UNITS IN HAHN–MAL'CEV–NEUMANN RINGS

BJORN POONEN

Abstract. We give a simpler proof that the units in a Hahn–Mal'cev–Neumann ring $A = R((G, \omega))$ (consisting of power series in which the exponents form a well-ordered subset of an ordered group) include all elements of the form $1 - \alpha$ such that all the exponents in α are positive. This is the most difficult step of the construction of Hahn–Mal'cev–Neumann division rings, which have applications to valuation theory and number theory. We also turn the standard proof on its head by showing that the invertibility result, applied to an auxiliary ring, implies the two lemmas about well-ordered subsets of groups required for that proof.

1. Introduction

There are two easy ways to prove that for any ring R, the units of the Laurent series ring $R((x))$ include all elements of the form $1 - \alpha$ with $\alpha \in xR[[x]]$:

Method 1. Show that $1 + \alpha + \alpha^2 + \alpha^3 + \cdots$ is a well-defined element of $R((x))$ and that it is an inverse of $1 - \alpha$.

Method 2. Write $\alpha = \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \cdots$, set up an equation

$$(1 - \alpha_1 x - \alpha_2 x^2 - \alpha_3 x^3 - \cdots)(\beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \cdots) = 1,$$

equate the coefficients of x^n on both sides, and solve for $\beta_0, \beta_1, \beta_2, \ldots$ in turn, to obtain "successive approximations" to the inverse of $1 - \alpha$.

The invertibility result holds more generally for Hahn–Mal'cev–Neumann rings $R((G; \omega))$, which consist of series α in which the exponent set $\operatorname{supp}(\alpha)$ is a well-ordered subset of an ordered group G (the optional ω twists the multiplication; see Section 2 for definitions):

Theorem 1.1 ([Mal48, Neu49]). *In a Hahn–Mal'cev–Neumann ring $R((G; \omega))$, if α is an element with strictly positive support, then $1 - \alpha$ is a unit.*

Remarks 1.2.

(i) Hahn in 1907 [Hah07] constructed these rings in the commutative case over a field R, and Mal'cev and Neumann generalized the construction to the noncommutative case.

(ii) Hahn–Mal'cev–Neumann rings arise in valuation theory: when R is a division ring, $R((G; \omega))$ is a spherically complete (also called maximally complete or maximal) division ring; cf. [Kru32, p. 193].

Date: January 14, 2026.

2020 *Mathematics Subject Classification.* Primary 13F25; Secondary 11D88, 12E15, 12J20, 13A18, 13F30.

Key words and phrases. Hahn series, Hahn–Mal'cev–Neumann ring.

This research was supported in part by National Science Foundation grant DMS-2101040 and Simons Foundation grants #402472 and #550033. It was partly carried out while B.P. was participating in the 2025 program "Definability, decidability, and computability" at the Hausdorff Research Institute in Bonn, funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy – EXC-2047/1 – 390685813.

(iii) The Hahn–Mal'cev–Neumann ring $k((x^{\mathbb{Q}}))$ is an explicit algebraically closed field extension of $k((x))$ for any algebraically closed field k.
(iv) In number theory, a variant gives an explicit algebraically closed field extension of $\mathbb{Q}_p$ for any prime p [Lam86, Poo93].

The original proofs of Theorem 1.1 follow Method 1 (see [Mal48, Neu49]), but they involve checking that $1+\alpha+\alpha^2+\cdots$ is well-defined, which requires two technical results that for general G are not immediate to verify directly:

- that $\bigcup_{n\geq 0}\operatorname{supp}(\alpha^n)$ is again well-ordered, and
- that each $g \in G$ belongs to $\operatorname{supp}(\alpha^n)$ for at most finitely many $n \geq 0$.

We give a simpler proof of Theorem 1.1 via a transfinite generalization of Method 2, which is closer to Hahn's original proof for the commutative case. (Neumann, in a note added in proof [Neu49, p. 203], writes that Daniel Zelinsky also found such a proof for the noncommutative case, but it seems that it was never published. Moreover, this seems to have been forgotten, since subsequent books by noncommutative ring experts have presented only the more complicated Method 1 proof; see [Pas77, Chapter 13, §2] and [Lam01, pp. 229–234].) Also, in Section 4 we turn Method 1 on its head, to show that Theorem 1.1 *implies* that $1+\alpha+\alpha^2+\cdots$ is well-defined.

2. Hahn–Mal'cev–Neumann rings

We follow the notation of [Lam01, pp. 229–234]. Let R be a (possibly noncommutative) ring. Let G be a (possibly nonabelian) ordered group. Let $\omega\colon G \to \operatorname{Aut} R$ be a homomorphism; this is the twist. The Hahn–Mal'cev–Neumann ring $A = R((G;\omega))$ is the set of formal sums $\alpha = \sum_{g\in G}\alpha_g g$ with $\alpha_g \in R$ such that the support $\operatorname{supp}(\alpha) := \{g \in G : \alpha_g \neq 0\}$ is well-ordered, equipped with the operations

$$\sum_{g\in G}\alpha_g g + \sum_{g\in G}\beta_g g := \sum_{g\in G}(\alpha_g+\beta_g)g$$
$$\sum_{g\in G}\alpha_g g \cdot \sum_{h\in G}\beta_h h := \sum_{u\in G}\Big(\sum_{g,h\in G\colon gh=u}\alpha_g\,\omega_g(\beta_h)\Big)u;$$

these operations are well-defined and make A a ring.

For nonzero $\beta \in A$, define $v(\beta) := \min\operatorname{supp}(\beta)$. Also, let ∞ be an element outside G that is greater than all elements of G and set $v(0) := \infty$. For $\beta \in A$ and $h \in G\cup\{\infty\}$, define $\beta_{<h} := \sum_{g<h}\beta_g g \in A$.

3. Existence of inverses

In this section, we give our new proof of Theorem 1.1.

Call a series $\beta \in A$ an approximate inverse of $1-\alpha$ if $(1-\alpha)\beta = 1+\epsilon$, where $v(\epsilon)$ is greater than all elements of $\operatorname{supp}(\beta)$. Let $\mathcal{F}$ be the set of approximate inverses of $1-\alpha$. Define a partial order on $\mathcal{F}$ as follows: $\beta' \preceq \beta$ if β' is an initial segment of β, that is, $\beta_{<h}$ for some $h \in G\cup\{\infty\}$.

Lemma 3.1. *Any chain C in $\mathcal{F}$ has an upper bound.*

Proof. For $g \in G$, define $\gamma_g := \beta_g$ if there is a $\beta \in C$ with $\beta_g \neq 0$, and $\gamma_g := 0$ otherwise. The series $\gamma := \sum_{g\in G}\gamma_g g$ has well-ordered support, since any descending sequence in $\operatorname{supp}(\gamma)$

is contained in $\operatorname{supp}(\beta)$ for some $\beta \in C$. Also, $\beta \preceq \gamma$ for every $\beta \in C$. Finally, $\gamma \in \mathcal{F}$: If $h \in \operatorname{supp}(\gamma)$, then $h \in \operatorname{supp}(\beta)$ for some $\beta \in C$, and if we ignore terms with exponent $> h$, then γ agrees with β, so $(1-\alpha)\gamma$ agrees with $(1-\alpha)\beta$, which agrees with 1. □

Lemma 3.2. *If $\beta \in \mathcal{F}$ and $(1-\alpha)\beta \neq 1$, then β is not a maximal element of $\mathcal{F}$.*

Proof. We have $(1-\alpha)\beta = 1+\epsilon$ for some nonzero ϵ. Let $\epsilon_g g$ be the initial term of ϵ. Since $\beta \in \mathcal{F}$, all elements of $\operatorname{supp}(\beta)$ are less than g. Let $\beta' = \beta - \epsilon_g g$, so elements of $\operatorname{supp}(\beta')$ are less than or equal to g. Then $(1-\alpha)\beta' = 1+\epsilon'$, where $\epsilon' := (\epsilon - \epsilon_g g) + \alpha \cdot \epsilon_g g$ has $v(\epsilon') > g$, so $\beta' \in \mathcal{F}$. By construction, $\beta \prec \beta'$. □

By Lemma 3.1 and Zorn's lemma, $\mathcal{F}$ has a maximal element β. By Lemma 3.2, $(1-\alpha)\beta = 1$. Similarly, $1-\alpha$ has a *left* inverse. This completes the proof of Theorem 1.1.

4. Explicit inverses

The main disadvantage of the proof in Section 3 is that it does not show that the inverse is $1+\alpha+\alpha^2+\cdots$. But once the existence of an inverse has been established, a short argument shows what it must be:

Theorem 4.1. *If the support of $\alpha \in A$ consists of strictly positive elements of G, then $1+\alpha+\alpha^2+\cdots$ is a well-defined element of A (when summed coefficientwise), and is the inverse of $1-\alpha$.*

Proof. Each automorphism σ of R extends to an automorphism of the polynomial ring $R[x]$ by letting σ fix x. So there is a natural embedding $\operatorname{Aut}(R) \subseteq \operatorname{Aut}(R[x])$, and we may consider ω as a homomorphism from G to $\operatorname{Aut}(R[x])$. Let $A' = R[x]((G;\omega))$ be the corresponding Hahn–Mal'cev–Neumann ring. Given an element of A', the degrees of the polynomials which are its coefficients may be unbounded, so A' does not naturally embed in $A[x]$, but we do have $A' \subseteq A[[x]]$. In $A[[x]]$, $(1-\alpha x)^{-1} = 1+\alpha x+\alpha^2 x^2+\cdots$. But $1-\alpha x$ is invertible in A' by Theorem 1.1, so $1+\alpha x+\alpha^2x^2+\cdots$ represents the inverse of $1-\alpha x$ in A'. The ring homomorphism $R[x] \to R$ fixing elements of R and mapping x to 1 induces a ring homomorphism $A' \to A$, under which $1-\alpha x$ maps to $1-\alpha$ and $1+\alpha x+\alpha^2x^2+\cdots$ maps to $1+\alpha+\alpha^2+\cdots$. Hence $1+\alpha+\alpha^2+\cdots$ is a well-defined element of A, and is the inverse of $1-\alpha$. □

References

[Hah07] H. Hahn, *Über die nichtarchimedischen Größensysteme.*, Wien. Ber. **116** (1907), 601–655 (German).

[Kru32] Wolfgang Krull, *Allgemeine Bewertungstheorie*, J. Reine Angew. Math. **167** (1932), 160–196. MR 1581334

[Lam86] David Lampert, *Algebraic p-adic expansions*, J. Number Theory **23** (1986), no. 3, 279–284. MR 846958

[Lam01] T. Y. Lam, *A first course in noncommutative rings*, second ed., Graduate Texts in Mathematics, vol. 131, Springer-Verlag, New York, 2001. MR 1838439

[Mal48] A. I. Mal'cev, *On the embedding of group algebras in division algebras*, Doklady Akad. Nauk SSSR (N.S.) **60** (1948), 1499–1501. MR 25457

[Neu49] B. H. Neumann, *On ordered division rings*, Trans. Amer. Math. Soc. **66** (1949), 202–252. MR 32593

[Pas77] Donald S. Passman, *The algebraic structure of group rings*, Pure and Applied Mathematics, Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1977. MR 470211

[Poo93] Bjorn Poonen, *Maximally complete fields*, Enseign. Math. (2) **39** (1993), no. 1-2, 87–106. MR 1225257

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139-4307, USA
Email address: poonen@math.mit.edu
URL: http://math.mit.edu/~poonen/