RANK STABILITY MAKES RINGS OF INTEGERS DIOPHANTINE

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ABSTRACT. The recent negative answer to Hilbert's tenth problem over rings of integers relies on a theorem that for every extension of number fields L/K, if there is an abelian variety A over K such that $0 < \operatorname{rank} A(K) = \operatorname{rank} A(L)$, then \mathcal{O}_K is \mathcal{O}_L -diophantine. We present an alternative proof of this theorem and review how it is used.

1. Introduction

1.1. **History.** Hilbert's tenth problem asked for an algorithm to decide, given a multivariable polynomial equation with integer coefficients, whether it has a solution in integers. By [DPR61, Mat70], there is no such algorithm.

For each number field K, replacing \mathbb{Z} with the ring of integers \mathcal{O}_K yields a new question. The negative answer for \mathbb{Z} implies a negative answer for \mathcal{O}_K if \mathbb{Z} is \mathcal{O}_K -diophantine; see Section 2 for the definition. This led Denef and Lipshitz [DL78] to conjecture that \mathbb{Z} is \mathcal{O}_K -diophantine for every number field K. Their conjecture was proved for many classes of number fields by using the structure of integer points on algebraic tori, specifically, Pell equations [Den75, DL78, Den80, Phe88, Shl89].

Starting with Denef, various authors [Den80, Poo02, CPZ05, Shl08, MRS24] showed that one could use elliptic curves or abelian varieties in place of algebraic tori, if certain rank conditions could be proven. The strongest of these results states, for an extension L/K of number fields, that if the condition

 $\mathcal{A}_{K,L}$: There exists an abelian variety A over K such that $0 < \operatorname{rank} A(K) = \operatorname{rank} A(L)$.

holds, then \mathcal{O}_K is \mathcal{O}_L -diophantine [MRS24, Theorem 1.1] (in fact, this result applies to some infinite algebraic extensions as well). Via such results, [MR10, MR18, MP18, GFP20, Pas23, SW23, KLS24, RW24] proved that \mathbb{Z} is \mathcal{O}_F -diophantine for many new number fields F.

Recently, [KP25], using an input from additive combinatorics, constructed elliptic curves proving $\mathcal{A}_{K,L}$ for enough degree 2 extensions L/K to prove the full Denef–Lipshitz conjecture. Soon thereafter, [ABHS25, Theorem 1.1] proved $\mathcal{A}_{K,L}$ for all degree 2 extensions L/K, constructing abelian varieties that were not necessarily elliptic curves. In contrast with [KP25], [ABHS25] requires no additive combinatorics beyond a number field analogue of Vinogradov's method from the 1930s. Later, [Zyw25, Theorem 1.2] proved the stronger theorem that for every degree 2 extension L/K, there exist infinitely many elliptic curves E over E over E with rank E (E) = rank E (E) = 1.

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- 1.2. **Outline.** The proof that \mathbb{Z} is \mathcal{O}_F -diophantine for all number fields F can be broken into four independent steps:
 - (i) If E is a totally real number field, then \mathbb{Z} is \mathcal{O}_E -diophantine [Den80].
 - (ii) $\mathcal{A}_{K,L}$ holds for all degree 2 extensions L/K [KP25, ABHS25, Zyw25].
- (iii) $\mathcal{A}_{K,L}$ implies that \mathcal{O}_K is \mathcal{O}_L -diophantine [MRS24, Theorem 1.1].
- (iv) If \mathbb{Z} is \mathcal{O}_E -diophantine for all totally real E, and \mathcal{O}_K is \mathcal{O}_L -diophantine for all degree 2 extensions L/K, then \mathbb{Z} is \mathcal{O}_F -diophantine for all number fields F. This reduction is due to Shlapentokh [MRS24, Theorem 4.8].

Remark 1.1. What is proved towards (ii) determines how strong a version of (iii) is needed. Specifically, [KP25] constructs elliptic curves and hence needs only [Shl08], whereas [ABHS25] needs the full abelian variety statement of [MRS24], and [Zyw25] needs only [Poo02].

We have nothing new to say about (i) and (ii). The main purpose of this note is to give an alternative proof of (iii); see Theorem 4.2(d). The key ideas are present in the earlier works, but we introduce several simplifications. In Section 5, we reproduce Shlapentokh's reduction argument (iv) since it is short.

1.3. **Notation.** Let K be a number field. Let \mathcal{O}_K be its ring of integers. Given $a \in K^{\times}$, there exist unique coprime ideals $I, J \subset \mathcal{O}_K$ such that (a) = I/J; define $\operatorname{num}(a) := I$ and $\operatorname{den}(a) := J$. If L is a finite extension of K, and $\alpha, \beta \in \mathcal{O}_L$ and I is a nonzero ideal of \mathcal{O}_K , the notation $\alpha \equiv \beta \pmod{I}$ means $I\mathcal{O}_L \mid \operatorname{num}(\alpha - \beta)$.

2. Diophantine sets

Let $R = \mathcal{O}_K$ for some K. For a finite-type R-scheme X, a subset $S \subset X(R)$ is R-diophantine if it is f(Y(R)) for some finite-type morphism $Y \to X$. It is not hard to show that a subset $S \subset R = \mathbb{A}^1(R)$ is R-diophantine if and only if $S = \{a \in R : (\exists x \in R^n) \ g(a,x) = 0\}$ for some $g \in R[t,x_1,\ldots,x_n]$. Finite unions of R-diophantine subsets are R-diophantine. Any morphism of finite-type R-schemes $X \to X'$ maps R-diophantine subsets of X(R) to R-diophantine subsets of X'(R). Applying this to $\mathbb{A}^1 \times \mathbb{A}^1 \xrightarrow{\text{sum}} \mathbb{A}^1$ shows that if $S,T \subset R = \mathbb{A}^1(R)$ are R-diophantine, then so is $S + T := \{s + t : s \in S, t \in T\}$.

Lemma 2.1 ([DL78, Proposition 1(b)]). The set $\mathcal{O}_K - \{0\}$ is \mathcal{O}_K -diophantine.

Proof. For any nonzero ideal $I \subset \mathcal{O}_K$, there exists $x \in \mathcal{O}_K$ with $(2x-1)(3x-1) \equiv 0 \pmod{I}$ (use the Chinese remainder theorem to reduce to the case where I is a power of a prime ideal). For $a \in \mathcal{O}_K$, taking I = (a) shows that

$$a \neq 0 \iff (\exists x, y \in \mathcal{O}_K) (2x - 1)(3x - 1) = ya.$$

Elements of K can be represented in the usual way as equivalence classes a/b of pairs (a, b) with $a, b \in \mathcal{O}_K$ and $b \neq 0$. Subsets of K can then be identified with certain subsets of \mathcal{O}_K^2 . Lemma 2.1 lets us

- use polynomial equations involving K-valued variables in diophantine definitions and
- extend the \mathcal{O}_K -diophantine notion to subsets of X(K) for finite-type K-schemes X. For a finite extension L/K,
 - choosing a finite presentation of \mathcal{O}_L as an \mathcal{O}_K -module lets us use polynomial equations involving \mathcal{O}_L -valued variables in constructing \mathcal{O}_K -diophantine sets, and

• if \mathcal{O}_K is \mathcal{O}_L -diophantine, then any \mathcal{O}_K -diophantine set is \mathcal{O}_L -diophantine.

Each ideal $I \subset \mathcal{O}_K$ can be represented as (i_1, i_2) for some $i_1, i_2 \in \mathcal{O}_K$. Conditions involving ideals can be expressed in diophantine terms in terms of the generators. For example:

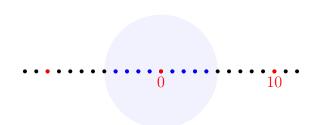
- $a \in I \iff (\exists x, y \in \mathcal{O}_K) \ a = xi_1 + yi_2;$
- $J|I \iff i_1, i_2 \in J;$
- $I = J \iff I|J \text{ and } J|I;$
- I, J are coprime $\iff (\exists i \in I)(\exists j \in J) \ i + j = 1;$
- for $s = a/b \in K$, we have $(s) = I/J \iff aJ = bI$;
- $I = \text{num}(s) \iff (\exists J) \ (s) = I/J \text{ and } I, J \text{ are coprime; and}$
- $a \equiv b \pmod{I} \iff a b \in I$.

3. Using congruences

Here is an example of how to use a congruence to force an algebraic integer to belong to a smaller ring of integers:

Example 3.1. If $\alpha \in \mathbb{Z}[i]$ is such that $|\alpha| < 5$ and $\alpha \equiv k \pmod{10\mathbb{Z}[i]}$ for some $k \in \mathbb{Z}$, then $\alpha \in \mathbb{Z}$. One proof: $\bar{\alpha} - \alpha \in 10\mathbb{Z}[i]$, but $|\bar{\alpha} - \alpha| < 5 + 5$, so $\bar{\alpha} = \alpha$; that is, $\alpha \in \mathbb{Z}$. (The dots below are the $\alpha \in \mathbb{Z}[i]$ congruent to an integer modulo $10\mathbb{Z}[i]$.)





To generalize to $\mathcal{O}_L \supset \mathcal{O}_K$ in place of $\mathbb{Z}[i] \supset \mathbb{Z}$, and an \mathcal{O}_K -ideal I in place of $10\mathbb{Z}[i]$, we use a condition $(\alpha - 1) \cdots (\alpha - n) \mid I$ to express that "I is much bigger than α ":

Lemma 3.2. Fix number fields $L \supset K$. There exists $n \ge 1$ such that for all $\alpha \in \mathcal{O}_L$, all nonzero ideals $I \subset \mathcal{O}_K$, and all $k \in K$,

$$(\alpha - 1) \cdots (\alpha - n) \mid I \text{ and } \alpha \equiv k \pmod{I} \implies \alpha \in \mathcal{O}_K.$$

Proof. Enlarge L to assume that L/\mathbb{Q} is Galois. Let $\ell = [L : \mathbb{Q}]$. Choose n such that $n > 23\ell$ and $10^{n-2\ell} > (4n)^{\ell}$. Below, j ranges over integers in [1, n], and τ ranges over elements of $\operatorname{Gal}(L/\mathbb{Q})$. Given $\alpha \in \mathcal{O}_L$, embed L in \mathbb{C} so that $|\alpha| \geq |\tau| \alpha|$ for all τ . Let $M = |\alpha|$.

Suppose that $(\alpha-1)\cdots(\alpha-n)\mid I$ and $\alpha\equiv k\pmod{I}$, but $\alpha\notin\mathcal{O}_K$. Choose $\sigma\in\mathrm{Gal}(L/K)$ with $\sigma\alpha\neq\alpha$. Applying σ to $\alpha\equiv k\pmod{I}$ and subtracting gives $I\mid(\sigma\alpha-\alpha)$, so $(\alpha-1)\cdots(\alpha-n)\mid(\sigma\alpha-\alpha)$. Apply the norm N: $L\to\mathbb{Q}$ and then $|\cdot|$:

$$\prod_{j,\tau} |\tau \alpha - j| \le \prod_{\tau} |\tau \alpha - \tau \alpha| \le (2M)^{\ell} \quad \text{(since } |\tau \alpha| \le |\alpha| = M \text{ for all } \tau\text{)}.$$

We will contradict the last line by proving that many terms on the left are large. For each τ , let $J_{\tau} = \{j : |^{\tau}\alpha - j| < 10\}$, so $\#J_{\tau} \leq 20$. Let $J_{0} := \bigcup_{\tau} J_{\tau}$, so $\#J_{0} \leq 20\ell$. Let $J_1 := \{1, \dots, n\} - J_0$, so $\#J_1 \ge n - 20\ell$.

- If $M \geq 2n$ (so in particular $M \geq 4$, so $(M/2)^3 \geq 2M$), then $\prod_{j \in J_1} |\alpha j| \geq 2M$ $(M/2)^{\#J_1} \geq (M/2)^{n-20\ell} > (M/2)^{3\ell} \geq (2M)^{\ell}$. If M < 2n, then $\prod_{j \in J_1} |\alpha - j| \geq 1$ $10^{\#J_1} \ge 10^{n-20\ell} > (4n)^{\ell} > (2M)^{\ell}$
- For $\tau \neq 1$, we have $\prod_{j \in J_1} |\tau \alpha j| \geq \prod_{j \in J_1} 10 \geq 1$. For $j \in J_0$, we use $\prod_{\tau} |\tau \alpha j| = |\mathrm{N}(\alpha j)| \geq 1$.

Multiplying these inequalities gives $\prod_{j,\tau} |^{\tau} \alpha - j| > (2M)^{\ell}$, a contradiction.

4. Weakly approximating Z

Let K be a number field. For each prime ideal $\mathfrak{p} \subset \mathcal{O}_K$, let $K_{\mathfrak{p}}$ be the completion. Let $S \subset K$. Say that S weakly approximates \mathbb{Z} if any of the following equivalent conditions holds:

- (i) \mathbb{Z} is contained in the closure of S in $\prod_{\mathfrak{p}} K_{\mathfrak{p}}$.
- (ii) for every $k \in \mathbb{Z}$ and primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ of \mathcal{O}_K , there is a sequence in S converging to kin $K_{\mathfrak{p}_i}$ simultaneously for every i;
- (iii) for every $k \in \mathbb{Z}$ and nonzero ideal $I \subset \mathcal{O}_K$, the congruence $x \equiv k \pmod{I}$ has a solution

Lemma 4.1. If $S \subset K$ weakly approximates \mathbb{Z} and $0 \neq \beta \in \mathcal{O}_K$, then there exists $s \in S$ with $\beta \mid \text{num}(s)$.

Proof. The congruence $x \equiv 0 \pmod{(\beta)}$ has a solution in S.

Theorem 4.2. For an extension of number fields L/K, if $A_{K,L}$ holds, then

- (a) there exists an infinite \mathcal{O}_L -diophantine subset $T \subset K$; and
- (b) there exists an \mathcal{O}_L -diophantine subset $S \subset K$ that weakly approximates \mathbb{Z} ;
- (c) there exists an \mathcal{O}_L -diophantine subset U with $\mathbb{Z} \subset U \subset \mathcal{O}_K$;
- (d) the subset \mathcal{O}_K is \mathcal{O}_L -diophantine.

Proof. Fix A as in $A_{K,L}$. Let r = (A(L) : A(K)). Then A(K) is a finite union of cosets of rA(L), so A(K) is \mathcal{O}_L -diophantine.

- (a) Choose a closed immersion $A \hookrightarrow \mathbb{P}^N_K$ for some N, and let T be the set of ratios of projective coordinates of the points in $A(K) \subset \mathbb{P}^N(K)$, excluding ratios with denominator 0. Since A(K) is infinite, T is infinite. By definition, T is \mathcal{O}_L -diophantine.
- (b) Let $y_1, \ldots, y_g \in K(A)$ be local parameters for A at 0. Define

$$S = \left\{ \frac{y(Q)}{y(P)} : P, Q \in A(K), \ y \in \sum_{i=1}^{g} Ty_i \right\} \quad \subset K;$$

we exclude ratios y(Q)/y(P) in which y(P) or y(Q) is undefined or in which y(P) = 0. By definition, S is \mathcal{O}_L -diophantine.

Let $k \in \mathbb{Z}$. Let p be a prime number. Let \mathfrak{p} be a prime of \mathcal{O}_K above p. If $R \to 0$ along a smooth analytic arc in the p-adic manifold $A(K_{\mathfrak{p}})$ and $y \in K(A)$ is a uniformizer at 0 along this arc, then $y(kR)/y(R) \to k$ in $K_{\mathfrak{p}}$ (l'Hôpital's rule). Let $a \in A(K)$ be a point of infinite order. Using formal group coordinates shows that $N!a \in A(K_p)$ tends to 0 along such an arc as $N \to \infty$, and the function $y := \sum t_i y_i$ is a uniformizer at 0 along the

- arc for any $(t_1, \ldots, t_g) \in K^g$ outside a hyperplane $H_{\mathfrak{p}} \subset \mathbb{A}^g_{K_{\mathfrak{p}}}$. Now, given finitely many primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ of \mathcal{O}_K , we can choose $(t_1, \ldots, t_g) \in T^g$ outside all of $H_{\mathfrak{p}_1}, \ldots, H_{\mathfrak{p}_m}$, since T is infinite; then $y(k(N!a))/y(N!a) \to k$ in $K_{\mathfrak{p}_i}$ for each $i \in \{1, \ldots, m\}$. Thus S weakly approximates \mathbb{Z} .
- (c) Let n be as in Lemma 3.2. Let U' be the set of $\alpha \in \mathcal{O}_L$ such that there exist $k \in S$ and I = num(s) for some $s \in S$ such that $(\alpha 1) \cdots (\alpha n) \mid I$ and $\alpha \equiv k \pmod{I}$. The end of Section 2 implies that U' is \mathcal{O}_L -diophantine. By Lemma 3.2, $U' \subset \mathcal{O}_K$.

If $\alpha \in \mathbb{Z} - \{1, \ldots, n\}$, Lemma 4.1 provides $s \in S$ such that $(\alpha - 1) \cdots (\alpha - n) \mid I := \text{num}(s)$. Since S weakly approximates \mathbb{Z} , there exists $k \in S$ such that $k \equiv \alpha \pmod{I}$. Thus $\alpha \in U'$.

Take $U := U' \cup \{1, ..., n\}$, which is \mathcal{O}_L -diophantine.

(d) Let b_1, \ldots, b_{κ} be a \mathbb{Z} -basis of \mathcal{O}_K . Then $\mathcal{O}_K = \sum_{i=1}^{\kappa} Ub_i$, which is \mathcal{O}_L -diophantine. \square

Remark 4.3. In order to guarantee that some y was a uniformizer along the arc for each of $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$, we let y range over all linear combinations of y_1, \ldots, y_g with coefficients in an infinite set T. But in fact, if we assume (as we may) that A is simple, then already y_1 suffices, because the p-adic analogue of Wüstholz's analytic subspace theorem implies that for every \mathfrak{p} , the \mathfrak{p} -adic logarithm $\log_{\mathfrak{p}} a \in \operatorname{Lie} A_{K_{\mathfrak{p}}}$ does not lie in any hyperplane defined over K; see [Mat10, Theorem 1] or [FP15, Proposition 2.5].

5. Shlapentokh's reduction

Theorem 5.1 (Shlapentokh). Assume that

- \mathbb{Z} is \mathcal{O}_E -diophantine for every totally real number field E, and
- \mathcal{O}_K is \mathcal{O}_L -diophantine for every degree 2 extension of number fields L/K.

Then \mathbb{Z} is \mathcal{O}_F -diophantine for every number field F.

Proof. If F'/F is a finite extension and \mathbb{Z} is $\mathcal{O}_{F'}$ -diophantine, then \mathbb{Z} is also \mathcal{O}_{F} -diophantine. Thus we may enlarge F to assume that F is Galois over \mathbb{Q} .

For each complex conjugation $\sigma \in \operatorname{Aut} F$ arising from a nonreal embedding $F \hookrightarrow \mathbb{C}$, we have $[F : F^{\sigma}] = 2$, so $\mathcal{O}_{F^{\sigma}}$ is \mathcal{O}_{F} -diophantine. Let $E = \bigcap_{\sigma} F^{\sigma}$. Then the intersection $\mathcal{O}_{E} = \bigcap_{\sigma} \mathcal{O}_{F^{\sigma}}$ is \mathcal{O}_{F} -diophantine. On the other hand, E is totally real, so \mathbb{Z} is \mathcal{O}_{E} -diophantine by assumption. By transitivity, \mathbb{Z} is \mathcal{O}_{F} -diophantine.

Remark 5.2. [KP25, Corollary 2.5] proved $\mathcal{A}_{K,L}$ (and hence that \mathcal{O}_K is \mathcal{O}_L -diophantine) not for all degree 2 extensions L/K, but only those satisfying all of the following additional assumptions: $K \subset \mathbb{R}$, L is Galois over \mathbb{Q} , and $L \supset L_0 := \mathbb{Q}(\sqrt{-1}, \sqrt{5}, \sqrt{7}, \sqrt{11}, \sqrt{13}, \sqrt{17}, \sqrt{19})$. But it is easy to adapt the proof of Theorem 5.1 to use its second hypothesis only for these extensions, by enlarging F to contain L_0 .

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