ALGEBRAIC DE RHAM COHOMOLOGY OF AN ELLIPTIC CURVE

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Abstract. Let X be an elliptic curve over a ring R. The goal of this note is to explain the claim at the bottom of page 163 in [Kat73] that the first de Rham cohomology group $\mathrm{H}^1_{\mathrm{dR}}(X)$ is isomorphic to the R-module of meromorphic 1-forms on X that are holomorphic except for at most a double pole along the zero section.

1. De Rham complex

Suppose that X is a scheme that is smooth of relative dimension n over a base ring R. Let $\mathscr{O} = \mathscr{O}_X$. Then the sheaf of 1-forms $\Omega := \Omega_{X/R}$ is a rank n vector bundle on X, i.e., a locally free $\mathscr{O}$-module of rank n. For $i \geq 0$, define $\Omega^i := \bigwedge^i \Omega$; in particular, $\Omega^0 = \mathscr{O}$. The R-linear homomorphism $d\colon \mathscr{O} \to \Omega$ extends to give the de Rham complex

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \cdots,$$

which is denoted $\Omega^\bullet$. For $i \geq 0$, the ith de Rham cohomology group of X is defined to be the hypercohomology of this complex:

$$\mathrm{H}^i(X) = \mathrm{H}^i_{\mathrm{dR}}(X) := \mathbb{H}^i(X, \Omega^\bullet).$$

Remark 1.1. In general, to compute the ith hypercohomology of a complex of abelian sheaves,

1. replace the complex by a quasi-isomorphic complex whose terms are *acyclic* sheaves;
2. take global sections to get a complex of abelian groups; and
3. take the ith cohomology (kernel modulo image at the ith position) of that complex.

2. Logarithmic de Rham complex

Suppose that D is a smooth divisor on X; that is, D is a closed subscheme of X that is smooth over R, and locally on X there are local coordinates $t_1, \ldots, t_n$ such that D is defined by $t_1 = 0$.

Let $\Omega(\log D)$ be the rank n vector bundle defined locally as the $\mathscr{O}$-span of $\dfrac{dt_1}{t_1}$, dt_2, …, dt_n inside the sheaf of meromorphic 1-forms on X. For $i \geq 0$, define $\Omega^i(\log D) := \bigwedge^i \Omega(\log D)$; in particular, $\Omega^0(\log D) = \mathscr{O}$. One obtains the logarithmic de Rham complex

$$\Omega^0(\log D) \xrightarrow{d} \Omega^1(\log D) \xrightarrow{d} \Omega^2(\log D) \xrightarrow{d} \cdots,$$

which is denoted $\Omega^\bullet(\log D)$. For $i \geq 0$, one may define

$$\mathrm{H}^i(X - D) = \mathrm{H}^i_{\mathrm{dR}}(X - D) := \mathbb{H}^i(X, \Omega^\bullet(\log D)).$$

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3. RESIDUES IN COMPLEX ANALYSIS (WARMUP)

If $\omega = \sum_{n \in \mathbb{Z}} a_n z^n \, dz$ is a meromorphic 1-form on a neighborhood of 0 in $\mathbb{C}$, then its residue at 0 is $\operatorname{Res} \omega := a_{-1}$. (It is more natural to speak of the residue of a differential form instead of the residue of a meromorphic function, because it is the former notion that is independent of the choice of analytic coordinate.)

Let $\mathbb{C}\{\{z\}\}$ be the ring of power series that converge on some unspecified neighborhood of 0; then

$$0 \longrightarrow \mathbb{C}\{\{z\}\} \, dz \longrightarrow \mathbb{C}\{\{z\}\} \frac{dz}{z} \xrightarrow{\operatorname{Res}} \mathbb{C} \longrightarrow 0 \tag{1}$$

$$f(z) \frac{dz}{z} \longmapsto f(0)$$

is exact.

4. RESIDUES ALONG A SMOOTH DIVISOR IN A SCHEME

Now let X and D be as in Sections 1 and 2. Let $j \colon D \to X$ be the inclusion. Any abelian sheaf $\mathscr{F}$ on D gives rise to a sheaf $j_* \mathscr{F}$ on X, and $\mathrm{H}^i(D, \mathscr{F}) \simeq \mathrm{H}^i(X, j_* \mathscr{F})$, so there is no harm in omitting the j_* in the notation; for example, we may write $\mathscr{O}_D$ and Ω^i_D when we mean the corresponding sheaves on X.

Residue along D is defined as the R-linear homomorphism of sheaves

$$\Omega(\log D) \xrightarrow{\operatorname{Res}} \mathscr{O}_D$$

$$f_1 \frac{dt_1}{t_1} + f_2 \, dt_2 + \cdots f_n \, dt_n \longmapsto f_1 \Big|_{t_1 = 0}$$

on X. The homomorphism Res fits into an exact sequence

$$0 \longrightarrow \Omega \longrightarrow \Omega(\log D) \xrightarrow{\operatorname{Res}} \mathscr{O}_D \longrightarrow 0$$

analogous to (1).

More generally, if $i \geq 1$, then a section of $\Omega^r(\log D)$ is locally of the form $\eta \wedge \dfrac{dt_1}{t_1} + \omega$ for some sections $\eta \in \Omega^{i-1}$ and $\omega \in \Omega^i$, and we define the residue homomorphism

$$\Omega^i(\log D) \xrightarrow{\operatorname{Res}} \Omega^{i-1}_D$$

$$\eta \wedge \frac{dt_1}{t_1} + \omega \longmapsto \eta \Big|_{t_1 = 0},$$

which fits in an exact sequence

$$0 \longrightarrow \Omega^i \longrightarrow \Omega^i(\log D) \xrightarrow{\operatorname{Res}} \Omega^{i-1}_D \longrightarrow 0.$$

The operator d (on differential forms on X, X, and D) maps each of these exact sequences to the next, so we obtain the following:

Theorem 4.1. *There is an exact sequence of complexes*

$$0 \longrightarrow \Omega^\bullet \longrightarrow \Omega^\bullet(\log D) \xrightarrow{\operatorname{Res}} \Omega^{\bullet - 1}_D \longrightarrow 0$$

of sheaves on X.

Taking hypercohomology yields

$$0 = \mathrm{H}^{-1}(D) \longrightarrow \mathrm{H}^1(X) \longrightarrow \mathrm{H}^1(X-D) \longrightarrow \mathrm{H}^0(D) \longrightarrow \mathrm{H}^2(X) \longrightarrow \cdots . \tag{2}$$

5. De Rham cohomology of a curve

Now specialize to the case in which $X \to \operatorname{Spec} R$ is smooth and proper with fibers that are geometrically integral curves, and D is given by a section (so if R is a field, then D is a single point). Let $t = t_1$. Let $\mathscr{I}_D$ be the ideal sheaf of $D \subset X$. Then the line bundles $\Omega(\log D)$, Ω, and $\mathscr{I}_D^{\otimes -1} \simeq \mathscr{O}(D)$ are locally generated by dt/t, dt, and $1/t$, respectively, so

$$\Omega(\log D) = \Omega \underset{\mathscr{O}}{\otimes} \mathscr{I}_D^{\otimes -1} = \Omega \underset{\mathscr{O}}{\otimes} \mathscr{O}(D) =: \Omega(D).$$

Also, it turns out that $\mathrm{H}^0(D) \to \mathrm{H}^2(X)$ is an isomorphism of free rank 1 R-modules, so (2) implies that

$$\mathrm{H}^1(X) \simeq \mathrm{H}^1(X-D).$$

In other words:

Lemma 5.1. *The inclusion of the de Rham complex* $\mathscr{O} \xrightarrow{d} \Omega$ *into the logarithmic de Rham complex* $\mathscr{O} \xrightarrow{d} \Omega(D)$ *induces an isomorphism on* $\mathbb{H}^1$.

On the other hand:

Lemma 5.2. *The inclusion of the complex* $\mathscr{O} \xrightarrow{d} \Omega(D)$ *into the complex* $\mathscr{O}(D) \xrightarrow{d} \Omega(2D)$ *is a quasi-isomorphism, so it too induces an isomorphism on* $\mathbb{H}^1$.

Proof. The quotient complex consists of the isomorphism of rank 1 $\mathscr{O}_D$-modules

$$\begin{aligned} \mathscr{O}(D)/\mathscr{O} &\xrightarrow{d} \Omega(2D)/\Omega(D) \\ \frac{1}{t} &\longmapsto -\frac{1}{t^2}\,dt \end{aligned}$$

(viewed as an isomorphism of sheaves on X), so it is exact. □

Theorem 5.3. *We have an isomorphism*

$$\mathrm{H}^1(X) = \mathbb{H}^1\left(\mathscr{O}(D) \xrightarrow{d} \Omega(2D)\right).$$

Proof. Combine the two lemmas. □

6. De Rham cohomology of an elliptic curve

Now specialize further to the case in which X is an elliptic curve over R, and D is the zero section.

Lemma 6.1. *We have* $\mathrm{H}^1(X, \mathscr{O}(D)) = 0$ *and* $\mathrm{H}^1(X, \Omega(2D)) = 0$.

Proof. If R is a field, then Serre duality states that $\mathrm{H}^1(X, \mathscr{O}(D))$ is dual to $\mathrm{H}^0(X, \Omega(-D))$, which is 0 since $\deg \Omega(-D) = 0 - 1 < 0$. The result $\mathrm{H}^1(X, \mathscr{O}(D)) = 0$ for a field R together with the cohomology and base change theorem [Har77, Theorem III.12.11(a)] implies the result for any noetherian R. The general case follows by taking a direct limit.

The proof that $\mathrm{H}^1(X, \Omega(2D)) = 0$ is similar. □

Lemma 6.2. *We have* $\Gamma(X, \mathscr{O}(D)) = R$.

Proof. First suppose that R is a field. A nonconstant element of $\Gamma(X, \mathscr{O}(D))$ would define a degree 1 rational map to $\mathbb{P}^1$, but an elliptic curve cannot be birational to $\mathbb{P}^1$, so $\Gamma(X, \mathscr{O}(D)) = R$. The general case follows from [Har77, Theorem III.12.11(a)] again. ☐

Theorem 6.3. *If X is an elliptic curve over R, then* $\mathrm{H}^1(X) \simeq \Gamma(X, \Omega(2D))$.

Proof. By Theorem 5.3,

$$\begin{aligned}
\mathrm{H}^1(X) &= \mathbb{H}^1\left(\mathscr{O}(D) \xrightarrow{d} \Omega(2D)\right) \\
&= h^1\left(\Gamma(X, \mathscr{O}(D)) \xrightarrow{d} \Gamma(X, \Omega(2D))\right) \\
&\qquad \text{(since } \mathscr{O}(D) \text{ and } \Omega(2D) \text{ are acyclic by Lemma 6.1)} \\
&= \operatorname{coker}\left(\Gamma(X, \mathscr{O}(D)) \xrightarrow{d} \Gamma(X, \Omega(2D))\right) \\
&= \operatorname{coker}\left(R \xrightarrow{0} \Gamma(X, \Omega(2D))\right) \quad \text{(by Lemma 6.2)} \\
&= \Gamma(X, \Omega(2D)).
\end{aligned}$$

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