form $c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n$ for some scalars $c_1, \dots, c_n$.

Definition 20.4. The **span** of $\mathbf{v}_1, \dots, \mathbf{v}_n$ is the set of *all* linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_n$:

$$\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_n) := \{\text{all vectors } c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n, \text{ where } c_1, \dots, c_n \text{ are scalars}\}.$$

The four examples below were not discussed in lecture.

Example 20.5. If $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ in $\mathbb{R}^2$, then $\text{Span}(\mathbf{v})$ is the set of all vectors $c \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ as c ranges over all real numbers, so $\text{Span}(\mathbf{v})$ is the line $y = x$.

Example 20.6. Similarly, $\text{Span}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}\right)$ is the set of vectors $c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ as c_1 and c_2 range over all real numbers, but this is still only the line $y = x$.

Example 20.7. Let $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ in $\mathbb{R}^2$. Is $\begin{pmatrix} 8 \\ 7 \end{pmatrix}$ in $\text{Span}(\mathbf{v}, \mathbf{w})$? Yes, it's $3\mathbf{v} + 2\mathbf{w}$. In fact, every vector of $\mathbb{R}^2$ is in the span: $\text{Span}(\mathbf{v}, \mathbf{w}) = \mathbb{R}^2$.

Example 20.8. If $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, then $\text{Span}(\mathbf{i}, \mathbf{j})$ is the set of all vectors of the form

$$c_1\mathbf{i} + c_2\mathbf{j} = \begin{pmatrix} c_1 \\ c_2 \\ 0 \end{pmatrix}.$$

These form the xy -plane in $\mathbb{R}^3$, whose equation is $z = 0$.

Problem 20.9. Explain the following statement:

If $\mathbf{v}$ and $\mathbf{w}$ are two vectors, then $\text{Span}(\mathbf{v}, \mathbf{w})$ is a vector space.

Solution:

0. The zero vector $\mathbf{0}$ is in $\text{Span}(\mathbf{v}, \mathbf{w})$ since $\mathbf{0} = 0\mathbf{v} + 0\mathbf{w}$.
1. Multiplying any linear combination of $\mathbf{v}$ and $\mathbf{w}$ by any scalar gives another linear combination of $\mathbf{v}$ and $\mathbf{w}$ (for example, $5(2\mathbf{v} + 3\mathbf{w}) = 10\mathbf{v} + 15\mathbf{w}$).
2. Adding any two linear combinations of $\mathbf{v}$ and $\mathbf{w}$ gives another linear combination of $\mathbf{v}$ and $\mathbf{w}$ (for example, $(2\mathbf{v} + 3\mathbf{w}) + (4\mathbf{v} + 5\mathbf{w}) = 6\mathbf{v} + 8\mathbf{w}$). $\square$

The same argument shows that *any* span is a vector space.

20.3. Nullspace.

Example 20.10 (Homogeneous linear system). For

$$A = \begin{pmatrix} 1 & 2 & 0 & 2 & 3 \\ 0 & -1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

we showed at the end of Section 19.9 that

$$\{\text{all solutions to } A\mathbf{x} = \mathbf{0}\} = \text{Span} \left(\begin{pmatrix} -8 \\ 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right).$$

This is a vector space (since any span is a vector space)!

The same happens for any matrix A :

Theorem 20.11. *For any homogeneous linear system $A\mathbf{x} = \mathbf{0}$, the set of all solutions is a vector space.*

This is analogous to the fact that the set of all solutions to a homogeneous linear ODE is a vector space!

Definition 20.12. The set of all solutions to $A\mathbf{x} = \mathbf{0}$ is called the **nullspace** of the matrix A , and denoted $\text{NS}(A)$.

Problem 20.13. Let $A = \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}$. Is $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ in $\text{NS}(A)$?

Solution: The question is asking whether $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ is a solution to $A\mathbf{x} = \mathbf{0}$. Is it true that

$$\begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}?$$

Yes! $\square$

Here is a more direct way to explain why the set of all solutions to $A\mathbf{x} = \mathbf{0}$ is a vector space, without computing it as a span:

0. The zero vector $\mathbf{0}$ is a solution since $A\mathbf{0} = \mathbf{0}$.

1. Multiplying any solution $\mathbf{v}$ by any scalar c gives another solution: given that $A\mathbf{v} = \mathbf{0}$, it follows that $A(c\mathbf{v}) = c(A\mathbf{v}) = c\mathbf{0} = \mathbf{0}$.

2. Adding any solutions gives another solution: given that $A\mathbf{v} = \mathbf{0}$ and $A\mathbf{w} = \mathbf{0}$, it follows that $A(\mathbf{v} + \mathbf{w}) = A\mathbf{v} + A\mathbf{w} = \mathbf{0} + \mathbf{0} = \mathbf{0}$.

To *compute* $\text{NS}(A)$, solve the system $A\mathbf{x} = \mathbf{0}$ by using Gaussian elimination and back-substitution. (Shortcut: For a homogeneous system, there is no need to keep track of an augmented column, because it would consist of zeros, and would stay that way even after row operations.)

20.4. Linearly dependent vectors, basis, dimension. This is review from 18.02.

Definition 20.14. Vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ are **linearly dependent** if one of them is a linear combination of the others.

Definition 20.15 (equivalent). Vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ are **linearly dependent** if there exist scalars $c_1, \dots, c_n$ *not all zero* such that $c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n = \mathbf{0}$.

Definition 20.16. Let S be a vector space (of vectors). A **basis** of S is a list of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots$ such that

1. $\text{Span}(\mathbf{v}_1, \mathbf{v}_2, \dots) = S$, and
2. $\mathbf{v}_1, \mathbf{v}_2, \dots$ are linearly **in**dependent.

Example 20.17. If S is the xy -plane in $\mathbb{R}^3$, then $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is a basis for S .

It turns out that every basis for a vector space has the same number of vectors.

Definition 20.18. Let S be a vector space. The **dimension** of S is the number of vectors in any basis of S .

Example 20.19. The line $x + 3y = 0$ in $\mathbb{R}^2$ is a vector space L . The vector $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ by itself is a basis for L , so the dimension of L is 1. (Not a big surprise!)

The dimension of $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_n)$ will be n if the vectors are linearly independent, and less than n if they are linearly dependent.

20.5. Basis of a nullspace.

Example 20.20. Let's solve $A\mathbf{x} = \mathbf{0}$ for

$$A = \begin{pmatrix} 1 & 2 & 0 & 2 & 3 \\ 0 & -1 & 2 & 3 & 1 \\ -2 & -4 & 0 & -4 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

Add 2 times the first row to the third row. This already puts the matrix in row-echelon form:

$$B = \begin{pmatrix} 1 & 2 & 0 & 2 & 3 \\ 0 & -1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This B corresponds to the homogeneous system already solved in Section 19.9: the result is

$$\text{NS}(A) = \text{NS}(B) = \text{Span} \left(\begin{pmatrix} -8 \\ 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right).$$

In other words, the general solution to $A\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x} = c_1 \begin{pmatrix} -8 \\ 3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -4 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -8c_1 - 4c_2 \\ 3c_1 + 2c_2 \\ c_2 \\ c_1 \\ 0 \end{pmatrix}.$$

Are there numbers c_1 and c_2 that make this combination $\mathbf{0}$? If that happens, then the blue entries on the right are 0, so $c_1 = 0$ and $c_2 = 0$. Thus

$$\begin{pmatrix} -8 \\ 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

are linearly independent. Since they span $\text{NS}(A)$ and are linearly independent, they form a *basis* of $\text{NS}(A)$. Since $\text{NS}(A)$ has a basis consisting of 2 vectors, $\dim \text{NS}(A) = 2$.

The same argument applies whenever we use Gaussian elimination and back-substitution to solve a homogeneous linear system $A\mathbf{x} = \mathbf{0}$: the list of solutions found spans $\text{NS}(A)$ and is automatically linearly independent, so it is a basis of $\text{NS}(A)$.

Steps to find a basis of $\text{NS}(A)$:

1. Perform Gaussian elimination on A to convert it to a matrix B in row-echelon form.
2. Use back-substitution to find the general solution to $B\mathbf{x} = \mathbf{0}$.
3. The general solution will be expressed as the general linear combination of a list of vectors; that list is a basis of $\text{NS}(A)$.

20.6. Dimension of a nullspace.

Theorem 20.21 (Formula for $\dim \text{NS}(A)$). Suppose that the result of putting a matrix A in row-echelon form is B . Then $\text{NS}(A) = \text{NS}(B)$, and

$$\dim \text{NS}(A) = \# \text{non-pivot columns of } B.$$

Proof. Row reductions do not change the solutions to a linear system, so $\text{NS}(A) = \text{NS}(B)$. Now

$$\begin{aligned} \dim \text{NS}(A) &= \dim \text{NS}(B) \\ &= \# \text{vectors in a basis of } \text{NS}(B) \\ &= \# \text{parameters in the general solution} \\ &= \# \text{free variables} \\ &= \# \text{non-pivot columns of } B. \end{aligned} \quad \square$$

Finding the *dimension* of $\text{NS}(A)$ is easier than finding a *basis* of $\text{NS}(A)$, because after finding a row-echelon form B , there is no need to do the back-substitution.

Steps to compute $\dim \text{NS}(A)$:

1. Perform Gaussian elimination on A to convert it to a matrix B in row-echelon form.
2. Identify the pivots of B .
3. Count the number of *non-pivot* columns of B ; that number is $\dim \text{NS}(A)$.

Question 20.22. The matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 2 & 3 \\ 0 & -1 & 2 & 3 & 1 \\ -2 & -4 & 0 & -4 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

of Example 20.20 has pivots in column 1 and column 2 only, so it has 3 non-pivot columns. Does this mean that $\dim \text{NS}(A) = 3$?

NO! You must convert to row-echelon form before even *talking* about non-pivot columns. Here's what we should have said:

$$\dim \text{NS}(A) = \# \text{non-pivot columns of } B = 2.$$

Warnings:

- You must put the matrix in row-echelon form before counting non-pivot columns!
- When solving a homogeneous system, *do not append an augmented column of zeros*, and even if you do, *do not include it in the count of non-pivot columns*. The augmented column does not correspond to a free variable, or any variable at all for that matter, so it should not be counted.

20.7. Checking whether vectors are linearly dependent.

Steps for testing whether given vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ are linearly dependent:

1. Create a matrix A whose columns are $\mathbf{v}_1, \dots, \mathbf{v}_n$.
2. Compute $\text{NS}(A)$.
3.
 - If $\text{NS}(A)$ contains any nonzero vector, then $\mathbf{v}_1, \dots, \mathbf{v}_n$ are linearly dependent.

In fact, any nonzero vector $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ in $\text{NS}(A)$ gives a linear dependence

$$a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{0}.$$

- If $\text{NS}(A) = \{\mathbf{0}\}$, then $\mathbf{v}_1, \dots, \mathbf{v}_n$ are linearly independent.

We'll soon explain why this works. But first, let's give an example:

Problem 20.23. Determine whether the vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$ are linearly dependent.

Solution:

1. Create $A = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$.
2. We convert A to row-echelon form. First add -2 times the first row to the second row, and add -3 times the first row to the third row, to get

$$\begin{pmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix}.$$

Now the first row is done. Add -2 times the second column to the third column to get

$$\begin{pmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix},$$

which is in row-echelon form. Solve the corresponding system

$$x + 4y + 7z = 0$$

$$-3y - 6z = 0$$

by back-substitution: $z = c_1$, $y = -2c_1$, $x = -4(-2c_1) - 7c_1 = c_1$, so the general solution to $A\mathbf{x} = \mathbf{0}$ is $\begin{pmatrix} c_1 \\ -2c_1 \\ c_1 \end{pmatrix}$,

$$\text{NS}(A) = \text{Span} \left(\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right).$$

3. Since $\text{NS}(A)$ contains a nonzero vector, the three given vectors are linearly dependent. $\square$

Actually, to answer the stated question, we could have stopped after finding the row-echelon form and seeing that there was a non-pivot column, since a non-pivot column means that there is a free variable, which means that there will be nonzero vectors in $\text{NS}(A)$.

The advantage of doing the back-substitution to actually *find* a nonzero vector in $\text{NS}(A)$ is that it tells us *which* combination of the three vectors is $\mathbf{0}$. In the example, $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ was in $\text{NS}(A)$, so

$$1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + (-2) \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + 1 \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = \mathbf{0}.$$

This is a nontrivial linear dependence showing that the three vectors are linearly dependent.

Although this is not necessary, we could also then solve for one of the vectors as a linear combination of the others:

$$\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = (-1) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 2 \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}.$$

This too shows that the three vectors are linearly dependent, using the other definition.

Question 20.24. Why does this algorithm work?

Answer: Checking for linear dependence is the same as searching for nonzero solutions to

$$a_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + a_2 \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + a_3 \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = \mathbf{0}.$$

By the interpretation of matrix-vector multiplication as a linear combination of columns, this equation is the same as

$$\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \mathbf{0},$$

so what we are really looking for is a nonzero vector $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ in the nullspace of A . $\square$

Question 20.25. Suppose that you have a system in row-echelon form, and you solve it using back-substitution to get a list of vectors. Do you then need to run the algorithm of this section to check that the list is independent?

Answer: No, it's not necessary, because a list of vectors arising that way is automatically linearly independent.

20.8. Review: solving a homogeneous linear system. Not done in lecture.

Problem 20.26. Find a basis of the vector space of solutions to the homogeneous linear system

$$2x + y - 3z + 4w = 0$$

$$4x + 2y - 2z + 3w = 0$$

$$2x + y - 7z + 9w = 0.$$

Solution: The system is $A\mathbf{x} = \mathbf{0}$ for

$$A := \begin{pmatrix} 2 & 1 & -3 & 4 \\ 4 & 2 & -2 & 3 \\ 2 & 1 & -7 & 9 \end{pmatrix}.$$

First we convert the matrix to row-echelon form. Add -2 times the first row to the second row, and -1 times the first row to the third row to get

$$\begin{pmatrix} 2 & 1 & -3 & 4 \\ 0 & 0 & 4 & -5 \\ 0 & 0 & -4 & 5 \end{pmatrix}$$

and then add the second row to the third row to get

$$B := \begin{pmatrix} \textcolor{red}{2} & 1 & -3 & 4 \\ 0 & 0 & \textcolor{red}{4} & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

which is in row-echelon form (the pivots are identified in **red**). Solve the corresponding system

$$2\textcolor{red}{x} + y - 3z + 4w = 0$$

$$4\textcolor{red}{z} - 5w = 0$$

$$0 = 0$$

by back-substitution:

$$w = c_1$$

$$z = \frac{5}{4}c_1$$

$$y = c_2$$

$$x = (-y + 3z - 4w)/2 = -\frac{1}{8}c_1 - \frac{1}{2}c_2,$$

$$\text{General solution: } \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -\frac{1}{8}c_1 - \frac{1}{2}c_2 \\ c_2 \\ \frac{5}{4}c_1 \\ c_1 \end{pmatrix} = c_1 \begin{pmatrix} -1/8 \\ 0 \\ 5/4 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1/2 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

$$\text{NS}(A) = \text{Span} \left(\begin{pmatrix} -1/8 \\ 0 \\ 5/4 \\ 1 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right).$$

$$\text{Basis for NS}(A): \begin{pmatrix} -1/8 \\ 0 \\ 5/4 \\ 1 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \text{ since these are automatically linearly independent.}$$

(As a check, plug each vector back into the original system.) $\square$

March 21

21. INHOMOGENEOUS LINEAR SYSTEMS

A linear system is called **consistent** if it has at least one solution, and **inconsistent** if there are no solutions.

- A homogeneous linear system is always consistent, since $\mathbf{0}$ is a solution.
- An inhomogeneous linear system can be consistent or inconsistent.

Problem 21.1. Find the general solution to the system with augmented matrix $\left(\begin{array}{ccc|c} 2 & 3 & 5 & 7 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 9 \end{array} \right)$.

Solution: The last equation says

$$0x + 0y + 0z = 9,$$

i.e., $0 = 9$, which cannot be satisfied. So there are no solutions! The system is inconsistent. $\square$

21.1. Algorithm to test if a linear system is consistent (has a solution).

Consider a linear system of m equations in n variables.

1. Construct the $m \times (n + 1)$ augmented matrix.
2. Put it in row-echelon form. Call this row-echelon form B .
3. Look for a row that is all zero except for a nonzero entry in the augmented column.
- 4a. If B has such a row, that row corresponds to an equation

$$0x_1 + \cdots + 0x_n = \underset{\text{nonzero number}}{b},$$

so the linear system is **inconsistent**.

- 4b. Otherwise, we can solve the system by back-substitution, so the linear system is **consistent**.
In a solution, the free variables may take any values, but in terms of these one can solve for the dependent variables in reverse order, so

$\# \text{parameters in general solution} = \underbrace{\# \text{non-pivot columns excluding the augmented column}}_{\text{\#free variables}}.$

21.2. Inhomogeneous linear systems: theory. For an inhomogeneous linear system $A\mathbf{x} = \mathbf{b}$, there are two possibilities:

1. There are no solutions.
2. There exists a solution. In this case, if $\mathbf{x}_p$ is a particular solution to $A\mathbf{x} = \mathbf{b}$, and $\mathbf{x}_h$ is the *general* solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$, then $\mathbf{x} := \mathbf{x}_p + \mathbf{x}_h$ is the general solution to $A\mathbf{x} = \mathbf{b}$.

Here is why: Suppose that a solution exists; let $\mathbf{x}_p$ be one, so $A\mathbf{x}_p = \mathbf{b}$. If $\mathbf{x}_h$ satisfies $A\mathbf{x}_h = \mathbf{0}$, adding the two equations gives $A(\mathbf{x}_p + \mathbf{x}_h) = \mathbf{b}$, so adding $\mathbf{x}_p$ to $\mathbf{x}_h$ produces a solution $\mathbf{x}$ to the inhomogeneous equation. Every solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{b}$ arises this way from some $\mathbf{x}_h$ (specifically, from $\mathbf{x}_h := \mathbf{x} - \mathbf{x}_p$, which satisfies $A\mathbf{x}_h = A\mathbf{x} - A\mathbf{x}_p = \mathbf{b} - \mathbf{b} = \mathbf{0}$).

Remark 21.2. To *solve* $A\mathbf{x} = \mathbf{b}$, however, don't use $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$. Instead use Gaussian elimination and back-substitution. The above is just to describe the shape of the solution.

21.3. Column space.

Problem 21.3. For which vectors $\mathbf{b} \in \mathbb{R}^2$ does the inhomogeneous linear system

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{b}$$

have a solution?

Answer: The left hand side can be rewritten as

$$\begin{pmatrix} x_1 + 2x_2 + 3x_3 \\ 2x_1 + 4x_2 + 6x_3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 4 \end{pmatrix} + x_3 \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

Thus, saying that the system has a solution is the same as saying that

$$\mathbf{b} \text{ is a linear combination of } \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \end{pmatrix},$$

or equivalently, that

$$\mathbf{b} \text{ is in the span of } \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \end{pmatrix}. \quad \square$$

Definition 21.4. The **column space** of a matrix A is the span of its columns. The notation for it is $\text{CS}(A)$. (It is also called the **image** of A , and written $\text{im}(A)$; the reason will be clearer when we talk about the geometric interpretation.)

Since $\text{CS}(A)$ is a span, it is a vector space.

Here is what happens in general for linear systems (the explanation is the same as in the example above):

Theorem 21.5. *The linear system $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is in $\text{CS}(A)$.*

For the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}$ in the problem above,

$$\text{CS}(A) = \text{the span of } \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \end{pmatrix},$$

which is the line $y = 2x$ in $\mathbb{R}^2$, a 1-dimensional vector space.

Steps to compute a basis for $\text{CS}(A)$:

1. Perform Gaussian elimination to convert A to a matrix B in row-echelon form.
2. Identify the pivot columns of B .
3. The *corresponding columns* of A are a basis for $\text{CS}(A)$.

Here is a summary of why this works (not discussed in lecture). Let C be the reduced row-echelon form of A . If

$$\text{fifth column} = 3(\text{first column}) + 7(\text{second column})$$

is true for a matrix, it will remain true after any row operation. Similarly, any linear relation between columns is preserved by row operations. So the linear relations between columns of A are the same as the linear relations between columns of C . The condition that certain numbered columns (say the first, second, and fourth) of a matrix form a basis is expressible in terms of which linear relations hold, so if certain columns form a basis for $\text{CS}(C)$, the same numbered columns will form a basis for $\text{CS}(A)$. Also, performing Gauss–Jordan reduction on B to obtain C in reduced row-echelon form does not change the pivot locations. Thus it will be enough to show that the pivot columns of C form a basis of $\text{CS}(C)$. Since C is in reduced row-echelon form, the pivot columns of C are the first r of the m standard basis vectors for $\mathbb{R}^m$, where r is

the number of nonzero rows of C . These columns are linearly independent, and every other column is a linear combination of them, since the entries of C below the first r rows are all zeros. Thus the pivot columns of C form a basis of $\text{CS}(C)$.

In particular,

$$\dim \text{CS}(A) = \# \text{pivot columns of } B.$$

Warning: Usually $\text{CS}(A) \neq \text{CS}(B)$.

Problem 21.6. Let A be the 3×5 matrix $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ -1 & -2 & 9 & 10 & 11 \\ 1 & 2 & 9 & 11 & 13 \end{pmatrix}$.

- (a) Find a basis for $\text{CS}(A)$.
- (b) What are $\dim \text{NS}(A)$ and $\dim \text{CS}(A)$?

Solution:

- (a) First we must find a row-echelon form. Add the first row to the second, and add -1 times the first row to the third:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 12 & 14 & 16 \\ 0 & 0 & 6 & 7 & 8 \end{pmatrix}.$$

Add $-1/2$ times the second row to the third:

$$B := \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 12 & 14 & 16 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This is in row-echelon form.

Basis for $\text{CS}(B)$: first and third columns (the pivot columns) of B , i.e., $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 12 \\ 0 \end{pmatrix}$.

This is not what was asked for!

Basis for $\text{CS}(A)$: first and third columns of A , i.e., $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 9 \\ 9 \end{pmatrix}$.

- (b)

$$\dim \text{NS}(A) = \# \text{non-pivot columns of } B = 3$$

$$\dim \text{CS}(A) = \# \text{pivot columns of } B = 2. \quad \square$$

21.4. Rank.

Definition 21.7. The **rank** of A is defined by

$$\text{rank}(A) := \dim \text{CS}(A).$$

Rank-nullity theorem. For any $m \times n$ matrix A ,

$$\boxed{\dim \text{NS}(A) + \text{rank}(A) = n}.$$

Proof.

$$\begin{aligned} \dim \text{NS}(A) + \text{rank}(A) &= (\# \text{non-pivot columns of } B) + (\# \text{pivot columns of } B) \\ &= \# \text{columns of } B \\ &= n. \end{aligned}$$

□

21.5. Computing a basis for a span.

Problem 21.8. Given vectors $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{R}^m$, how can one compute a basis of $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_n)$?

Solution:

1. Form the matrix A whose columns are $\mathbf{v}_1, \dots, \mathbf{v}_n$.
2. Find a basis for $\text{CS}(A)$ as above (columns of A corresponding to pivot columns of B).

21.6. Example: a projection. Let $\mathbf{f}$ be the function from $\mathbb{R}^3$ to $\mathbb{R}^3$ that projects all of $\mathbb{R}^3$ onto the xy -plane:

$$\mathbf{f} \begin{pmatrix} x \\ y \\ z \end{pmatrix} := \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}.$$

Problem 21.9. What is the matrix A that represents f ?

Solution: The matrix A is a 3×3 matrix such that

$$\begin{aligned} (\text{first column of } A) &= \mathbf{f} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ (\text{second column of } A) &= \mathbf{f} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ (\text{third column of } A) &= \mathbf{f} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

Thus $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. $\square$

$\text{NS}(A)$ is a subspace of the *input* space:

$$\begin{aligned} \text{NS}(A) &= \{\text{solutions to } A\mathbf{x} = \mathbf{0}\} \\ &= \{\text{solutions to } \mathbf{f} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0}\} \\ &= \left\{ \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} : z \in \mathbb{R} \right\} \\ &= \text{the } z\text{-axis in the input space } \mathbb{R}^3. \end{aligned}$$

The image $\text{CS}(A)$ is a subspace of the *output* space:

$$\begin{aligned} \text{CS}(A) &= \{\text{values of } A\mathbf{x}\} \\ &= \left\{ \text{values of } \mathbf{f} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\} \\ &= \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} : x, y \in \mathbb{R} \right\} \\ &= \text{the } xy\text{-plane in the output space } \mathbb{R}^3. \end{aligned}$$

Here $\text{rank}(A) = \dim \text{CS}(A) = 2$.

The linear transformation $\mathbf{f}$ crushes $\text{NS}(A)$ to the point $\mathbf{0}$ in the output space, and it flattens the whole input space $\mathbb{R}^3$ onto $\text{CS}(A)$ in the output space. Of the 3 input dimensions, 1 is crushed, so $3 - 1 = 2$ dimensions are left.

(Mathematically, “there are m crushed dimensions” means just that $\text{NS}(A)$ is m -dimensional.)

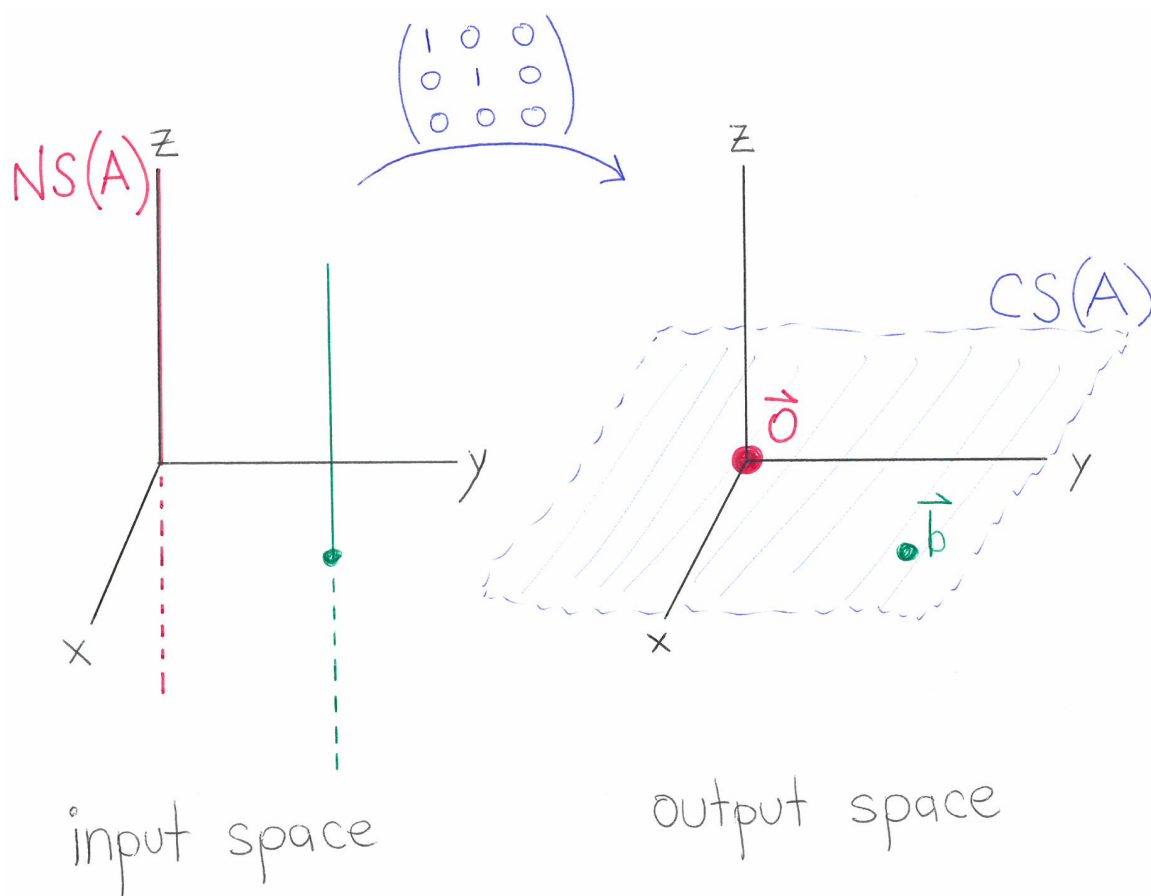
In general, for a linear transformation $\mathbf{f}: \mathbb{R}^n \rightarrow \mathbb{R}^m$ represented by an $m \times n$ matrix A , of the n input dimensions, $\dim \text{NS}(A)$ of them are crushed, leaving an image of dimension $n - \dim \text{NS}(A)$. This explains geometrically why

$$\dim \text{CS}(A) = n - \dim \text{NS}(A),$$

which is the same as the rank-nullity theorem

$$\dim \text{NS}(A) + \text{rank}(A) = n$$

we stated earlier.



Back to the example: What does the solution set to $A\mathbf{x} = \mathbf{b}$ look like?

- If $\mathbf{b}$ is not in $CS(A)$, then there are no solutions.

- If $\mathbf{b}$ is in $CS(A)$, say $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ 0 \end{pmatrix}$, then

$$\begin{aligned} \{\text{solutions to } A\mathbf{x} = \mathbf{b}\} &= \left\{ \text{solutions to } \mathbf{f} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ 0 \end{pmatrix} \right\} \\ &= \left\{ \begin{pmatrix} b_1 \\ b_2 \\ z \end{pmatrix} : z \in \mathbb{R} \right\} \end{aligned}$$

= a vertical line parallel to the z -axis in the input space $\mathbb{R}^3$.

The set of solutions to the homogeneous system $A\mathbf{x} = \mathbf{0}$ is the line $NS(A)$. To get from $NS(A)$ to the set of solutions to $A\mathbf{x} = \mathbf{b}$, choose a particular solution vector to $A\mathbf{x} = \mathbf{b}$ and add it to every vector in $NS(A)$.

Problem 21.10. What is the volume scaling factor?

Solution 1: It's the absolute value of the determinant:

$$\left| \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right| = |0| = 0.$$

Solution 2: The linear transformation $\mathbf{f}$ takes any unit cube of volume 1 to a flat object of volume 0, so volume is getting multiplied by 0.

22. SQUARE MATRICES

22.1. Determinants. Review of 18.02 material.

To each *square* matrix A is associated a number called the **determinant**:

$$\det \begin{pmatrix} a \end{pmatrix} := a$$

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} := ad - bc$$

$$\det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} := a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - c_1 b_2 a_3 - c_2 b_3 a_1 - c_3 b_1 a_2.$$

Alternative notation for determinant: $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$. (This is a scalar, not a matrix!)

Geometric meaning: The *absolute value* of $\det A$ is the **area scaling factor** (or **volume scaling factor** or...).

Laplace expansion (along the first row) for a 3×3 determinant:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = +a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

The general rule leading to the formula above is this:

- (1) Move your finger along the entries in a row.
- (2) At each position, compute the **minor**, defined as the smaller determinant obtained by crossing out the row and the column through your finger; then multiply the minor by the number you are pointing at, and adjust the sign according to the checkerboard pattern

$$\begin{array}{ccc} + & - & + \\ - & + & - \\ + & - & + \end{array}$$

(the pattern always starts with + in the upper left corner).

(3) Add up the results.

There is a similar expansion for a determinant of any size, computed along any row or column.

The **diagonal** of a matrix consists of the entries a_{ij} with $i = j$.

A **diagonal matrix** is a matrix that has zeros everywhere outside the diagonal:

$$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$$

(it may have some zeros along the diagonal too).

An **upper triangular matrix** is a matrix whose entries strictly below the diagonal are all 0:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$$

(the entries on or above the diagonal may or may not be 0).

Example 22.1. Any square matrix in row-echelon form is upper triangular.

Theorem 22.2. *The determinant of an upper triangular matrix equals the product of the diagonal entries.*

For example,

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} = a_{11} a_{22} a_{33}.$$

Why is the theorem true? The Laplace expansion along the first column shows that the determinant is a_{11} times a upper triangular minor with diagonal entries $a_{22}, \dots, a_{nn}$.

Properties of determinants:

1. Interchanging two rows changes the sign of $\det A$.
2. Multiplying an entire row by a scalar c multiplies $\det A$ by c .
3. Adding a multiple of a row to another row does not change $\det A$.
4. If one of the rows is all 0, then $\det A = 0$.
5. $\det(AB) = \det(A) \det(B)$ (assuming that A, B are square matrices of the same size).

In particular, row operations multiply $\det A$ by nonzero scalars, but do not change whether $\det A = 0$.

Question 22.3. Suppose that A is a 3×3 matrix such that $\det A = 5$. Doubling every entry of A gives a matrix $2A$. What is $\det(2A)$?

Solution: Each time we multiply a row by 2, the determinant gets multiplied by 2. We need to do this three times to double the whole matrix A , so the determinant gets multiplied by $2 \cdot 2 \cdot 2 = 8$. Thus $\det(2A) = 8\det(A) = 40$.

March 31

22.2. Inverse matrices.

Example 22.4. Suppose that A is a square matrix and $\det A \neq 0$. Then $\text{RREF}(A)$ has nonzero determinant too, but is now upper triangular, so its diagonal entries are nonzero. In fact, these diagonal entries are 1 since they are pivots of a RREF matrix. Moreover, all non-diagonal entries are 0, by definition of RREF. So $\text{RREF}(A) = I$.

Now imagine solving $A\mathbf{x} = \mathbf{b}$. Gauss–Jordan elimination converts the augmented matrix $[A|\mathbf{b}]$ to $[I|\mathbf{c}]$ for some vector $\mathbf{c}$. Thus $A\mathbf{x} = \mathbf{b}$ has the same solutions as $I\mathbf{x} = \mathbf{c}$; the unique solution is $\mathbf{c}$.

What if instead we wanted to solve many equations with the same A , say, $A\mathbf{x}_1 = \mathbf{b}_1, \dots, A\mathbf{x}_p = \mathbf{b}_p$? Use many augmented columns! Gauss–Jordan elimination converts $[A|\mathbf{b}_1 \dots \mathbf{b}_p]$ to $[I|\mathbf{c}_1 \dots \mathbf{c}_p]$, and $\mathbf{c}_1, \dots, \mathbf{c}_p$ are the solutions.

In other words, to solve an equation $AX = B$ to find the unknown matrix X , convert $[A|B]$ to RREF $[I|C]$. Then C is the solution. $\square$

We included the previous example to help us compute inverse matrices; let's define these now.

Definition 22.5. The *inverse* of an $n \times n$ matrix A is an $n \times n$ matrix A^{-1} such that

$$AA^{-1} = I \quad \text{and} \quad A^{-1}A = I.$$

It exists if and only if $\det A \neq 0$.

Suppose that A represents the linear transformation $\mathbf{f}$. Then A^{-1} exists if and only if an inverse function $\mathbf{f}^{-1}$ exists; in that case, A^{-1} represents $\mathbf{f}^{-1}$.

Problem 22.6. Does the rotation matrix $R := \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ have an inverse? If so, what is it?

Solution: The inverse linear transformation is rotation by $-\theta$, so

$$R^{-1} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.$$

(As a check, try multiplying R by this matrix, in either order.) $\square$

Problem 22.7. Does the projection matrix $A := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ have an inverse? If so, what is it?

Solution: The associated linear transformation $\mathbf{f}$ is not a 1-to-1 correspondence, because it maps more than one vector to $\mathbf{0}$ (it maps the whole z -axis to $\mathbf{0}$). Thus $\mathbf{f}^{-1}$ does not exist, so A^{-1} does not exist. $\square$

Suppose that $\det A \neq 0$. In 18.02, you learned a formula for the inverse of a 2×2 matrix:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

For larger square matrices A , there is also a formula for A^{-1} , in terms of the “cofactor matrix”, but using this formula is not the fastest way to compute A^{-1} , especially for big matrices. Here is the better way:

Steps to compute A^{-1} :

1. Form the $n \times 2n$ augmented matrix $[A|I]$.
2. Convert to RREF; the result will be $[I|B]$ for some $n \times n$ matrix B .
3. Then $A^{-1} = B$.

This is a special case of Example 22.4 since A^{-1} is the solution to $AX = I$.

22.3. Conditions for invertibility. There are two types of square matrices A :

- those with $\det A \neq 0$ (called **nonsingular** or **invertible**), and
- those with $\det A = 0$ (called **singular**).

The answer to the one question “Is $\det A = 0$?” determines a lot about the geometry of A and about solving systems $A\mathbf{x} = \mathbf{b}$, as we’ll now explain.

22.3.1. Nonsingular matrices.

Theorem 22.8. *For a square $n \times n$ matrix A , the following are equivalent:*

1. $\det A \neq 0$ (scaling factor is positive)
2. $\text{NS}(A) = \{\mathbf{0}\}$ (the only solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{0}$)
3. $\text{rank}(A) = n$ (image is n -dimensional)
4. $\text{CS}(A) = \mathbb{R}^n$ (image is the whole space $\mathbb{R}^n$)
5. For each vector $\mathbf{b}$, the system $A\mathbf{x} = \mathbf{b}$ has exactly one solution.
6. A^{-1} exists.
7. $\text{RREF}(A) = I$.

So if you have a matrix A for which one of these conditions holds, then all of the conditions hold for A .

Let's explain the consequences of $\det A \neq 0$. Suppose that $\det A \neq 0$. Then the volume scaling factor is not 0, so the input space $\mathbb{R}^n$ is not flattened by A . This means that there are no “crushed dimensions”, so $\text{NS}(A) = \{\mathbf{0}\}$. Since no dimensions were crushed, the image $\text{CS}(A)$ has the same dimension as the input space, namely n . By definition, $\text{rank}(A) = \dim \text{CS}(A) = n$. (Alternatively, this follows from $\dim \text{NS}(A) + \text{rank}(A) = n$.) The only n -dimensional subspace of $\mathbb{R}^n$ is $\mathbb{R}^n$ itself, so $\text{CS}(A) = \mathbb{R}^n$. Thus every $\mathbf{b}$ is in $\text{CS}(A)$, so $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$. The system $A\mathbf{x} = \mathbf{b}$ has the same number of solutions as $A\mathbf{x} = \mathbf{0}$ (they are just shifted by adding a particular solution $\mathbf{x}_p$); that number is 1 (the only solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{0}$). To say that $A\mathbf{x} = \mathbf{b}$ has exactly one solution for each $\mathbf{b}$ means that the associated linear transformation $\mathbf{f}$ is a 1-to-1 correspondence, so $\mathbf{f}^{-1}$ exists, so A^{-1} exists. (Moreover, we showed how to find A^{-1} by applying Gauss–Jordan elimination to $[A|I]$.) We have $\text{RREF}(A) = I$ as explained earlier, since I is the only RREF square matrix with nonzero determinant.

22.3.2. *Singular matrices.* The same theorem can be stated in terms of the opposite conditions (it's essentially the same theorem, so this is really just review):

Theorem 22.9. *For a square matrix A , the following are equivalent:*

1. $\det A = 0$ (scaling factor is 0)
2. $\text{NS}(A)$ is larger than $\{\mathbf{0}\}$ (i.e., $A\mathbf{x} = \mathbf{0}$ has a nonzero solution)
3. $\text{rank}(A) < n$ (image has dimension less than n)
4. $\text{CS}(A)$ is smaller than $\mathbb{R}^n$ (image is not the whole space $\mathbb{R}^n$)
5. The system $A\mathbf{x} = \mathbf{b}$ has no solutions for some vectors $\mathbf{b}$, and infinitely many solutions for other vectors $\mathbf{b}$.
6. A^{-1} does not exist.
7. $\text{RREF}(A) \neq I$.

Now let's explain the consequences of $\det A = 0$. **To be done in 4/1/25 recitation.**

Suppose that $\det A = 0$. Then the volume scaling factor is 0, so the input space is flattened by A . This means that some input dimensions are getting crushed, so $\text{NS}(A)$ is larger than $\{\mathbf{0}\}$ (at least 1-dimensional), and the image is smaller than the n -dimensional input space: $\text{rank}(A) < n$. In particular, the image $\text{CS}(A)$ is not all of $\mathbb{R}^n$.

- If $\mathbf{b} \notin \text{CS}(A)$, then $A\mathbf{x} = \mathbf{b}$ has no solution.
- If $\mathbf{b} \in \text{CS}(A)$, then $A\mathbf{x} = \mathbf{b}$ has the same number of solutions as $A\mathbf{x} = \mathbf{0}$, i.e., infinitely many since $\dim \text{NS}(A) \geq 1$.

The associated linear transformation $\mathbf{f}$ is not a 1-to-1 correspondence (it maps many vectors to $\mathbf{0}$); thus $\mathbf{f}^{-1}$ does not exist, so A^{-1} does not exist. Row operations do not change the

condition $\det A = 0$, so $\det \text{RREF}(A) = 0$, so definitely $\text{RREF}(A) \neq I$. (In fact, $\text{RREF}(A)$ must have at least one 0 along the diagonal.)

Problem 22.10. Devise a test for deciding whether a homogeneous square system $A\mathbf{x} = \mathbf{0}$ has a nonzero solution.

Solution: Compute $\det A$. If $\det A = 0$, there exists a nonzero solution. If $\det A \neq 0$, then $A\mathbf{x} = \mathbf{0}$ has only the zero solution. $\square$

22.4. Characteristic polynomial, eigenvalues, eigenvectors, again. Suppose that A is an $n \times n$ matrix.

Characteristic polynomial: $\det(\lambda I - A)$. (If instead you use $\det(A - \lambda I)$, you will need to change the sign when n is odd.)

Expanding out the determinant shows that the characteristic polynomial has the form

$$\lambda^n - (\text{tr } A)\lambda^{n-1} + \cdots \pm \det A$$

(the $\pm$ is $+$ if n is even, and $-$ if n is odd).

Conclusion:

- For a 2×2 matrix A , there is a shortcut for computing the characteristic polynomial: it is

$$\lambda^2 - (\text{tr } A)\lambda + \det A.$$

- For 3×3 and larger, the trace and determinant tell you only *some* of the coefficients; to compute the whole characteristic polynomial, you must compute $\det(\lambda I - A)$.

Eigenvalue: a scalar λ such that $A\mathbf{v} = \lambda\mathbf{v}$ for some nonzero vector $\mathbf{v}$.

To compute the eigenvalues, find the roots of the characteristic polynomial.

If $\lambda_1, \dots, \lambda_n$ are all the (possibly complex) eigenvalues, listed with multiplicity, then the characteristic polynomial is

$$(\lambda - \lambda_1) \cdots (\lambda - \lambda_n) = \lambda^n - (\lambda_1 + \cdots + \lambda_n)\lambda^{n-1} + \cdots \pm \lambda_1 \cdots \lambda_n.$$

Comparing coefficients with the previous displayed equation shows that

$$\text{tr } A = \lambda_1 + \lambda_2 + \cdots + \lambda_n \quad (\text{the sum of the eigenvalues})$$

$$\det A = \lambda_1 \lambda_2 \cdots \lambda_n \quad (\text{the product of the eigenvalues}).$$

Problem 22.11. Find the eigenvalues of the upper triangular matrix $A := \begin{pmatrix} 2 & 3 & 5 \\ 0 & 2 & 7 \\ 0 & 0 & 6 \end{pmatrix}$.

Solution: The characteristic polynomial is

$$\det(\lambda I - A) = \det \begin{pmatrix} \lambda - 2 & -3 & -5 \\ 0 & \lambda - 2 & -7 \\ 0 & 0 & \lambda - 6 \end{pmatrix} = (\lambda - 2)(\lambda - 2)(\lambda - 6),$$

so the eigenvalues, listed with multiplicity, are 2, 2, 6. $\square$

In general, for any upper triangular or lower triangular matrix, the eigenvalues are the diagonal entries.

Eigenvector associated to λ : a vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$.

22.5. Eigenspaces. For each eigenvalue λ of A , define

$$\begin{aligned} \text{eigenspace of } \lambda &:= \{\text{all eigenvectors associated to } \lambda, \text{ and } \mathbf{0}\} \\ &= \{\text{all solutions to } (A - \lambda I)\mathbf{v} = \mathbf{0}\} \\ &= \text{NS}(A - \lambda I). \end{aligned}$$

There is one eigenspace for each eigenvalue. Each eigenspace is a vector space, so it can be described as the span of a basis. To compute the eigenspace of λ , compute $\text{NS}(A - \lambda I)$ by Gaussian elimination and back-substitution.

Problem 22.12. (Skipped) Find the eigenvalues and eigenvectors of the 90° counterclockwise rotation matrix $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Solution: Since $\text{tr } R = 0$ and $\det R = 1$, the characteristic polynomial of R is $\lambda^2 + 1$. Its roots are i and $-i$; these are the eigenvalues.

The eigenspace of i is $\text{NS}(R - iI)$. Converting

$$R - iI = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}$$

to row-echelon form gives $\begin{pmatrix} -i & -1 \\ 0 & 0 \end{pmatrix}$, so we solve $-ix - y = 0$ by back-substitution and get

the general solution $c \begin{pmatrix} i \\ 1 \end{pmatrix}$. Thus the eigenvectors having eigenvalue i are the nonzero scalar multiples of $\begin{pmatrix} i \\ 1 \end{pmatrix}$.

Applying complex conjugation to the entire universe shows that the eigenvectors having eigenvalue $-i$ are the nonzero scalar multiples of $\begin{pmatrix} -i \\ 1 \end{pmatrix}$. $\square$

22.5.1. Dimension of an eigenspace.

Theorem 22.13. Let λ be an eigenvalue of an $n \times n$ matrix A . Let m be the multiplicity of λ as a root of the characteristic polynomial. Then

$$1 \leq (\text{dimension of eigenspace of } \lambda) \leq m.$$

Given λ , the dimension of the eigenspace of λ is also the maximum number of linearly independent eigenvectors of eigenvalue λ that can be found. This dimension is at least 1 since A has at least one nonzero eigenvector of eigenvalue λ (otherwise λ would not have been an eigenvalue). That this dimension is at most m requires more work to prove, and we're not going to do it in this class.

Problem 22.14. A 9×9 matrix has characteristic polynomial $(\lambda - 2)^3(\lambda - 5)^6$. What are the possibilities for the dimension of the eigenspace of 2?

Solution: The multiplicity of the eigenvalue 2 is $m = 3$, so the dimension is 1, 2, or 3. $\square$

Definition 22.15. The eigenspace of λ is called **complete** if its dimension equals the multiplicity m of λ , and **deficient** if its dimension is less than m . **Warning:** Different authors use different terminology here.

Example 22.16. If the multiplicity is 1, then the dimension of the eigenspace is sandwiched between 1 and 1, so the eigenspace is complete.

Definition 22.17. A matrix is **complete** if *all* its eigenspaces are complete. A matrix is **deficient** if at least one of its eigenspaces is deficient.

For the application to solving linear systems of ODEs, given an $n \times n$ matrix A we will want to find as many linearly independent eigenvectors as possible. To do this, we choose a basis of each eigenspace, and concatenate these lists of eigenvectors; it turns out that the resulting long list is linearly independent.

How many eigenvectors are in this list? Well,

$$\begin{aligned} \# \text{eigenvectors} &= \sum_{\lambda} (\# \text{eigenvectors from eigenspace of } \lambda) \\ &= \sum_{\lambda} \dim(\text{eigenspace of } \lambda) \\ &\leq \sum_{\lambda} (\text{multiplicity of } \lambda) \\ &= \# \text{roots of the characteristic polynomial counted with multiplicity} \\ &= \deg(\text{characteristic polynomial}) \quad (\text{by the fundamental theorem of algebra}) \\ &= n, \end{aligned}$$

and the $\leq$ is $=$ if and only if all the eigenspaces are complete.

Conclusion:

- If A is complete, we get n eigenvectors forming a basis of $\mathbb{R}^n$ (or $\mathbb{C}^n$ if complex).
- If A is deficient, we get less than n eigenvectors, not enough for a basis.

Why does concatenating the bases produce a linearly independent list? The vectors within each basis are linearly independent, and there are no linear relations involving eigenvectors from different eigenspaces because of the following:

Theorem 22.18. *Fix a square matrix A . Eigenvectors corresponding to distinct eigenvalues are linearly independent.*

Proof. Suppose that $\mathbf{v}_1, \dots, \mathbf{v}_n$ are eigenvectors corresponding to distinct eigenvalues $\lambda_1, \dots, \lambda_n$. Suppose that there were a linear relation

$$c_1 \mathbf{v}_1 + \dots + c_n \mathbf{v}_n = \mathbf{0}.$$

Apply $A - \lambda_1 I$ to both sides; this gets rid of the first summand on the left. Next apply $A - \lambda_2 I$, and so on, up to $A - \lambda_{n-1} I$. This shows that some nonzero number times $c_n \mathbf{v}_n$ equals $\mathbf{0}$. But $\mathbf{v}_n \neq \mathbf{0}$, so $c_n = 0$. Similarly each c_i must be 0. Thus only the trivial relation between $\mathbf{v}_1, \dots, \mathbf{v}_n$ exists, so they are linearly independent. $\square$

22.5.2. *Examples.* Here are three examples showing all the situations that can arise for a 2×2 matrix.

Example 22.19. Let $A := \begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix}$. The characteristic polynomial is $(\lambda - 2)(\lambda + 1)$, so the eigenvalues (2 and -1) each have multiplicity 1, so the eigenspaces are automatically **complete**. Calculation shows that the eigenspace of 2 has basis $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and the eigenspace of -1 has basis $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$; together, these two vectors form a basis for $\mathbb{R}^2$.

Example 22.20. Let $B = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$. Since B is upper triangular (even diagonal), the eigenvalues are 5, 5. The eigenspace of 5 is $\text{NS}(B - 5I)$, which is the set of solutions to $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} = \mathbf{0}$, which is the entire space $\mathbb{R}^2$. Its dimension (namely, 2) matches the multiplicity of the eigenvalue 5, so this eigenspace is **complete**. Every vector is an eigenvector with eigenvalue 5. So it is easy to find two linearly independent eigenvectors: for example, take $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Example 22.21. Let $C = \begin{pmatrix} 5 & 3 \\ 0 & 5 \end{pmatrix}$. Again the eigenvalues are 5, 5. The eigenspace of 5 is $\text{NS}(C - 5I)$, which is the set of solutions to $\begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}$. This system consists of a single nontrivial equation $3y = 0$. Thus the eigenspace is the set of vectors of the form $\begin{pmatrix} c \\ 0 \end{pmatrix}$; it is only 1-dimensional, even though the multiplicity of the eigenvalue 5 is still 2. This means that this eigenspace is **deficient**, and hence C is **deficient**. It is impossible to find two linearly independent eigenvectors.

April 2

22.6. Diagonalization. Suppose that A is a 2×2 matrix such that $\mathbb{R}^2$ has a basis of eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ having eigenvalues λ_1, λ_2 (in other words, we are assuming that A is complete). (**Warning:** For what we are about to do, the eigenvectors must be listed in the same order as their eigenvalues.)

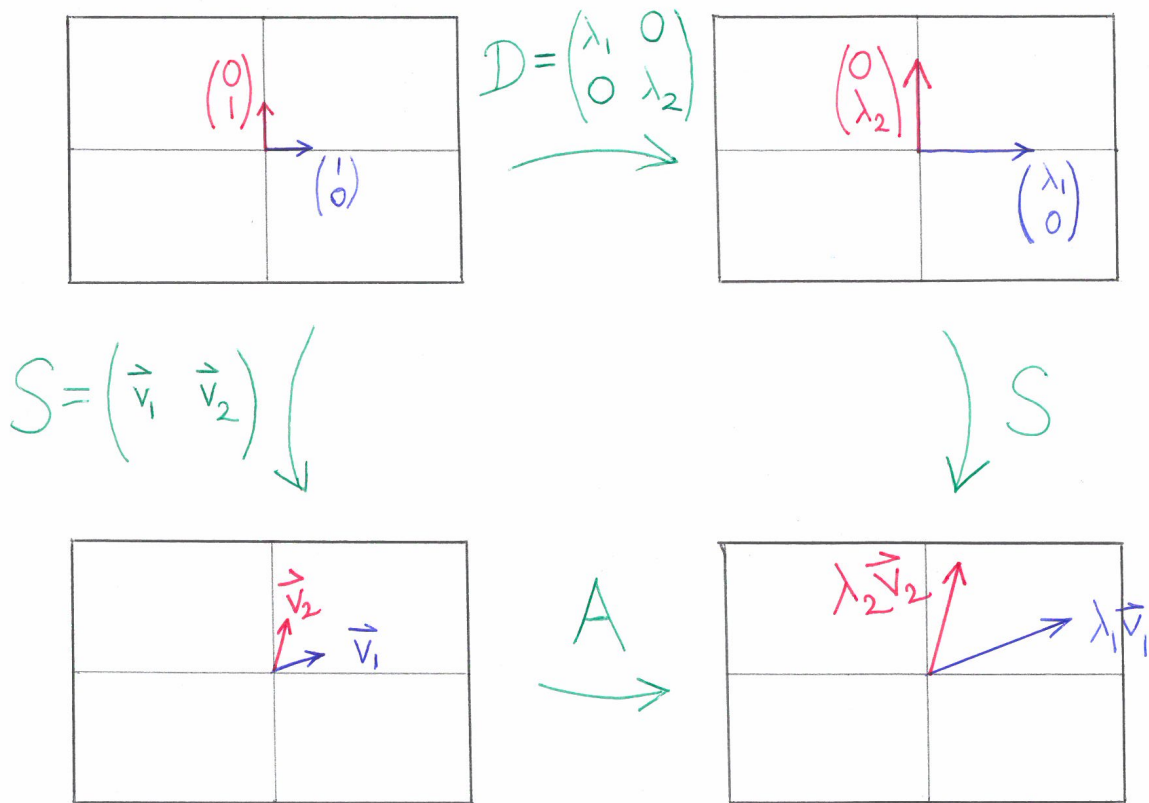
Use the eigenvalues to define a diagonal matrix

$$D := \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

The matrix mapping $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to $\mathbf{v}_1, \mathbf{v}_2$, respectively, is the matrix

$$S := \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix}$$

whose columns are the eigenvectors $\mathbf{v}_1, \mathbf{v}_2$.



Question 22.22. What is the 2×2 matrix that maps $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to $\lambda_1 \mathbf{v}_1$, $\lambda_2 \mathbf{v}_2$, respectively?

Answer 1: AS , because applying AS means that $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are mapped by S to $\mathbf{v}_1$, $\mathbf{v}_2$, which are then mapped by A to $\lambda_1 \mathbf{v}_1$, $\lambda_2 \mathbf{v}_2$.

Answer 2: SD , because $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are mapped by D to $\begin{pmatrix} \lambda_1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ \lambda_2 \end{pmatrix}$ which are then mapped by S to $\lambda_1 \mathbf{v}_1$, $\lambda_2 \mathbf{v}_2$.

Conclusion: $\boxed{AS = SD}$. (Memory aid: Look at where A , S , D are on your keyboard.)

Multiply by S^{-1} on the right to get another way to write it: $\boxed{A = SDS^{-1}}$.

Writing the matrix A like this is called **diagonalizing** A . Think of S as a “coordinate-change matrix” or “change-of-basis matrix” that

- relates the standard basis $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to the basis of eigenvectors of A , and
- relates the easy matrix D scaling the standard basis vectors to the original matrix A scaling the original eigenvectors.

Diagonalization of an $n \times n$ matrix A is possible if and only if A is complete (we need A to have n independent eigenvectors).

Steps to diagonalize a $n \times n$ matrix A (will succeed if and only if A is complete):

1. Find the eigenvalues of A , and list them with multiplicity: $\lambda_1, \dots, \lambda_n$.
2. Find a basis of each eigenspace.
3. If any eigenspace is deficient, then A is not diagonalizable.
4. Otherwise, we have found a total of n linearly independent eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$, enough to form a basis of $\mathbb{R}^n$ (or $\mathbb{C}^n$). Make sure that they are ordered so that $\mathbf{v}_i$ is associated to λ_i .
5. Set $D := \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$.
6. Set $S := \begin{pmatrix} \vdots & \vdots \\ \mathbf{v}_1 & \mathbf{v}_n \\ \vdots & \vdots \end{pmatrix}$ (the matrix whose columns are the eigenvectors).
7. Write $A = SDS^{-1}$.

Remark 22.23 (Using diagonalization to compute matrix powers). Suppose that $A = SDS^{-1}$. Then

$$A^3 = SD \underbrace{S^{-1}S}_{\text{cancels}} D \underbrace{S^{-1}S}_{\text{cancels}} DS^{-1} = SD^3S^{-1}.$$

More generally, for any integer $n \geq 0$,

$$\boxed{A^n = SD^nS^{-1}}. \quad (11)$$

Later we'll use diagonalization also to compute e^A for a matrix A !

23. HOMOGENEOUS LINEAR SYSTEMS OF ODEs, AGAIN

23.1. Solving a homogeneous linear system of ODEs.

Steps to find a basis of solutions to $\dot{\mathbf{x}} = A\mathbf{x}$, given a complete $n \times n$ constant matrix A :

1. Compute the eigenvalues (the roots of the characteristic polynomial $\det(\lambda I - A)$).
2. For each eigenvalue λ ,
 - compute a basis of the eigenspace $\text{NS}(A - \lambda I)$;
 - for each eigenvector $\mathbf{v}$ in this basis, write down the vector-valued function $e^{\lambda t}\mathbf{v}$.

The total number of functions written down is the sum of the dimensions of the eigenspaces, which is n , provided that A really was complete. These functions form the basis.

Remark 23.1. These n solutions will automatically be linearly independent, since their values at $t = 0$ are the eigenvectors, which are linearly independent. (The chosen eigenvectors

within each eigenspace are linearly independent, and there is no linear dependence between eigenvectors with different eigenvalues.)

Remark 23.2. If some of the eigenvalues are complex, they must be included (if you ignore them, you won't find enough eigenvectors). In this case, you may wish to find a new basis of real-valued functions (replace each pair $\boxed{\mathbf{x}, \bar{\mathbf{x}}}$ in the basis by $\boxed{\operatorname{Re} \mathbf{x}, \operatorname{Im} \mathbf{x}}$).

Remark 23.3. If some eigenspace has dimension less than the multiplicity of the eigenvalue (that is, A is deficient), then this method fails: it does not produce enough functions to form a basis.

23.2. Fundamental matrix.

23.2.1. *Definition.* Consider a homogeneous linear system of n ODEs $\dot{\mathbf{x}} = A\mathbf{x}$. (We'll assume that A is constant, but everything in this section remains true even if A is replaced by a matrix-valued function $A(t)$.) The dimension theorem says that the set of solutions is an n -dimensional vector space. Let $\mathbf{x}_1, \dots, \mathbf{x}_n$ be a basis of solutions. Write $\mathbf{x}_1, \dots, \mathbf{x}_n$ as column vectors side-by-side to form a matrix

$$X(t) := \begin{pmatrix} \vdots & \vdots \\ \mathbf{x}_1 & \mathbf{x}_n \\ \vdots & \vdots \end{pmatrix}.$$

(It's really a matrix-valued *function*, since each $\mathbf{x}_i$ is a vector-valued function of t .) Any such $X(t)$ is called a **fundamental matrix** for $\dot{\mathbf{x}} = A\mathbf{x}$. (There are many possible bases, so there are many possible fundamental matrices.)

23.2.2. *General solution in terms of a fundamental matrix.*

What is the point of putting the solutions in a fundamental matrix?

$$\text{The general solution to } \dot{\mathbf{x}} = A\mathbf{x} \text{ is } c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n = \begin{pmatrix} \vdots & \vdots \\ \mathbf{x}_1 & \mathbf{x}_n \\ \vdots & \vdots \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}.$$

Conclusion: If $X(t)$ is a fundamental matrix, then the general solution is $X(t)\mathbf{c}$, where $\mathbf{c}$ ranges over constant vectors.

23.2.3. *Solving a homogeneous system of ODEs with initial conditions.*

Problem 23.4. The matrix $A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$ has

- an eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ with eigenvalue 2 and
- an eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ with eigenvalue 3.

- (a) Find a fundamental matrix for $\dot{\mathbf{x}} = A\mathbf{x}$.
- (b) Use it to find the solution to $\dot{\mathbf{x}} = A\mathbf{x}$ satisfying the initial condition $\mathbf{x}(0) = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$.

Solution:

- (a) The functions

$$e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ e^{2t} \end{pmatrix} \quad \text{and} \quad e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix}$$

are a basis of solutions, so one fundamental matrix is

$$X(t) = \begin{pmatrix} 2e^{2t} & e^{3t} \\ e^{2t} & e^{3t} \end{pmatrix}.$$

- (b) The solution will be $X(t)\mathbf{c}$ for some constant vector $\mathbf{c}$. Thus

$$\mathbf{x} = \begin{pmatrix} 2e^{2t} & e^{3t} \\ e^{2t} & e^{3t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

for some c_1, c_2 to be determined. Set $t = 0$ and use the initial condition to get

$$\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

In other words,

$$\begin{aligned} 2c_1 + c_2 &= 4 \\ c_1 + c_2 &= 5. \end{aligned}$$

Solving leads to $c_1 = -1$ and $c_2 = 6$, so

$$\mathbf{x} = \begin{pmatrix} 2e^{2t} & e^{3t} \\ e^{2t} & e^{3t} \end{pmatrix} \begin{pmatrix} -1 \\ 6 \end{pmatrix} = \begin{pmatrix} -2e^{2t} + 6e^{3t} \\ -e^{2t} + 6e^{3t} \end{pmatrix}. \quad \square$$

23.2.4. Criterion for a matrix to be a fundamental matrix. To say that each column of $X(t)$ is a solution is the same as saying that $\dot{X} = AX$, because the matrix multiplication can be done column-by-column.

For a $n \times n$ matrix whose columns are solutions, to say that the columns form a basis is equivalent to saying that they are linearly independent (the space of solutions is n -dimensional, so if n solutions are linearly independent, their span is the entire space). By the existence and uniqueness theorem, linear independence of solutions is equivalent to linear independence of their initial values at $t = 0$, i.e., to linear independence of the columns of $X(0)$. So it is equivalent to say that $X(0)$ is a nonsingular matrix.

Conclusion:

Theorem 23.5. A matrix-valued function $X(t)$ is a fundamental matrix for $\dot{\mathbf{x}} = A\mathbf{x}$ if and only if

- $\dot{X} = AX$ and
- the matrix $X(0)$ is nonsingular.

23.3. Matrix exponential.

23.3.1. *Definition.* Inspired by the formula for a real (or complex) number x ,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$

define, for any square matrix A ,

$$e^A := I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots.$$

So e^A is another square matrix of the same size as A .

23.3.2. *Properties.*

- $e^{\mathbf{0}} = I$ (here $\mathbf{0}$ is the zero matrix)
(Proof: $e^{\mathbf{0}} = I + \mathbf{0} + \frac{\mathbf{0}^2}{2!} + \cdots = I$.)
- $\frac{d}{dt}e^{At} = Ae^{At}$
(Proof: Take the derivative of e^{At} term by term.)
- If $AB = BA$, then $e^{A+B} = e^Ae^B$. (**Warning:** This can fail if $AB \neq BA$.)
(Proof: Skipped.)

- If $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, then $e^D = \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix}$.

(Proof: $D^2 = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix}$, $D^3 = \begin{pmatrix} \lambda_1^3 & 0 \\ 0 & \lambda_2^3 \end{pmatrix}$, and so on. Thus

$$e^D = I + D + \frac{D^2}{2!} + \cdots = \begin{pmatrix} 1 + \lambda_1 + \frac{\lambda_1^2}{2!} + \cdots & 0 \\ 0 & 1 + \lambda_2 + \frac{\lambda_2^2}{2!} + \cdots \end{pmatrix} = \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix}.$$

A similar statement holds for diagonal matrices of any size.)

23.3.3. *Exponential of a diagonalizable matrix.*

Suppose that A is diagonalizable: $A = SDS^{-1}$. Substituting $A^n = SD^nS^{-1}$ for each term in the power series definition of e^A leads to

$$e^A = Se^DS^{-1}.$$

Use this formula to compute e^A ! (It works whenever A is diagonalizable (complete).)

23.3.4. Matrix exponential and systems of ODEs.

Theorem 23.6. The function e^{At} is a fundamental matrix for the system $\dot{\mathbf{x}} = A\mathbf{x}$.

Proof. The function e^{At} satisfies $\dot{X} = AX$ and $\underbrace{\text{its value at } 0}_I$ is nonsingular. □

Consequence: The general solution to $\dot{\mathbf{x}} = A\mathbf{x}$ is $e^{At} \mathbf{c}$.

Compare:

The solution to $\dot{x} = ax$ satisfying the initial condition $x(0) = c$ is $e^{at}c$.

The solution to $\dot{\mathbf{x}} = A\mathbf{x}$ satisfying the initial condition $\mathbf{x}(0) = \mathbf{c}$ is $e^{At} \mathbf{c}$.

Question 23.7. If the solution is as simple as $e^{At} \mathbf{c}$, why did we bother with the method involving eigenvalues and eigenvectors?

Answer: Because computing e^{At} is usually hard! (In fact, the standard method for computing it involves finding the eigenvalues and eigenvectors of A .)

Problem 23.8. Use the matrix exponential to find the solution to the system

$$\begin{aligned}\dot{x} &= 2x + y \\ \dot{y} &= 2y\end{aligned}$$

satisfying $x(0) = 5$ and $y(0) = 7$.

Solution: This is $\dot{\mathbf{x}} = A\mathbf{x}$ with $A := \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}}_D + \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_N$. Then $N^2 = 0$, so

$$e^{Nt} = I + Nt + 0 + 0 + \cdots = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Also, Dt and Nt commute (a scalar times I commutes with any matrix of the same size), so

$$\begin{aligned}
 e^{At} &= e^{Dt+Nt} \\
 &= e^{Dt}e^{Nt} \\
 &= \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix} \\
 \begin{pmatrix} x \\ y \end{pmatrix} &= e^{At} \begin{pmatrix} 5 \\ 7 \end{pmatrix} \\
 &= \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 5 \\ 7 \end{pmatrix} \\
 &= \begin{pmatrix} 5e^{2t} + 7te^{2t} \\ 7e^{2t} \end{pmatrix}. \quad \square
 \end{aligned}$$

April 4

24. INHOMOGENEOUS LINEAR SYSTEMS OF ODES

24.1. Diagonalization and decoupling.

24.1.1. *Solving a decoupled system.* The system $\dot{\mathbf{x}} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x}$ is the same as

$$\begin{aligned}
 \dot{x} &= 3x \\
 \dot{y} &= 2y.
 \end{aligned}$$

This is a **decoupled** system, consisting of two ODEs that can be solved separately. More generally, if D is a *diagonal* matrix of any size and $\mathbf{q}$ is a vector-valued function of t , the inhomogeneous system

$$\dot{\mathbf{x}} = D\mathbf{x} + \mathbf{q}$$

consists of first-order linear ODEs that can be solved separately.

Plan: develop a method to transform other systems into this form.

24.1.2. *Decoupling.* Here is a slightly silly way to solve

$$\dot{\mathbf{x}} = A\mathbf{x}$$

for a complete matrix $A = SDS^{-1}$. Substitute $\mathbf{x} = S\mathbf{y}$, and rewrite the system in terms of $\mathbf{y}$:

$$\begin{aligned} S\dot{\mathbf{y}} &= AS\mathbf{y} && \text{(since } S \text{ is constant)} \\ S\dot{\mathbf{y}} &= SD\mathbf{y} && \text{(since } AS = SD) \\ \dot{\mathbf{y}} &= D\mathbf{y} && \text{(we multiplied by } S^{-1} \text{ on the left).} \end{aligned}$$

This is decoupled! So solve for each coordinate function of $\mathbf{y}$, and then compute $\mathbf{x} = S\mathbf{y}$. $\square$

Why is this silly? Because finding D and S requires finding eigenvalues and eigenvectors, and we already know how to solve $\dot{\mathbf{x}} = A\mathbf{x}$ when we have eigenvalues and a basis of eigenvectors!

But... the same decoupling method lets us solve also an *inhomogeneous* linear system, and that's not silly:

Steps to solve $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{q}$ by decoupling (when A is complete):

1. Find the eigenvalues of A with multiplicity, and put them in a diagonal matrix D .
2. Find a basis of each eigenspace. If the total number of independent eigenvectors found is less than n , then a more complicated method (not discussed here) is required. Put the eigenvectors as columns of a matrix S .
3. Substitute $\mathbf{x} = S\mathbf{y}$ to get

$$\begin{aligned} S\dot{\mathbf{y}} &= AS\mathbf{y} + \mathbf{q} \\ S\dot{\mathbf{y}} &= SD\mathbf{y} + \mathbf{q} \\ \dot{\mathbf{y}} &= D\mathbf{y} + S^{-1}\mathbf{q}. \end{aligned}$$

(You may skip to the last of these equations.) This is a decoupled system of inhomogeneous linear ODEs.

4. Solve for each coordinate function of $\mathbf{y}$.
5. Compute $S\mathbf{y}$; the result is $\mathbf{x}$.

Problem 24.1. Find a particular solution to $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{q}$, where $A := \begin{pmatrix} -4 & -3 \\ 6 & 5 \end{pmatrix}$ and

$$\mathbf{q} = \begin{pmatrix} 0 \\ \cos t \end{pmatrix}.$$

Solution: We will instead solve

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{r} \quad (12)$$

with $\mathbf{r} := \begin{pmatrix} 0 \\ e^{it} \end{pmatrix}$ in place of $\mathbf{q}$ (complex replacement), and take the real part of the solution at the very end.

Step 1. We have $\text{tr } A = 1$ and $\det A = -20 - (-18) = -2$.

Characteristic polynomial: $\lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$.

Eigenvalues: 2, -1. Therefore define

$$D := \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}.$$

Step 2. Calculating eigenspaces in the usual way leads to corresponding eigenvectors $\begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, so define

$$S := \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}.$$

Now $A = SDS^{-1}$.

Step 3. The result of substituting $\mathbf{x} = S\mathbf{y}$ into (12) is

$$\dot{\mathbf{y}} = D\mathbf{y} + S^{-1}\mathbf{r}.$$

We have

$$S^{-1} = \frac{1}{\det S} \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix},$$
$$S^{-1}\mathbf{r} = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ e^{it} \end{pmatrix} = \begin{pmatrix} -e^{it} \\ e^{it} \end{pmatrix}$$

so the decoupled system is

$$\begin{aligned} \dot{y}_1 &= 2y_1 - e^{it} \\ \dot{y}_2 &= -y_2 + e^{it}. \end{aligned}$$

Step 4. Solving each equation with ERF gives particular solutions

$$\begin{aligned} y_1 &= \frac{-1}{i-2} e^{it} = \left(\frac{2}{5} + \frac{1}{5}i \right) e^{it} \\ y_2 &= \frac{1}{i+1} e^{it} = \left(\frac{1}{2} - \frac{1}{2}i \right) e^{it}. \end{aligned}$$

to the decoupled system.

Step 5. Then

$$\begin{aligned}\mathbf{x} &= S\mathbf{y} \\ &= \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} \frac{2}{5} + \frac{1}{5}i \\ \frac{1}{2} - \frac{1}{2}i \end{pmatrix} e^{it} \\ &= \begin{pmatrix} \frac{9}{10} - \frac{3}{10}i \\ -\frac{13}{10} + \frac{1}{10}i \end{pmatrix} (\cos t + i \sin t).\end{aligned}$$

is a solution to (12).

Final step: Take the real part to get a particular solution to the original system:

$$\mathbf{x} = \begin{pmatrix} \frac{9}{10} \cos t + \frac{3}{10} \sin t \\ -\frac{13}{10} \cos t - \frac{1}{10} \sin t \end{pmatrix}.$$

24.2. Variation of parameters.

Long ago (in Section 6.2) we learned how to use variation of parameters to solve inhomogeneous linear ODEs

$$\dot{y} + p(t)y = q(t).$$

Now we're going to use the same idea to solve an inhomogeneous linear *system* of ODEs such as

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{q}, \tag{13}$$

where $\mathbf{q}$ is a vector-valued function of t . First find a basis of solutions to the corresponding homogeneous system

$$\dot{\mathbf{x}} = A\mathbf{x}, \tag{14}$$

and put them together to form a fundamental matrix X (a matrix-valued function of t). We know that $X\mathbf{c}$, where $\mathbf{c}$ ranges over constant vectors, is the general solution to the homogeneous system (14). Replace $\mathbf{c}$ by a vector-valued function $\mathbf{u}$: try $\mathbf{x} = X\mathbf{u}$ in the original system (13):

$$\begin{aligned}\dot{\mathbf{x}} &= A\mathbf{x} + \mathbf{q} \\ \dot{X}\mathbf{u} + X\dot{\mathbf{u}} &= AX\mathbf{u} + \mathbf{q} \\ AX\mathbf{u} + X\dot{\mathbf{u}} &= AX\mathbf{u} + \mathbf{q} \\ X\dot{\mathbf{u}} &= \mathbf{q} \\ \dot{\mathbf{u}} &= X^{-1}\mathbf{q}.\end{aligned}$$

Steps to solve $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{q}$ by variation of parameters:

1. Find a fundamental matrix X for the homogeneous system $\dot{\mathbf{x}} = A\mathbf{x}$ (e.g., by using eigenvalues and eigenvectors to find a basis of solutions).

2. Substitute $\mathbf{x} = X\mathbf{u}$ for a vector-valued function $\mathbf{u}$; this eventually leads to

$$\dot{\mathbf{u}} = X^{-1}\mathbf{q}$$

(and you may jump right to this if you want).

3. Compute the right hand side and integrate each component function to find $\mathbf{u}$.

(The indefinite integral will have a $+\mathbf{c}$.)

4. Then $\mathbf{x} = X\mathbf{u}$ is the general solution to the inhomogeneous equation.

(It is a *family* of vector-valued functions because of the $+\mathbf{c}$ in $\mathbf{u}$.)

Remark 24.2. **Not mentioned in lecture.** One choice of X is e^{At} , in which case $\dot{\mathbf{u}} = e^{-At}\mathbf{q}$ and

$$\mathbf{x} = e^{At}\mathbf{u} = e^{At} \int e^{-At}\mathbf{q} dt.$$

25. COORDINATES

25.1. Coordinates with respect to a basis.

Problem 25.1. The vectors $\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ form a basis for $\mathbb{R}^2$, so any vector in $\mathbb{R}^2$ is a linear combination of them. Find the numbers c_1 and c_2 such that

$$c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}. \quad (15)$$

(These c_1, c_2 are called the **coordinates** of $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ with respect to the basis. There is only one

solution, since if there were two different linear combinations giving $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$, subtracting them would give a nontrivial linear combination giving $\mathbf{0}$, which is impossible since the basis vectors are linearly independent.)

Solution: Multiply out (15) to get

$$2c_1 - c_2 = 2$$

$$c_1 + c_2 = 4$$

or equivalently, in matrix form,

$$\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}.$$

Solving gives $c_1 = 2$ and $c_2 = 2$.

“Coordinates with respect to a basis” make sense also in vector spaces of *functions*.

Problem 25.2. Let V be the vector space with basis consisting of the three functions 1 , $t - 3$, $(t - 3)^2$. Find the coordinates of the function t^2 with respect to this basis.

Solution: We need to find numbers c_1, c_2, c_3 such that

$$c_1(1) + c_2(t - 3) + c_3(t - 3)^2 = t^2,$$

or, equivalently,

$$c_3 t^2 + (c_2 - 6c_3)t + (c_1 - 3c_2 + 9c_3) = t^2. \quad (16)$$

Two polynomials are equal *as functions* (i.e., for all values of t) if and only if their coefficients match, so (16) is equivalent to the system

$$c_3 = 1$$

$$c_2 - 6c_3 = 0$$

$$c_1 - 3c_2 + 9c_3 = 0.$$

Solving leads to $(c_1, c_2, c_3) = (9, 6, 1)$. $\square$

April 7

25.2. Orthogonal basis and orthonormal basis.

Consider a basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ of $\mathbb{R}^n$.

- If every two vectors in the list are orthogonal (that is, perpendicular: $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for all distinct i and j), then the basis is called an **orthogonal basis**.
- If every two vectors in the list are orthogonal *and* every $\mathbf{v}_i$ has length 1, then the basis is called an **orthonormal basis**.

(The vectors in it are neither abnormally long nor abnormally short!)

Example 25.3. The vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ form an orthonormal basis for $\mathbb{R}^3$.

Question: What is true of the list of vectors $\mathbf{v}_1 := \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and $\mathbf{v}_2 := \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ in $\mathbb{R}^2$?

Possible answers:

- This is not a basis of $\mathbb{R}^2$.
- This is a basis, but not an orthogonal basis.
- This is an orthogonal basis, but not an orthonormal basis.
- This is an orthonormal basis.

Answer: It is an orthogonal basis, but not an orthonormal basis. To test, use dot products: $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$, so they are orthogonal. But $\mathbf{v}_1 \cdot \mathbf{v}_1 \neq 1$, so $\mathbf{v}_1$ does not have length 1. $\square$

25.3. Shortcuts for finding coordinates.

Question 25.4. Suppose that $\mathbf{v}_1, \dots, \mathbf{v}_n$ is an *orthonormal* basis of $\mathbb{R}^n$. How can we find the coordinates $c_1, \dots, c_n$ of a vector $\mathbf{w}$ with respect to this basis?

Answer: We need to solve

$$\mathbf{w} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_n \mathbf{v}_n$$

for $c_1, \dots, c_n$. Trick: dot both sides with $\mathbf{v}_1$ to get

$$\mathbf{w} \cdot \mathbf{v}_1 = c_1(1) + 0 + \cdots + 0.$$

Get

$$\boxed{c_1 = \mathbf{w} \cdot \mathbf{v}_1}, \quad \cdots, \quad \boxed{c_n = \mathbf{w} \cdot \mathbf{v}_n}. \quad \square$$

Question 25.5. Suppose that $\mathbf{v}_1, \dots, \mathbf{v}_n$ is only an *orthogonal* basis of $\mathbb{R}^n$. How can we find the coordinates $c_1, \dots, c_n$ of a vector $\mathbf{w}$ with respect to this basis?

Answer: The same trick leads to

$$\mathbf{w} \cdot \mathbf{v}_1 = c_1 \mathbf{v}_1 \cdot \mathbf{v}_1,$$

so we get

$$\boxed{c_1 = \frac{\mathbf{w} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}}, \quad \cdots, \quad \boxed{c_n = \frac{\mathbf{w} \cdot \mathbf{v}_n}{\mathbf{v}_n \cdot \mathbf{v}_n}}. \quad \square \tag{17}$$

26. INTRODUCTION TO FOURIER SERIES

26.1. Periodic functions. Because

$$\sin(t + 2\pi) = \sin t, \quad \cos(t + 2\pi) = \cos t,$$

hold for all t , the functions $\sin t$ and $\cos t$ are called periodic with period 2π . In general, “ $f(t)$ is **periodic of period P** ” means that $f(t + P) = f(t)$ for all t (or at least all t for which either side is defined).

There are many such functions beyond the sinusoidal functions. To construct one, divide the real line into intervals of length P , start with any function defined on one such interval $[t_0, t_0 + P)$, and then copy its values in the other intervals. The entire graph consists of horizontally shifted copies of the width P graph.

Today: $P = 2\pi$, interval $[-\pi, \pi)$.

Question 26.1. Is $\sin 3t$ periodic of period 2π ?

Answer: The *shortest* period is $2\pi/3$, but $\sin 3t$ is also periodic with period any positive integer multiple of $2\pi/3$, including $3(2\pi/3) = 2\pi$:

$$\sin(3(t + 2\pi)) = \sin(3t + 6\pi) = \sin 3t.$$

So the answer is yes.

26.2. Square wave. Define

$$\text{Sq}(t) := \begin{cases} 1, & \text{if } 0 < t < \pi, \\ -1 & \text{if } -\pi < t < 0. \end{cases}$$

and extend it to a periodic function of period 2π , called a **square wave**. The function $\text{Sq}(t)$ has jump discontinuities, for example at $t = 0$. If you must define $\text{Sq}(0)$, compromise between the upper and lower values: $\text{Sq}(0) := 0$. The graph is usually drawn with vertical segments at the jumps (even though this violates the vertical line test).

It turns out that

$$\text{Sq}(t) = \frac{4}{\pi} \left(\sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \cdots \right).$$

We'll explain later today where this comes from.

Try the “Fourier Coefficients” mathlet

<http://mathlets.org/mathlets/fourier-coefficients/>

26.3. Fourier series. A linear combination like $2 \sin 3t - 4 \sin 7t$ is another periodic function of period 2π .

Definition 26.2. A **Fourier series** is a linear combination of the infinitely many functions $\cos nt$ and $\sin nt$ as n ranges over integers:

$$\begin{aligned} f(t) = \frac{a_0}{2} + a_1 \cos t + a_2 \cos 2t + a_3 \cos 3t + \cdots \\ + b_1 \sin t + b_2 \sin 2t + b_3 \sin 3t + \cdots \end{aligned}$$

(Terms like $\cos(-2t)$ are redundant since $\cos(-2t) = \cos 2t$. Also $\sin 0t = 0$ produces nothing new. But $\cos 0t = 1$ is included; the first term is the coefficient $a_0/2$ times the function 1. We'll explain later why we write $a_0/2$ instead of a_0 .)

Written using sigma-notation,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt.$$

Recall that, for example, $\sum_{n=1}^{\infty} b_n \sin nt$ means the sum of the series whose n^{th} term is obtained by plugging in the positive integer n into the expression $b_n \sin nt$, so

$$\sum_{n=1}^{\infty} b_n \sin nt = b_1 \sin t + b_2 \sin 2t + b_3 \sin 3t + \cdots.$$

The sum could also be written as $\sum_{n \geq 1} b_n \sin nt$.

Any Fourier series as above is periodic of period 2π . (Later we'll extend to the definition of Fourier series to include functions of other periods.) The numbers a_n and b_n are called the **Fourier coefficients** of f . Each summand ($a_0/2$, $a_n \cos nt$, or $b_n \sin nt$) is called a **Fourier component** of f .

Fourier's theorem. *“Every” periodic function f of period 2π “is” a Fourier series, and the Fourier coefficients are uniquely determined by f .*

(The word “Every” has to be taken with a grain of salt: The function has to be “reasonable”. Piecewise differentiable functions with jump discontinuities are reasonable, as are virtually all other functions that arise in physical applications.

The word “is” has to be taken with a grain of salt: If f has a jump discontinuity at τ , then the Fourier series might disagree with f there; the value of the Fourier series at τ is always the average of the left limit $f(\tau^-)$ and the right limit $f(\tau^+)$, regardless of the actual value of $f(\tau)$.)

In other words, the functions

$$1, \cos t, \cos 2t, \cos 3t, \dots, \sin t, \sin 2t, \sin 3t, \dots$$

form a basis for the vector space of “all” periodic functions of period 2π .

Question 26.3. Given f , how do you find the Fourier coefficients a_n and b_n ?

In other words, how do you find the coordinates of f with respect to the basis of cosines and sines?

26.4. A “dot product” for functions. If $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^n$, then

$$\mathbf{v} \cdot \mathbf{w} := \sum_{i=1}^n v_i w_i.$$

Can one define the dot product of two functions? Sort of.

Definition 26.4. If f and g are real-valued periodic functions with period 2π , then their **inner product** is

$$\langle f, g \rangle := \int_{-\pi}^{\pi} f(t)g(t) dt$$

(It acts like a dot product $f \cdot g$, but don't write it that way because $\cdot$ could be misinterpreted as multiplication.)

Example 26.5. By definition,

$$\langle 1, \cos t \rangle = \int_{-\pi}^{\pi} \cos t dt = 0.$$

Thus the functions 1 and $\cos t$ are orthogonal.

In fact, calculating all the inner products shows that

$1, \cos t, \cos 2t, \cos 3t, \dots, \sin t, \sin 2t, \sin 3t, \dots$ is an <i>orthogonal</i> basis!
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Question 26.6. Is it an orthonormal basis?

Answer: No, since $\langle 1, 1 \rangle = \int_{-\pi}^{\pi} 1 \, dt = 2\pi \neq 1$. $\square$

Example 26.7.

$$\begin{aligned}\langle \sin t, \sin t \rangle &= \int_{-\pi}^{\pi} \sin^2 t \, dt = ? \\ \langle \cos t, \cos t \rangle &= \int_{-\pi}^{\pi} \cos^2 t \, dt = ?\end{aligned}$$

Since $\cos t$ is just a shift of $\sin t$, the answers are going to be the same. Also, the two answers add up to

$$\int_{-\pi}^{\pi} \underbrace{(\sin^2 t + \cos^2 t)}_{\text{this is 1}} \, dt = 2\pi,$$

so each is π .

The same idea works to show that

$$\boxed{\langle \cos nt, \cos nt \rangle = \pi} \quad \text{and} \quad \boxed{\langle \sin nt, \sin nt \rangle = \pi}$$

for each positive integer n .

26.5. Fourier coefficient formulas. Given f , how do you find the a_n and b_n such that

$$\begin{aligned}f(t) &= \frac{a_0}{2} + a_1 \cos t + a_2 \cos 2t + a_3 \cos 3t + \cdots \\ &\quad + b_1 \sin t + b_2 \sin 2t + b_3 \sin 3t + \cdots?\end{aligned}$$

Answer: By the shortcut formula (17) for an orthogonal basis,

$$a_n = \frac{\langle f, \cos nt \rangle}{\langle \cos nt, \cos nt \rangle} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt,$$

and the coefficient of 1 is

$$\frac{a_0}{2} = \frac{\langle f, 1 \rangle}{\langle 1, 1 \rangle} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \, dt,$$

so multiplying by 2 gives

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos 0t \, dt.$$

(The unexplained point of why we write the constant term of a Fourier series as $a_0/2$ is now explained: it ensures that the formula for a_n for $n > 0$ works also for $n = 0$.)

A similar formula holds for b_n .

Conclusion: Given f , its Fourier coefficients can be calculated as follows:

$$\begin{aligned}\boxed{a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt} &\quad \text{for all } n \geq 0, \\ \boxed{b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt} &\quad \text{for all } n \geq 1.\end{aligned}$$

26.6. **Meaning of the constant term.** The constant term of the Fourier series of f is

$$\frac{a_0}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt,$$

which is the average value of f on $(-\pi, \pi)$.

26.7. **Even and odd symmetry.**

- A function $f(t)$ is **even** if $f(-t) = f(t)$ for all t .
- A function $f(t)$ is **odd** if $f(-t) = -f(t)$ for all t .

If

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt,$$

then substituting $-t$ for t gives

$$f(-t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} (-b_n) \sin nt.$$

The right hand sides match if and only if $b_n = 0$ for all n .

Conclusion: The Fourier series of an even function f has only cosine terms (including the constant term):

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt.$$

Similarly, the Fourier series of an odd function f has only sine terms:

$$f(t) = \sum_{n=1}^{\infty} b_n \sin nt.$$

Example 26.8. The square wave $\text{Sq}(t)$ is an odd function, so

$$\text{Sq}(t) = \sum_{n=1}^{\infty} b_n \sin nt$$

for some numbers b_n . The Fourier coefficient formula says

$$\begin{aligned}
 b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{\text{Sq}(t) \sin nt}_{\text{even}} dt \\
 &= \frac{2}{\pi} \int_0^{\pi} \text{Sq}(t) \sin nt dt \quad (\text{the two halves of the integral are equal, by symmetry}) \\
 &= \frac{2}{\pi} \int_0^{\pi} \sin nt dt \quad (\text{since } \text{Sq}(t) = 1 \text{ whenever } 0 < t < \pi) \\
 &= \left. \frac{2(-\cos nt)}{\pi n} \right|_0^{\pi} \\
 &= \frac{2}{\pi n} (-\cos n\pi + \cos 0) \\
 &= \begin{cases} \frac{4}{\pi n}, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases}
 \end{aligned}$$

Thus

$$b_1 = \frac{4}{\pi}, \quad b_3 = \frac{4}{3\pi}, \quad b_5 = \frac{4}{5\pi}, \dots$$

and all other Fourier coefficients are 0.

Conclusion:

$$\text{Sq}(t) = \frac{4}{\pi} \left(\sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \dots \right). \quad \square$$

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26.8. Finding a Fourier series representing a function on an interval.

Problem 26.9. Suppose that $f(t)$ is a (reasonable) function defined only on the interval $(0, \pi)$. Find numbers $a_0, a_1, \dots$ such that

$$f(t) = \frac{a_0}{2} + a_1 \cos t + a_2 \cos 2t + \dots$$

for all $t \in (0, \pi)$.