ALGEBRAIC DIFFERENTIAL EQUATIONS: AN INTRODUCTION TO LOCAL SYSTEMS, CONNECTIONS, AND p-CURVATURE

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1. Introduction

We aim to give a down-to-earth introduction to the theory of linear differential equations on complex manifolds, and the algebraic analogue of this theory. Along the way, we introduce complex manifolds, local systems, vector bundles, derivations, connections, the Riemann–Hilbert correspondence, p-curvature, and the Grothendieck–Katz p-curvature conjecture.

There are excellent sources for this material, such as [Del70], [Kat70], and [Kat72], containing much more than we cover here. The present article is intended to be a warm-up for reading such accounts. To make the article accessible to a broad audience, we provide extra background and discuss more examples to motivate definitions. The main prerequisites are topology, single-variable complex analysis, sheaves, $\mathscr{O}_X$-modules, and schemes.

2. Complex manifolds

Equip $\mathbb{C}$ with the usual topology, defined by the absolute value. Let $n \in \mathbb{Z}_{\geq 0}$. Give $\mathbb{C}^n$ the product topology. Let $U \subset \mathbb{C}^n$ be an open subset. A function $f \colon U \to \mathbb{C}$ is holomorphic if it is locally given by a power series; more explicitly, f is holomorphic if U has an open cover $(U_i)_{i \in I}$ such that for each i, the restriction $f|_{U_i}$ is given by a convergent power series in n variables centered at some point. Let $\mathscr{O}(U)$ be the ring of holomorphic functions $U \to \mathbb{C}$.

The rings $\mathscr{O}(V)$ for open $V \subset \mathbb{C}^n$, with the restriction maps, form a sheaf $\mathscr{O}$ on $\mathbb{C}^n$. Then $(\mathbb{C}^n, \mathscr{O})$ is a locally ringed space. Restricting to any open subset $U \subset \mathbb{C}^n$ defines a locally ringed space $(U, \mathscr{O}_U)$.

An n-dimensional complex manifold is a locally ringed space $(X, \mathscr{O}_X)$ that is locally isomorphic to one of the form $(U, \mathscr{O}_U)$ as above; that is, X has an open cover $(X_i)_{i \in I}$ such that for each i, there exists an open subset $U_i \subset \mathbb{C}^n$ and an isomorphism of locally ringed spaces

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$(X_i, \mathscr{O}_X|_{X_i}) \to (U_i, \mathscr{O}_{U_i})$. (By using a different sheaf of rings, one can similarly define C^∞ real manifolds, topological manifolds, and so on.)

For $x \in X$, define the local ring $\mathscr{O}_{(x)}$ as the stalk of $\mathscr{O}_X$ at x, let $\mathfrak{m}_x \subset \mathscr{O}_{(x)}$ be its maximal ideal, and let $k_x = \mathscr{O}_{(x)}/\mathfrak{m}_x \simeq \mathbb{C}$. (The local ring is more commonly denoted by $\mathscr{O}_x$ or $\mathscr{O}_{X,x}$, but we follow the notation of [Del70, I.2.1] in order to distinguish stalks from fibers for a vector bundle in Section 5.)

3. Review of linear differential equations

Let $U \subset \mathbb{C}$ be a simply connected open subset. Let $a \in \mathscr{O}(U)$. Then the equation $f' = af$, in which $f \in \mathscr{O}(U)$ is the function to be solved for, is an ordinary differential equation (ODE) as opposed to a partial differential equation (PDE), because f is a function of only *one* variable. It is linear, meaning that every term is a function of the input variable(s) times f or (one of) its derivative(s); this implies that the set of holomorphic solutions on U is a $\mathbb{C}$-*subspace* of $\mathscr{O}(U)$. It is first-order, since the highest derivative of f that appears is the first derivative. Let $u \in U$ and $b \in \mathbb{C}$. The *existence and uniqueness theorem* says that there exists a unique holomorphic function $f \colon U \to \mathbb{C}$ satisfying the ODE $f' = af$ with initial condition $f(u) = b$. (In fact, separation of variables leads to the explicit solution, $f(z) = b \exp(\int_u^z a(w)\, dw)$.)

The following theorem is a version involving a *tuple* of unknown functions, or equivalently a function valued in $\mathbb{C}^n$ (but there might not be an explicit formula for the solution anymore):

Theorem 3.1 (Existence and uniqueness for a system of linear ODEs)**.** *Fix $n \geq 0$. Let $U \subset \mathbb{C}$ be a simply connected open subset. Let $u \in U$. Let $A \in \mathrm{M}_n(\mathscr{O}(U))$. Let $b \in \mathbb{C}^n$. Then there exists a unique $f \in \mathscr{O}(U)^n$ satisfying $f' = Af$ and $f(u) = b$.*

Remark 3.2 (C^∞ version). Theorem 3.1 remains true if one replaces "holomorphic" by "infinitely differentiable" (C^∞) and $\mathbb{C}$ by $\mathbb{R}$ everywhere.

Remark 3.3 (Not algebraic). Existence can fail in the algebraic context. For example, the solution of the algebraic differential equation $f' = z^2 f$ with $f(0) = 1$ on $U = \mathbb{C}$ is $e^{z^3/3}$, which is not algebraic.

Remark 3.4 (Nonlinear DEs). There is an existence and uniqueness theorem for *nonlinear* differential equations, but the solutions need not exist on the whole domain. For example, on $\mathbb{C}$, the differential equation $f' = f^2$ with initial condition $f(0) = 1$ has a holomorphic solution in a neighborhood of 0, namely $1/(1-z)$, but the solution does not extend to a holomorphic function on all of $\mathbb{C}$.

☡ *Warning* 3.5 (Simply connected requirement). One cannot remove "simply connected" in Theorem 3.1. For example, $f' = \frac{1}{2z} f$ (think $d(\log f) = \frac{1}{2} d(\log z)$) has a nonzero solution on any simply connected subset of $\mathbb{C}^\times$ (for instance, a branch of $\sqrt{z}$), but no nonzero holomorphic solution on $\mathbb{C}^\times$ itself.

Remark 3.6 (Higher-order DEs). Higher-order differential equations can be rewritten as first-order systems by introducing new unknown function variables to represent the intermediate derivatives. For example, any branch of $\log z$ on an open subset of $\mathbb{C}^\times$ is a solution to $(zf')' = 0$, which is equivalent to the second-order equation

$$f'' + (1/z)f' = 0.$$

Introduce $g = f'$ to obtain the equivalent first-order system $f' = g$ and $g' = (-1/z)g$, which in the format of Theorem 3.1 is

$$\begin{pmatrix} f' \\ g' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -1/z \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}. \tag{3.7}$$

We lose nothing by focusing only on *first-order* systems from now on.

Remark 3.8 (PDEs). Theorem 7.20, to be discussed later, is a version of Theorem 3.1 that applies to functions of many variables, defined on a simply connected open subset of $\mathbb{C}^m$, say, but for $m \geq 2$ an additional "integrability hypothesis" is needed to ensure that solutions exist.

- As a nonlinear example, no holomorphic function $F(x, y)$ on a nonempty open subset of $\mathbb{C}^2$ satisfies $\frac{\partial F}{\partial x} = y$ and $\frac{\partial F}{\partial y} = -x$, because for any holomorphic $F(x, y)$, the y-derivative of $\frac{\partial F}{\partial x}$ must equal the x-derivative of $\frac{\partial F}{\partial y}$.
- As a linear example, no nonzero holomorphic function $f(x, y)$ satisfies $\frac{\partial f}{\partial x} = yf$ and $\frac{\partial f}{\partial y} = -xf$, since on any open ball where f is nonvanishing, a branch of $\log f$ would be a solution F to the previous system.

4. LOCAL SYSTEMS

In this section, X is a topological space.

4.1. **Constant sheaves.** Let $n \in \mathbb{Z}_{\geq 0}$. The constant presheaf $\mathbb{C}^n_{X,\mathrm{pre}}$ is the presheaf such that

(i) for every open subset $U \subset X$, one has $\mathbb{C}^n_{X,\mathrm{pre}}(U) = \mathbb{C}^n$, and
(ii) the restriction maps are the identity maps.

The constant sheaf $\mathbb{C}^n_X$ is the sheafification of $\mathbb{C}^n_{X,\mathrm{pre}}$. Thus, for every open subset $U \subset X$, the space $\mathbb{C}^n_X(U)$ is the $\mathbb{C}$-vector space of *locally* constant functions $U \to \mathbb{C}^n$. An automorphism of $\mathbb{C}^n_X$ as sheaf of $\mathbb{C}$-vector spaces is given by a locally constant function $X \to \mathrm{GL}_n(\mathbb{C})$; if X is connected, this function is constant. If $\phi\colon Y \to X$ is a continuous map, then $\phi^{-1}\mathbb{C}^n_X = \mathbb{C}^n_Y$.

4.2. **The definition of local system.** A local system $\mathcal{L}$ on X is a sheaf of $\mathbb{C}$-vector spaces that is locally isomorphic to a constant sheaf $\mathbb{C}^n_X$; this means that there exist an open covering (U_i) of X, nonnegative integers n_i, and isomorphisms $\phi_i\colon \mathbb{C}^{n_i}_{U_i} \to \mathcal{L}|_{U_i}$ of sheaves of $\mathbb{C}$-vector spaces. If all the $n_i = n$ for all i, then $\mathcal{L}$ is called an n-dimensional local system.

Remark 4.1. If X is connected, then all the n_i are necessarily equal.

Remark 4.2. A more general definition of local system allows constant sheaves other than $\mathbb{C}^n_X$, but in this article we stick to local systems of $\mathbb{C}$-vector spaces.

Local systems form a full subcategory of the category of sheaves of $\mathbb{C}$-vector spaces. In fact, the category of local systems on X is a *rigid tensor category*, because it has operations $\oplus$, $\otimes$, $\mathscr{H}om$ and an identity object $\mathbb{C}_X$, all behaving as expected.

4.3. **Examples of local systems.**

Example 4.3. A local system isomorphic to $\mathbb{C}^n_X$ for some n is called constant.

Example 4.4 (Solutions to a differential equation)**.** Let $X = \mathbb{C}^\times$. Consider the differential equation

$$f' = \frac{1}{2z} f \tag{4.5}$$

mentioned in Warning 3.5. For each open subset $U \subset X$, let $\mathcal{L}(U)$ be the $\mathbb{C}$-vector space of holomorphic solutions $f\colon U \to \mathbb{C}$ to (4.5). These form a sheaf of $\mathbb{C}$-vector spaces $\mathcal{L}$ on X. We claim that $\mathcal{L}$ is a 1-dimensional local system.

Suppose that U is simply connected. Analytic continuation constructs a branch of $\sqrt{z}$ on U, and it is an everywhere nonvanishing solution to (4.5); call it $\sqrt{z}$. Any other holomorphic function on an open subset of U is $g\sqrt{z}$ for some g, and $g\sqrt{z}$ is a solution to (4.5) if and only if $g' = 0$. Thus $\mathcal{L}|_U = \mathbb{C}_U \sqrt{z}$.

Since X is covered by its simply connected subsets, $\mathcal{L}$ is a local system. The only holomorphic solution to (4.5) on X is 0, so the only global section of $\mathcal{L}$ is 0, so $\mathcal{L} \not\simeq \mathbb{C}_X$.

Example 4.6. Likewise, for any system of linear ODEs $f' = Af$ as in Theorem 3.1, the solutions form an n-dimensional local system.

Let $\mathcal{L}$ be a local system on X. The fiber of $\mathcal{L}$ at a point $x \in X$ is the stalk $\mathcal{L}_x$, which is a finite-dimensional $\mathbb{C}$-vector space. If $\mathcal{L}$ is an n-dimensional local system, then $\mathcal{L}_x$ is an n-dimensional $\mathbb{C}$-vector space.

Remark 4.7 (Visualizing a local system: the total space). Let us construct a topological space L with a map $p\colon L \to X$ such that $p^{-1}x = \mathcal{L}_x$ for each $x \in X$. If $\mathcal{L} \simeq \mathbb{C}^n_X$, give $L := \coprod_{x \in X} \mathcal{L}_x \simeq \coprod_{x \in X} \mathbb{C}^n = \mathbb{C}^n \times X$ the product topology. In general, equip $L := \coprod_{x \in X} \mathcal{L}_x$ with the topology such that for each open subset U with trivialization $\mathbb{C}^n_U \to \mathcal{L}|_U$, the subset $\coprod_{x \in U} \mathcal{L}_x$ is open with the product topology as above. Call L the total space of the local system $\mathcal{L}$. If X is a complex manifold, then L has a natural structure of complex manifold as well. A section of $L \to X$ above U is a continuous map $s\colon U \to p^{-1}U$ such that $ps = 1_U$; call s locally constant if s is locally constant with respect to every trivialization of $\mathcal{L}$ on every open subset of U. Then $\mathcal{L}$ is the sheaf of locally constant sections of $L \to X$.

Example 4.8 (Relative Betti cohomology)**.** Let X be a compact C^∞ manifold. Fix $q \in \mathbb{Z}_{\geq 0}$. The qth Betti cohomology group (or singular cohomology group) of X with complex coefficients is a finite-dimensional vector space $\mathrm{H}^q(X, \mathbb{C})$. The global sections functor $\Gamma(X, -)$ can be viewed as π_* for the map $X \xrightarrow{\pi} \{\mathrm{pt}\}$; taking derived functors lets one view $\mathrm{H}^q(X, \mathbb{C})$ as $\mathrm{R}^q\pi_*\mathbb{C}_X$. What happens in a *family* of such manifolds?

Let $X \xrightarrow{\pi} B$ be a proper submersion of C^∞ manifolds (submersion in differential geometry is the analogue of smooth morphism in algebraic geometry). For each $b \in B$, the fiber $X_b := \pi^{-1}b$ is a compact C^∞ manifold; then $X \to B$ may be viewed as the *family* of these manifolds X_b, parametrized by B. The relative Betti cohomology is $\mathrm{R}^q\pi_*\mathbb{C}_X$, a sheaf of $\mathbb{C}$-vector spaces on B. Ehresmann's fibration theorem states that every $b \in B$ has neighborhood U such that $\pi^{-1}U \to U$ is isomorphic to a constant family $Y \times U \to U$, for some Y (isomorphic to X_b); then $(\mathrm{R}^q\pi_*\mathbb{C}_X)|_U$ is a constant sheaf on U (with fibers isomorphic to $\mathrm{H}^q(Y, \mathbb{C})$). Thus $\mathrm{R}^q\pi_*\mathbb{C}_X$ is a local system on B. The fiber of $\mathrm{R}^q\pi_*\mathbb{C}_X$ above any point b is $\mathrm{H}^q(X_b, \mathbb{C})$ by the proper base change theorem.

Warning 4.9. Ehresmann's fibration theorem is a C^∞ phenomenon with no holomorphic analogue. For example, let $\mathfrak{h}$ be the upper half-plane $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. There is a proper holomorphic submersion $X \xrightarrow{\pi} \mathfrak{h}$ whose fiber above $\tau \in \mathfrak{h}$ is the elliptic curve $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$. But nearby fibers are usually not isomorphic as complex manifolds: the j-invariant is varying in this family. What one can say is that *as C^∞ manifolds*, the fibers are all isomorphic to $\mathbb{R}^2/\mathbb{Z}^2$, so each space $\mathrm{H}^1(X_\tau, \mathbb{C})$ is 2-dimensional, and $\mathrm{R}^1\pi_*\mathbb{C}_X$ is a 2-dimensional local system.

If $\phi\colon X \to Y$ is a continuous map and $\mathcal{Y}$ is a local system on Y, then $\phi^{-1}\mathcal{Y}$ is a local system on X, and its fiber at any $x \in X$ is $(\phi^{-1}\mathcal{Y})_x \simeq \mathcal{Y}_{\phi(x)}$.

4.4. **Local systems on an interval.**

Proposition 4.10. *Let $\mathcal{L}$ be a local system on the real interval $[0,1]$. Then*

(a) *$\mathcal{L}$ is constant.*

(b) *There is a* canonical *isomorphism of fibers $\mathcal{L}_0 \xrightarrow{\sim} \mathcal{L}_1$.*

Proof of (a). Use the following two facts:

(i) Every open covering of $[0,1]$ can be refined to one consisting of a finite list of intervals, each intersecting the next. (Proof: $[0,1]$ is compact.)

(ii) If X is a topological space covered by open sets U and V with $U \cap V$ connected, then any local system on X that is constant on U and constant on V is constant. (Proof: The trivializations on U and V differ on $U \cap V$ by a locally constant function $\alpha\colon U \cap V \to \mathrm{GL}_n(\mathbb{C})$. Since $U \cap V$ is connected, α is constant. Thus, by composing the trivialization on V with an element of $\mathrm{GL}_n(\mathbb{C})$, it can be made to agree with the trivialization on U. Gluing gives a trivialization on X.) □

Proof of (b). A trivialization $\mathcal{L} \simeq \mathbb{C}^n_{[0,1]}$ identifies both $\mathcal{L}_0$ and $\mathcal{L}_1$ with $\mathbb{C}^n$, and hence with each other. Changing the trivialization amounts to composing it with an automorphism of $\mathbb{C}^n_{[0,1]}$, which is given by an element of $\mathrm{GL}_n(\mathbb{C})$ since $[0,1]$ is connected, so the identification $\mathcal{L}_0 \overset{\sim}{\to} \mathcal{L}_1$ is unchanged. □

Remark 4.11. A similar proof shows that a local system on $[0,1]^2$ is constant, and its fibers are canonically isomorphic.

Remark 4.12. The same proof shows that a local system on an irreducible quasi-compact topological space is constant. For this reason, local systems on an irreducible algebraic variety with the *Zariski topology* are not interesting — one needs a finer topology like the analytic topology on a complex manifold, or the étale topology.

4.5. **The fundamental group.** Let X be a topological space. Let $x, y \in X$. A path from x to y is a continuous function $\gamma \colon [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(y)$; if $x = y$, then γ is a loop based at x. If α is a path from x to y, and β is a path from y to z, then traversing α followed by β (and renormalizing the domain to make it $[0,1]$ again) gives a path $\beta\alpha$ from x to z. Paths γ_0 and γ_1 from x to y are homotopic if there is a family of paths $(\gamma_t)_{t\in[0,1]}$ from x to y, such that γ_0 and γ_1 are the given ones, and such that $(t,u) \mapsto \gamma_t(u)$ is a continuous function $[0,1]^2 \to X$. Let $\pi_1(X, x, y)$ be the set of homotopy classes of paths from x to y. For $x, y, z \in X$, concatenation induces $\pi_1(X, y, z) \times \pi_1(X, x, y) \to \pi_1(X, x, z)$. The fundamental group $\pi_1(X, x)$ is $\pi_1(X, x, x)$, the group of homotopy classes of loops based at x; the group operation is concatenation.

4.6. **Local systems and representations of $\boldsymbol{\pi_1}$.** Let $\mathcal{L}$ be a local system on X. Let $x, y \in X$. Let γ be a path from x to y. By Proposition 4.10, $\gamma^{-1}\mathcal{L}$ is constant, and its fibers above 0 and 1 are canonically isomorphic: $\mathcal{L}_x \simeq \mathcal{L}_y$; this isomorphism is called parallel transport along γ. If two paths are homotopic via $(\gamma_t)_{t\in[0,1]}$, then every γ_t induces the *same* isomorphism $\mathcal{L}_x \to \mathcal{L}_y$, because of Remark 4.11. Thus we obtain

$$\pi_1(X, x, y) \times \mathcal{L}_x \longrightarrow \mathcal{L}_y.$$

Taking $x = y$ gives an action of the group $\pi_1(X, x)$ on the $\mathbb{C}$-vector space $\mathcal{L}_x$. In other words, the fiber $\mathcal{L}_x$ becomes a representation of $\pi_1(X, x)$; call it the monodromy representation. The image of $\pi_1(X, x) \to \mathrm{GL}(\mathcal{L}_x)$ is called the monodromy group.

Example 4.13. Let $X = \mathbb{C}^\times$ and $x = 1$. Then $\pi_1(\mathbb{C}^\times, 1) \simeq \mathbb{Z}$, generated by the class $[\gamma]$ of a loop γ going once counterclockwise around 0. Let $\mathcal{L}$ be the local system of Example 4.4. Then $\mathcal{L}_x \simeq \mathbb{C}$. Analytically continuing a local solution $\sqrt{z}$ near 1 along γ returns to the solution $-\sqrt{z}$, so the monodromy representation

$$\pi_1(\mathbb{C}^\times, 1) \longrightarrow \mathrm{GL}(\mathcal{L}_x) \simeq \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$$

sends $[\gamma]$ to -1. The monodromy group is $\{\pm 1\}$.

Example 4.14. Let $X = \mathbb{C}^\times$. In Remark 3.6 we encountered the system

$$f' = \begin{pmatrix} 0 & 1 \\ 0 & -1/z \end{pmatrix} f. \tag{4.15}$$

Let $U \subset \mathbb{C}^\times$ be a disk centered at 1. By Theorem 3.1, there exist unique solutions to (4.15) on U taking the values $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ at 1; these are $f_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $f_2 := \begin{pmatrix} \operatorname{Log} z \\ 1/z \end{pmatrix}$, where $\operatorname{Log} z$ is the principal branch of the complex logarithm. Analytically continuing f_1 and f_2 along the loop γ defined in Example 4.13, we find that they return to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} \operatorname{Log} z + 2\pi i \\ 1/z \end{pmatrix}$. Thus the monodromy representation sends $[\gamma]$ to $\begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$. The monodromy group is the group $\begin{pmatrix} 1 & 2\pi i\mathbb{Z} \\ 0 & 1 \end{pmatrix} \simeq \mathbb{Z}$.

Theorem 4.16. *Let X be a connected and locally simply connected topological space. Fix $x \in X$. Then the functor*

$$\begin{aligned} \{\text{local systems on } X\} \quad &\longleftrightarrow \quad \{\text{finite-dimensional } \mathbb{C}\text{-representations of } \pi_1(X,x)\} \\ \mathcal{L} \quad &\longmapsto \quad (\mathcal{L}_x \text{ with the monodromy action}) \end{aligned}$$

is an equivalence of tensor categories.

Sketch of proof. Let us describe the inverse functor. Let $G = \pi_1(X,x)$, and let $\rho\colon G \to \mathrm{GL}(V)$ be a representation. Turn the left G-action on V into a right action by letting g act as $v \mapsto \rho(g)^{-1}v$. Let $\tilde{X}$ be the universal cover of X; a point of $\tilde{X}$ is a pair (y,α) where $y \in X$ and $\alpha \in \pi_1(X,x,y)$. The covering map $\tilde{X} \overset{c}{\to} X$ defined by $(y,\alpha) \mapsto y$ is a Galois cover with group G acting on $\tilde{X}$ on the right by composing with α. Let $\mathcal{L}(X)$ be the space of G-equivariant locally constant functions $\tilde{X} \to V$. Likewise, for each open subset $U \subset X$, let $\tilde{U} = c^{-1}U \subset \tilde{X}$, and let $\mathcal{L}(U)$ be the space of G-equivariant locally constant functions $\tilde{U} \to V$. Then $\mathcal{L}$ is a local system on X with monodromy representation ρ. The associated total space $L \to X$ (see Remark 4.7) is $(V \times \tilde{X})/G \longrightarrow \tilde{X}/G = X$. □

For more details, see [Sza09, Chapter 2].

5. VECTOR BUNDLES

Let X be a complex manifold. Let $m = \dim X$. Let $\mathscr{O} = \mathscr{O}_X$ be the sheaf of holomorphic functions on X.

A vector bundle on X is a locally free $\mathscr{O}$-module $\mathcal{V}$ that is locally of finite rank; this means that there exist an open covering (U_i) of X, nonnegative integers n_i, and isomorphisms $\phi_i \colon \mathscr{O}_{U_i}^{n_i} \to \mathcal{V}|_{U_i}$ of $\mathscr{O}_{U_i}$-modules. If all the n_i equal one nonnegative integer n, then $\mathcal{V}$ is called a rank n vector bundle. Vector bundles form a full subcategory of the category of $\mathscr{O}_X$-modules.

Conventions:

- $v \in \mathcal{V}$ means that $v \in \mathcal{V}(U)$ for some open subset $U \subset X$.
- $\mathscr{H}om$ or $\otimes$ of vector bundles always denotes $\mathscr{H}om_{\mathscr{O}}$ or $\otimes_{\mathscr{O}}$, respectively.
- The dual of $\mathcal{V}$ is $\mathcal{V}^\vee := \mathscr{H}om(\mathcal{V}, \mathscr{O})$, a vector bundle of the same rank as $\mathcal{V}$.

Here are some examples of vector bundles:

vector bundle	notation	rank
sheaf of holomorphic functions	$\mathscr{O}$	1
tangent bundle	$\mathcal{T}$	m
sheaf of holomorphic 1-forms = cotangent bundle	$\Omega^1 := \mathcal{T}^\vee$	m
sheaf of holomorphic p-forms	$\Omega^p := \bigwedge^p \Omega^1$	$\binom{m}{p}$
sheaf of 1-jets	J^1	$m+1$

Let $\mathcal{V}$ be a vector bundle on X. For $x \in X$, we have

- the stalk $\mathcal{V}_{(x)}$, which is a finite free $\mathscr{O}_{(x)}$-module, and
- the fiber $\mathcal{V}_x := \mathcal{V} \underset{\mathscr{O}_X}{\otimes} k_x = \mathcal{V}_{(x)}/\mathfrak{m}_x \mathcal{V}_{(x)}$, which is a finite-dimensional $\mathbb{C}$-vector space.

Example 5.1. The fiber of J^1 at a point $x \in X$ is the k_x-vector space of possible first-order Taylor expansions of holomorphic functions defined in a neighborhood of x. “Taking the constant term” defines a k_x-linear map $(J^1)_x \to k_x$. These are the fiber maps of an $\mathscr{O}$-linear map $J^1 \to \mathscr{O}$ fitting in an exact sequence

$$0 \longrightarrow \Omega^1 \longrightarrow J^1 \longrightarrow \mathscr{O} \longrightarrow 0. \tag{5.2}$$

Tensoring (5.2) with another vector bundle $\mathcal{V}$ yields the Atiyah exact sequence

$$0 \longrightarrow \Omega^1 \otimes \mathcal{V} \longrightarrow J^1 \otimes \mathcal{V} \longrightarrow \mathcal{V} \longrightarrow 0. \tag{5.3}$$

As in Remark 4.7, we construct a total space V, a complex manifold with a holomorphic map $p \colon V \to X$ such that $p^{-1}x = \mathcal{V}_x$ for each $x \in X$.

Remark 5.4 (Transition functions). Let $\mathcal{V}$ be a rank n vector bundle, so there exist an open covering (U_i) of X and isomorphisms $\phi_i \colon \mathscr{O}_{U_i}^n \to \mathcal{V}|_{U_i}$. For any i and j, on $U_{ij} = U_i \cap U_j$, the map $\phi_j^{-1}\phi_i$ is an $\mathscr{O}_{U_{ij}}$-linear automorphism of $\mathscr{O}_{U_{ij}}^n$, hence an element $\alpha_{ij} \in \mathrm{GL}_n(\mathscr{O}(U_{ij}))$, called a transition function. Moreover, on each triple intersection $U_{ijk} = U_i \cap U_j \cap U_k$, one has the cocycle condition $\alpha_{jk}\alpha_{ij} = \alpha_{ik}$. Conversely, given X, an open covering (U_i), a nonnegative integer n, and elements $\alpha_{ij} \in \mathrm{GL}_n(\mathscr{O}(U_{ij}))$ satisfying the cocycle condition, one can use the α_{ij} to glue the sheaves $\mathscr{O}_{U_i}^n$ to get a vector bundle on X.

One can describe local systems on X in the same way, except that now the entries of each α_{ij} are required to be *locally constant* functions on U_i instead of arbitrary holomorphic functions.

Example 5.5. Suppose that $\mathcal{L}$ is a local system on X. Then $\mathcal{V} := \mathscr{O} \underset{\mathbb{C}}{\otimes} \mathcal{L}$ is a vector bundle on X. If $\mathcal{L}$ is an n-dimensional local system, then $\mathcal{V}$ is a vector bundle of rank n. A trivialization of $\mathcal{L}$ on an open cover determines a trivialization of $\mathcal{V}$, and the (locally constant) transition functions for $\mathcal{L}$ are also (holomorphic) transition functions for $\mathcal{V}$.

View $\mathcal{L}$ as a subsheaf of $\mathcal{V}$. For each $x \in X$, we have $\mathcal{L}_x = \mathcal{V}_x$. Thus the total spaces L and V are the same manifold over X! But $\mathcal{L}$ is the sheaf of locally constant sections, while $\mathcal{V}$ is the sheaf of all holomorphic sections.

6. DERIVATIONS

Let A be a $\mathbb{C}$-algebra. A derivation of A is a $\mathbb{C}$-linear map $D\colon A \to A$ satisfying the Leibniz rule (product rule)

$$D(fg) = D(f)\,g + f\,D(g) \quad \text{for all } f, g \in A.$$

Example 6.1. The map $\frac{\partial}{\partial x}\colon \mathbb{C}[x,y] \to \mathbb{C}[x,y]$ is a derivation.

Example 6.2. The map $y^2\frac{\partial}{\partial x}\colon \mathbb{C}[x,y] \to \mathbb{C}[x,y]$ defined by $f \mapsto y^2\frac{\partial f}{\partial x}$ is another derivation.

Now let $\mathscr{A}$ be a sheaf of $\mathbb{C}$-algebras on X. A derivation of $\mathscr{A}$ is a morphism $D\colon \mathscr{A} \to \mathscr{A}$ of sheaves of $\mathbb{C}$-vector spaces such that $D(fg) = D(f)\,g + f\,D(g)$ for all sections $f, g \in \mathscr{A}$. Let $\mathrm{Der}(\mathscr{A})$ be the $\mathbb{C}$-vector space of all derivations of $\mathscr{A}$. Let $\mathscr{D}er(\mathscr{A})$ be the sheaf $U \mapsto \mathrm{Der}(\mathscr{A}_U)$. In the special case that $\mathscr{A}$ is an $\mathscr{O}$-algebra, $\mathscr{D}er(\mathscr{A})$ is an $\mathscr{O}$-module. Let $\mathscr{D}er = \mathscr{D}er(\mathscr{O})$.

Example 6.3. Let X be an open subset of $\mathbb{C}^m$ with coordinates $z_1, \ldots, z_m$. Then

$$\mathscr{D}er = \mathscr{O}\frac{\partial}{\partial z_1} + \cdots + \mathscr{O}\frac{\partial}{\partial z_m}.$$

Now fix any complex manifold X. For every vector field $t \in \mathcal{T}$ and function $f \in \mathscr{O}$, let $D_t f \in \mathscr{O}$ be the function whose value at x is the directional derivative of f in the direction given by the tangent vector $t(x)$. Then $D_t \in \mathscr{D}er$. One can show that $t \mapsto D_t$ defines an isomorphism $\mathcal{T} \simeq \mathscr{D}er$. We identify vector fields with derivations from now on.

7. CONNECTIONS

7.1. Motivation: derivatives of functions.

Let X be a complex manifold. The pairing

$$\begin{aligned} \mathcal{T} \times \mathscr{O} &\longrightarrow \mathscr{O} \\ t, f &\longmapsto D_t f \end{aligned}$$

is $\mathscr{O}$-linear in t, but only $\mathbb{C}$-linear in f (for example, $\left(e^z \frac{d}{dz}\right) f = e^z \left(\frac{d}{dz} f\right)$ by definition, but $\frac{d}{dz}(e^z f) \neq e^z \left(\frac{d}{dz} f\right)$ in general). The pairing induces a $\mathbb{C}$-linear map

$$\mathscr{O} \longrightarrow \mathscr{H}om(\mathcal{T}, \mathscr{O}) = \Omega^1,$$

which equals the map

$$\begin{aligned} d\colon \mathscr{O} &\longrightarrow \Omega^1 \\ f &\longmapsto df. \end{aligned}$$

It satisfies $d(fg) = df\, g + f\, dg$ (we view $df\, g$ as synonymous with $g\, df$).

7.2. **Derivatives of sections: the definition of connection.** Now let $\mathcal{V}$ be a vector bundle on X. We would like to take derivatives of sections of $\mathcal{V}$, but the following example shows that there is no *canonical* way to do so.

Example 7.1. Let $X = \mathbb{C}$. Let $\mathcal{V}$ be a free rank 1 vector bundle on X. If we choose an identification $\mathcal{V} \simeq \mathscr{O}$, then we can use the derivation $\frac{d}{dz}$ on $\mathscr{O}$ to obtain a rule for differentiating sections of $\mathcal{V}$. For example, if $v \in \mathcal{V}(X)$ corresponds to $1 \in \mathscr{O}(X)$, then $\frac{d}{dz} v = 0$. But if we change the identification by composing $\mathcal{V} \xrightarrow{\sim} \mathscr{O}$ with the isomorphism $\mathscr{O} \to \mathscr{O}$ defined by $f \mapsto e^z f$, then the same v now corresponds to e^z, whose derivative is e^z, so we would instead define $\frac{d}{dz} v = v \neq 0$. Thus there is no canonical way to differentiate sections of $\mathcal{V}$.

Given any vector bundle $\mathcal{V}$ on a complex manifold X, to equip $\mathcal{V}$ with a rule for taking directional derivatives of sections of $\mathcal{V}$, we should specify a pairing

$$\begin{aligned} \mathcal{T} \times \mathcal{V} &\longrightarrow \mathcal{V} \\ t, v \quad &\longmapsto \nabla_t\, v \end{aligned}$$

that is $\mathscr{O}$-linear in t and $\mathbb{C}$-linear in v, or equivalently a $\mathbb{C}$-linear map

$$\nabla\colon \mathcal{V} \longrightarrow \mathscr{H}om(\mathcal{T}, \mathcal{V}) \simeq \Omega^1 \otimes \mathcal{V}.$$

This motivates the following definition: A connection on $\mathcal{V}$ is a $\mathbb{C}$-linear map

$$\nabla\colon \mathcal{V} \longrightarrow \Omega^1 \otimes \mathcal{V}$$

satisfying the Leibniz rule

$$\nabla(fv) = df \otimes v + f\, \nabla v$$

for all functions $f \in \mathscr{O}$ and sections $v \in \mathcal{V}$.

Given $(\mathcal{V}, \nabla)$, each $D \in \mathscr{D}er(X) = \mathcal{T}(X) = \operatorname{Hom}(\Omega^1, \mathscr{O})$ induces a $\mathbb{C}$-linear map

$$\nabla_D\colon\ \mathcal{V} \xrightarrow{\quad \nabla \quad} \Omega^1 \otimes \mathcal{V} \xrightarrow{\quad D\otimes 1 \quad} \mathscr{O} \otimes \mathcal{V} = \mathcal{V}$$

satisfying a Leibniz rule, and likewise each $D \in \mathscr{D}er(U)$ for $U \subset X$ gives $\nabla_D \colon \mathcal{V}|_U \to \mathcal{V}|_U$. Giving such a collection of maps ∇_D indexed by D satisfying suitable conditions is the same as giving ∇.

7.3. **Examples of connections.**

Example 7.2. The map d is a connection on $\mathscr{O}$.

Example 7.3. Let $\omega \in \Omega^1(X)$. Interpret ω as the "multiplication-by-ω" map $\mathscr{O} \longrightarrow \Omega^1$ defined by $f \mapsto f\omega$. Then

$$\begin{aligned} d+\omega \colon \mathscr{O} &\longrightarrow \Omega^1 \\ f &\longmapsto df + f\omega \end{aligned}$$

is a connection on $\mathscr{O}$.

Proposition 7.4. *Every connection on $\mathscr{O}$ is $d+\omega$ for some $\omega \in \Omega^1(X)$.*

Proof. Let ∇ be a connection on $\mathscr{O}$. Then

$$\begin{aligned} \nabla(fg) &= df\, g + f\, \nabla g \\ d(fg) &= df\, g + f\, dg \\ (\nabla - d)(fg) &= \qquad\quad f\, (\nabla - d) g. \end{aligned}$$

Thus $\nabla - d$ is an $\mathscr{O}$-*linear* map $\mathscr{O} \to \Omega^1$, so it is $f \mapsto f\omega$ for some $\omega \in \Omega^1(X)$. □

In general, if ∇ is one connection on $\mathcal{V}$, all the others are $\nabla + h$ for a global section h of

$$\mathscr{H}om(\mathcal{V}, \Omega^1 \otimes \mathcal{V}) \quad \simeq \quad \Omega^1 \otimes (\mathscr{E}nd\, \mathcal{V}).$$

Corollary 7.5. *Every connection on $\mathscr{O}^n$ is $d+\omega$, where d acts coordinate-wise, and ω is some $n \times n$ matrix of global 1-forms.*

Remark 7.6. One can also define C^∞ connections in the context of C^∞ vector bundles on C^∞ real manifolds. The definitions are essentially the same.

7.4. **Operations on vector bundles with connections.**

One has the same constructions for vector bundles with connections as one has for representations of a fixed group. Suppose that $(\mathcal{V}_1, \nabla_1)$ and $(\mathcal{V}_2, \nabla_2)$ are vector bundles with connections on X. Then one has the following.

Direct sum: $\mathcal{V}_1 \oplus \mathcal{V}_2$ with ∇ defined by

$$\nabla(v_1 + v_2) = \nabla_1 v_1 + \nabla_2 v_2.$$

Tensor product: $\mathcal{V}_1 \otimes \mathcal{V}_2$ with ∇ defined by the Leibniz rule

$$\nabla(v_1 \otimes v_2) = \nabla_1 v_1 \otimes v_2 + v_1 \otimes \nabla_2 v_2.$$

Identity for tensor product: $\mathscr{O}$ with d.

Internal hom: $\mathscr{H}om(\mathcal{V}_1,\mathcal{V}_2)$ with ∇ defined implicitly by the Leibniz rule

$$\nabla_2(\phi v_1) = (\nabla\phi)(v_1) + \phi(\nabla_1 v_1), \tag{7.7}$$

for all $\phi \in \mathscr{H}om(\mathcal{V}_1,\mathcal{V}_2)$ and $v_1 \in \mathcal{V}_1$
(the right side of (7.7) is an abbreviation for $(\nabla\phi)(1\otimes v_1) + (1\otimes\phi)(\nabla_1 v_1)$).

We also have

Pullback: Let $\phi\colon Y \to X$ be a holomorphic map of complex manifolds. Let $(\mathcal{V},\nabla)$ be a vector bundle with connection on X. Then $\phi^*\mathcal{V}$ with the composition

$$\begin{tikzcd} \phi^*\mathcal{V} \arrow[r] & \phi^*(\Omega^1_X \underset{\mathscr{O}_X}{\otimes} \mathcal{V}) \arrow[r, "\sim"] & \phi^*\Omega^1_X \underset{\mathscr{O}_Y}{\otimes} \phi^*\mathcal{V} \arrow[r, "(d\phi)^\vee\otimes 1"] & \Omega^1_Y \underset{\mathscr{O}_Y}{\otimes} \phi^*\mathcal{V} \end{tikzcd}$$

is a vector bundle with connection on Y, denoted $\phi^*(\mathcal{V},\nabla)$.

7.5. **Horizontal sections and solutions to differential equations.** Let $(\mathcal{V},\nabla)$ be a vector bundle with connection. A section v of $\mathcal{V}$ is called horizontal if $\nabla v = 0$. Let $\mathcal{V}^\nabla = \ker\nabla \subset \mathcal{V}$ be the subsheaf of horizontal sections.

Example 7.8. Let $U \subset \mathbb{C}$ be an open subset. Let $\mathcal{V} = \mathscr{O}_U^n$. Let $A \in \mathrm{M}_n(\mathscr{O}(U))$. Let $\nabla = d - A\,dz$. Then the horizontal sections of $\mathcal{V}$ are the solutions to the system $f' = Af$ of Theorem 3.1.

Given *any* $(\mathcal{V},\nabla)$, suppose that we choose a trivialization $\mathcal{V}|_U \simeq \mathscr{O}_U^n$ on some open subset U; then the equation $\nabla v = 0$ restricted to U amounts to a system of linear differential equations (ODEs if $\dim X = 1$, PDEs if $\dim X \geq 2$). The horizontal sections on U are the solutions to the system. In general, $\nabla v = 0$ can be viewed as a coordinate-free version of a system of linear differential equations.

Example 7.9. Let $X = \mathbb{C}$. Let $\mathcal{V} = \mathscr{O}$ and $\nabla = d - z^2\,dz$. Then $\nabla f = df - fz^2\,dz$, so $\nabla f = 0$ if and only if f is a solution to $f' = z^2 f$. Thus $\mathcal{V}^\nabla$ is $\mathbb{C}_X \cdot e^{z^3/3}$, the sheaf of locally constant multiples of the function $e^{z^3/3}$.

Example 7.10. Let $X = \mathbb{C}^\times$. Let $\mathcal{V} = \mathscr{O}^2$ and $\nabla = d - \begin{pmatrix} 0 & 1 \\ 0 & -1/z \end{pmatrix} dz$. Then $\mathcal{V}^\nabla$ is the sheaf of solutions to (4.15). The linearly independent solutions f_1 and f_2 on the disk U in Example 4.14 must have linearly independent *values* in $\mathbb{C}^2$ at every point $u \in U$, since otherwise uniqueness in Theorem 3.1 with initial condition at u would be violated. Thus $\mathcal{V}^\nabla|_U = \mathbb{C}_U f_1 \oplus \mathbb{C}_U f_2$. Likewise, there exists a basis of 2 solutions on *any* disk in X, so $\mathcal{V}^\nabla$ is a 2-dimensional local system on X.

Proposition 7.11. *Suppose that* $\dim X = 1$. *Let* $(\mathcal{V},\nabla)$ *be a rank* n *vector bundle with connection on* X. *Then* $\mathcal{V}^\nabla$ *is an* n*-dimensional local system on* X.

Proof. Apply the existence and uniqueness theorem (Theorem 3.1) locally, as in Example 7.10. $\square$

Remark 7.12 (Parallel transport). Proposition 7.11 holds also for C^∞ connections on a 1-dimensional real manifold. Now suppose that X is a C^∞ manifold of arbitrary dimension, $\mathcal{V}$ is a C^∞ vector bundle, and ∇ is a C^∞ connection. Given a C^∞ path $\gamma\colon [0,1] \to X$ from x to y, the pullback of $(\mathcal{V}, \nabla)$ to $[0,1]$ corresponds to a trivial local system, so we get a parallel transport isomorphism $\mathcal{V}_x \to \mathcal{V}_y$ of vector spaces. To find the image of $v \in \mathcal{V}_x$, "follow the horizontal section along γ". In this way, ∇ "connects" the fiber $\mathcal{V}_x$ to the fiber $\mathcal{V}_y$ (via γ); that is why it is called a connection.

7.6. **Curvature and integrable connections.** We return to the setting of holomorphic vector bundles on a complex manifold X. If $\dim X \geq 2$, the conclusion of Proposition 7.11 does not hold for every connection ∇, because of the issue mentioned in Remark 3.8. But it will hold if ∇ satisfies an integrability condition, expressed by the vanishing of its curvature; we next explain what this means.

A connection ∇ on $\mathcal{V}$ induces an infinite sequence of $\mathbb{C}$-linear maps

$$\mathcal{V} \xrightarrow{\nabla} \Omega^1 \otimes \mathcal{V} \xrightarrow{\nabla_1} \Omega^2 \otimes \mathcal{V} \xrightarrow{\nabla_2} \Omega^3 \otimes \mathcal{V} \xrightarrow{\nabla_3} \cdots, \tag{7.13}$$

in which ∇_i is defined by combining $d\colon \Omega^i \to \Omega^{i+1}$ and $\nabla\colon \mathcal{V} \to \Omega^1 \otimes \mathcal{V}$ using a Leibniz-like rule:

$$\nabla_i(\omega \otimes v) := d\omega \otimes v + (-1)^i \omega \wedge \nabla v$$

for all sections $\omega \in \Omega^i$ and $v \in \mathcal{V}$.

The curvature of ∇ is

$$K := \nabla_1 \circ \nabla \colon \quad \mathcal{V} \longrightarrow \Omega^2 \otimes \mathcal{V}.$$

A priori, K is only $\mathbb{C}$-linear, but a calculation shows that K is $\mathscr{O}$-linear, so K is a global section of

$$\mathscr{H}\!om(\mathcal{V}, \Omega^2 \otimes \mathcal{V}) \simeq \Omega^2 \otimes (\mathscr{E}\!nd\, \mathcal{V}).$$

One calls ∇ an integrable connection (or flat connection) if $K = 0$.

Example 7.14. If $\dim X = 1$, then $\Omega^2 = 0$, so $K = 0$ automatically! (Note: We have been discussing a *holomorphic* connection. If instead we consider a C^∞ connection on a C^∞ vector bundle on a 1-dimensional complex manifold X, then X functions as a 2-*dimensional* real manifold, so K could be nonzero.)

Example 7.15. If $(V, \nabla) = (\mathscr{O}, d)$, then (7.13) is the usual holomorphic de Rham complex

$$\mathscr{O} \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \Omega^3 \xrightarrow{d} \cdots,$$

denoted $\Omega^\bullet$. We have $d \circ d = 0$ starting at any term (that is what it means to be a complex). In particular, $K = 0$.

Example 7.16. If $(V, \nabla) = (\mathscr{O}, d + \omega)$ for some $\omega \in \Omega^1(X)$, then

$$K(1) = \nabla_1(\nabla 1) = \nabla_1 \omega = d\omega,$$

so $K = d\omega$ as a global section of $\Omega^2 \otimes (\mathscr{E}nd\, \mathcal{V}) \simeq \Omega^2$. Thus

$$\nabla \text{ is integrable } \iff \omega \text{ is a } \textit{closed} \text{ 1-form.}$$

Example 7.17. Let X be $\mathbb{C}^2$ with coordinates x, y. Let $\mathcal{V} = \mathscr{O}$. Let

$$\omega \;=\; -y\,dx + x\,dy \;\in\; \Omega^1(X).$$

Let $\nabla = d + \omega$. By Example 7.16, the curvature of ∇ is

$$K \;=\; d\omega \;=\; -dy \wedge dx + dx \wedge dy \;=\; 2dx \wedge dy \;\neq\; 0.$$

Thus ∇ is not integrable.

The sections of $\mathcal{V}^\nabla$ are the solutions to $df + \omega f = 0$, which is equivalent to the system of linear PDEs $\partial f/\partial x = yf$ and $\partial f/\partial y = -xf$. As explained in Remark 3.8, there are no nonzero solutions. Conclusion: $\mathcal{V}^\nabla = 0$.

Remark 7.18. All the operations in Section 7.4 respect integrability [Kat70, p. 180].

7.7. **The Riemann–Hilbert correspondence for holomorphic connections.** Let $(\mathcal{V}_1, \nabla_1)$ and $(\mathcal{V}_2, \nabla_2)$ be vector bundles with connection on X. Let Then $\phi\colon \mathcal{V}_1 \to \mathcal{V}_2$ be an $\mathscr{O}$-linear map. Call ϕ horizontal if it is horizontal as a section of $\mathscr{H}om(\mathcal{V}_1, \mathcal{V}_2)$ with respect to the connection defined in Section 7.4; explicitly, ϕ is horizontal if and only if ϕ is compatible with the connections:

$$\phi(\nabla_1 v_1) = \nabla_2(\phi v_1) \qquad \text{for all } v_1 \in \mathcal{V}_1.$$

For fixed X, we get a category with

- *objects*: $(\mathcal{V}, \nabla)$, where $\mathcal{V}$ is a vector bundle on X,
  and ∇ is an integrable connection on $\mathcal{V}$,
- *morphisms*: horizontal $\mathscr{O}$-linear maps.

Example 7.19. Given $\omega, \omega' \in \Omega^1(X)$, when is $(\mathscr{O}, d + \omega) \simeq (\mathscr{O}, d + \omega')$? An isomorphism of vector bundles $\phi\colon \mathscr{O} \to \mathscr{O}$ is multiplication by some $g \in \mathscr{O}(X)^\times$. The following are equivalent:

- ϕ is an isomorphism $(\mathscr{O}, d + \omega) \simeq (\mathscr{O}, d + \omega')$;
- ϕ is compatible with the connections;
- $g(d + \omega)f = (d + \omega')(gf)$ for all $f \in \mathscr{O}$;
- $g\,df + g\omega f = g\,df + dg\,f + \omega' g f$ for all $f \in \mathscr{O}$;
- $\omega = \omega' + \frac{dg}{g}$.

Thus $(\mathscr{O}, d + \omega) \simeq (\mathscr{O}, d + \omega')$ if and only if $\omega = \omega' + \frac{dg}{g}$ for some $g \in \mathscr{O}(X)^\times$.

Theorem 7.20 (Riemann–Hilbert correspondence for holomorphic connections [Del70, I.2.17])**.**

Fix a complex manifold X. Then the functors

$$\begin{array}{rcl} \{\text{local systems on } X\} & \longleftrightarrow & \{\text{vector bundles with integrable connection}\} \\ \mathcal{L} & \longmapsto & (\mathscr{O} \underset{\mathbb{C}}{\otimes} \mathcal{L}, d \otimes 1) \\ \mathcal{V}^{\nabla} & \longleftarrow\!\shortmid & (\mathcal{V}, \nabla) \end{array}$$

define an equivalence of tensor categories.

Sketch of proof. In checking that the functors are defined and that their composition in either order is isomorphic to the identity, we may work locally on X. Thus, starting from $\mathcal{L}$, we may assume that $\mathcal{L} = \mathbb{C}_X^n$; then $(\mathscr{O} \underset{\mathbb{C}}{\otimes} \mathcal{L}, d \otimes 1) \simeq (\mathscr{O}^n, d)$, which is a vector bundle with integrable connection since $d \circ d = 0$, and taking the sheaf of horizontal sections recovers $\mathbb{C}_X^n$.

On the other hand, starting from $(\mathcal{V}, \nabla)$, we may assume that X is a simply connected open subset of $\mathbb{C}^m$ and $\mathcal{V} \simeq \mathscr{O}^n$; then the main point is the following

Claim: For every $x \in X$, the evaluate-at-x map $\mathcal{V}^{\nabla}(X) \to \mathcal{V}_x$
is an isomorphism of $\mathbb{C}$-vector spaces.

Given this, a $\mathbb{C}$-basis of $\mathcal{V}_x$ gives a basis of $\mathcal{V}^{\Delta}(X)$, and those basis solutions have $\mathbb{C}$-independent values at each $y \in X$ (by the claim at y), so they form both an $\mathscr{O}$-basis of $\mathcal{V}$ and a $\mathbb{C}_X$-basis of $\mathcal{V}^{\nabla}$, so we have an isomorphism $(\mathcal{V}, \nabla) \simeq (\mathscr{O}^n, d)$ restricting to $\mathcal{V}^{\nabla} \simeq \mathbb{C}_X^n$, and everything follows.

The claim is an existence and uniqueness theorem for solutions of a system of linear PDEs with an initial condition (existence is surjectivity of the evaluation map, and uniqueness is injectivity). When $\dim X = 1$, it is a restatement of Theorem 3.1. When $\dim X \geq 2$, one still has uniqueness, since Theorem 3.1 provides a unique solution along each 1-dimensional submanifold, and every point of X can be reached along a chain of such submanifolds. Existence is what requires that ∇ be integrable; to construct solutions, one uses a theorem from differential topology, the Frobenius theorem on involutive distributions [Lee13, Chapter 19]. $\square$

Remark 7.21. For a local system $\mathcal{L}$, the connection $d \otimes 1$ is called the canonical connection on $\mathscr{O} \underset{\mathbb{C}}{\otimes} \mathcal{L}$ since it is the only one whose sheaf of horizontal sections is $\mathcal{L}$.

Corollary 7.22. *For a rank n vector bundle $\mathcal{V}$ with connection ∇, the following are equivalent:*

- $\mathcal{V}^{\nabla}$ *is an n-*dimensional *local system* ($\mathcal{V}^{\nabla}$ *is locally isomorphic to* $\mathbb{C}_X^n$).
- $(\mathcal{V}, \nabla) \simeq (\mathscr{O} \underset{\mathbb{C}}{\otimes} \mathcal{L}, d \otimes 1)$ *for some local system* $\mathcal{L}$.
- $(\mathcal{V}, \nabla)$ *is locally isomorphic to* $(\mathscr{O}^n, d)$.
- $\mathcal{V}^{\nabla}$ *spans* $\mathcal{V}$ *as an* $\mathscr{O}$*-module.*
- $\mathscr{O} \underset{\mathbb{C}}{\otimes} \mathcal{V}^{\nabla} \longrightarrow \mathcal{V}$ *is an isomorphism.*
- ∇ *is integrable* (*its curvature is* 0).

If these hold, one says that the system of differential equations $\nabla v = 0$ has a full set of solutions.

7.8. **The Riemann–Hilbert correspondence for algebraic connections.** There are algebraic analogues of vector bundles, connections, curvature, and integrable connections. The definitions are the same, except starting with a smooth variety X over a field k, and its structure sheaf $\mathscr{O}$. One can also define the notion of connection on a quasi-coherent sheaf.

In the case where $k = \mathbb{C}$, there are analytification functors taking the algebraic objects to their analytic counterparts:

- a smooth variety X gives rise to a complex manifold X^{an},
- a vector bundle $\mathcal{V}$ on X gives rise to a vector bundle $\mathcal{V}^{\text{an}}$ on X^{an}, and
- a connection ∇ on $\mathcal{V}$ gives rise to a connection ∇^{an} on $\mathcal{V}^{\text{an}}$.

Serre's GAGA paper [Ser56] proves that if X is proper, then the analytification functor

$$\{\text{vector bundles on } X\} \longrightarrow \{\text{vector bundles on } X^{\text{an}}\}$$

is an equivalence of categories. For a vector bundle $\mathcal{V}$ on a smooth proper $\mathbb{C}$-variety X, it is also true that the analytification functor

$$\begin{aligned}&\{\text{vector bundles with integrable connection on } X\}\\ &\qquad \longrightarrow \{\text{vector bundles with integrable connection on } X^{\text{an}}\}\end{aligned} \tag{7.23}$$

is an equivalence of categories, though GAGA cannot be applied directly, since connections are not $\mathscr{O}$-linear; instead, one way to prove (7.23) is to relate connections to $\mathscr{O}$-linear splittings of the Atiyah exact sequence (5.3) before applying GAGA [Del70, I.2.3].

In the rest of Section 7.8, we assume that X is a smooth $\mathbb{C}$-variety, but *not necessarily proper*.

Warning 7.24. If X is *non-proper*, one cannot expect (7.23) to be an equivalence. For example, suppose that $X = \mathbb{A}^1$. Let $\omega \in \Omega^1(X) = \mathbb{C}[z]\,dz$; suppose that $\omega \neq 0$. Then $(\mathscr{O}, d + \omega)$ and $(\mathscr{O}, d)$ are not isomorphic algebraically, as can be checked by using Example 7.19: there is no $g \in \mathscr{O}(X)^\times = \mathbb{C}[z]^\times$ satisfying $\frac{dg}{g} = \omega$. But their analytifications *are* isomorphic, since we can choose $F \in \mathscr{O}(X) = \mathbb{C}[z]$ with $dF = \omega$ and use the nonvanishing *holomorphic* function $g = e^F$. Thus the functor (7.23) is not full, hence not an equivalence of categories.

The problem is that there are too many algebraically-non-isomorphic objects on the left side of (7.23). To obtain an equivalence, we replace the category on the left by a full subcategory consisting of $(\mathcal{V}, \nabla)$ with at worst *regular* singularities, in a sense to be defined. To motivate the definition, consider the following example.

Example 7.25. Let $X = \mathbb{A}^1 - \{0\}$. We claim that any $(\mathscr{O}, d + \omega)$ on X^{an} is isomorphic to $(\mathscr{O}, d + c\frac{dz}{z})$ for some $c \in \mathbb{C}$. Write $\omega = \sum_{n \in \mathbb{Z}} a_n z^n\,dz$ for some $a_n \in \mathbb{C}$. Let $c = a_{-1}$. Then the residue of $\omega - c\frac{dz}{z}$ at 0 vanishes, so there exists a single-valued holomorphic $F \in \mathscr{O}(X^{\mathrm{an}})$ with $dF = \omega - c\frac{dz}{z}$. By Example 7.19, $g := e^F$ defines an isomorphism.

Remark 7.26. In Example 7.25, if $c \neq 0$, then $c\frac{dz}{z}$ does not extend to a holomorphic 1-form on the compactification $\mathbb{P}^1$, but it is meromorphic, with simple poles at the boundary points 0 and ∞. One says that $c\frac{dz}{z}$ has *logarithmic poles* along the boundary, since $\frac{dz}{z} = d(\log z)$.

Remark 7.27 (Moderate growth). Every solution to $(d + c\frac{dz}{z})f = 0$ is, up to a constant, a branch of z^{-c}. If $c \notin \mathbb{Z}$, these solutions are not meromorphic, since they are not even single-valued on $\mathbb{C}^\times$. But each branch shares a property with meromorphic functions, having moderate growth: each branch is $O(z^{-N})$ for some $N \in \mathbb{Z}_{\geq 0}$ as $z \to 0$ inside the domain of the branch. In contrast, if $\omega = z^{-2}\,dz$ (worse than a simple pole), then the solutions to $(d+\omega)f = 0$ are multiples of $e^{-1/z}$, which has an essential singularity, not of moderate growth.

Remark 7.28 (Compactification). A smooth curve X has a canonical compactification $\overline{X}$. For smooth $\mathbb{C}$-varieties X of arbitrary dimension, Nagata's compactification theorem [Nag62, Nag63] (see [Con07] for a modern treatment) identifies X with an open subscheme of a proper $\mathbb{C}$-scheme $\overline{X}$ (not unique), and Hironaka's work on resolution of singularities [Hir64] shows that one can blow up $\overline{X}$ to assume that $\overline{X}$ is smooth and that $X = \overline{X} - D$ for a normal crossings divisor $D \subset \overline{X}$; here, normal crossings divisor means that at each point of $\overline{X}$, there are local analytic coordinates $z_1, \ldots, z_m$ such that D is given by $z_1 \cdots z_r = 0$ for some $r \leq m$ (that is, analytically locally, D looks like a union of r smooth hyperplanes intersecting transversely).

Definition 7.29. Consider $(\mathcal{V}, \nabla)$ on X. Call ∇ regular if we can write $X = \overline{X} - D$ as above such that each point of $\overline{X}$ has an analytic open neighborhood U with local analytic coordinates $z_1, \ldots, z_m$ such that

- D is given by $z_1 z_2 \cdots z_r = 0$ for some $r \leq m$, and
- we have a trivialization $\mathcal{V}|_{U-D} \simeq \mathscr{O}^n|_{U-D}$ with respect to which

$$\nabla^{\mathrm{an}} = d + \sum_{i=1}^{r} A_i \frac{dz_i}{z_i} + \sum_{j=r+1}^{m} A_j\, dz_j \tag{7.30}$$

  for some matrices of holomorphic functions $A_1, \ldots, A_m \in \mathrm{M}_n(\mathscr{O}(U))$ (thus the 1-forms in (7.30) have logarithmic poles along D).

Remark 7.31. Regular is equivalent to the horizontal sections being of moderate growth along D, in a sense defined in [Del70, II.2]; see [Del70, II.4.1].

Remark 7.32. Deligne in [Del70, II.4.2] defines regular in the more general context of a *holomorphic* vector bundle $\mathcal{V}$ instead of an algebraic one; this requires $\mathcal{V}$ to be equipped with a fixed meromorphic structure in the sense of [Del70, II.1.14], since otherwise one could change basis of $\mathcal{V}$ using functions with essential singularities along D, and this would destroy regularity.

Now, if one restricts the left side of (7.23) to vector bundles with *regular* integrable connection, then (7.23) is an equivalence of categories, even when X is non-proper [Del70, II.5.9]. Combining this deep result with Theorems 4.16 and 7.20 gives the following:

Theorem 7.33 (Riemann–Hilbert correspondence)**.** *Let X be a connected smooth variety over $\mathbb{C}$. Let $x \in X(\mathbb{C})$. Then the following tensor categories are equivalent:*

- $\{$finite-dimensional $\mathbb{C}$-representations of $\pi_1(X^{\mathrm{an}}, x)\}$
- $\{$local systems on $X^{\mathrm{an}}\}$
- $\{$vector bundles on X^{an} with integrable connection$\}$
- $\{$vector bundles on X with regular integrable connection$\}$.

(Note: An object of the last category is just a $(\mathcal{V}, \nabla)$ satisfying the regularity condition: $\overline{X}$, D, and so on must *exist*, but are not part of the data of the object.)

Example 7.34. In Theorem 7.33, the trivial 1-dimensional representation of $\pi_1(X^{\mathrm{an}}, x)$ corresponds to the constant local system $\mathbb{C}_X$, which corresponds to $(\mathscr{O}, d)$ on X^{an} or X.

Corollary 7.35. *Suppose that $(\mathcal{V}, \nabla)$ has regular singularities. Let $\mathcal{L} := (\mathcal{V}^{\mathrm{an}})^{\nabla}$ be the associated local system. Let $\rho\colon \pi_1(X^{\mathrm{an}}, x) \to \mathrm{GL}_n(\mathcal{L}_x)$ be the representation associated to $\mathcal{L}$. Then the following are equivalent:*

(i) *The representation ρ has finite image.*
(ii) *The monodromy group of $\mathcal{L}$ is finite.*
(iii) *There exists a finite unramified cover $\pi\colon \mathcal{Y} \to X^{\mathrm{an}}$ such that the pullback $\pi^*(\mathcal{V}^{\mathrm{an}}, \nabla)$ has a basis of (holomorphic) horizontal sections in $\mathscr{O}(\mathcal{Y})$.*
(iv) *There exists a finite étale cover $\pi\colon Y \to X$ such that $\pi^*(\mathcal{V}, \nabla)$ has a basis of horizontal sections in $\mathscr{O}(Y)$.*

Proof. Finite-index subgroups of $\pi_1(X^{\mathrm{an}}, x)$ correspond to finite unramified covers of X^{an}, which are analytifications of finite étale covers of X by the Riemann existence theorem. The four conditions are equivalent ways of saying that $(\mathcal{V}, \nabla)$, after pullback to a finite cover, is a direct sum of copies of the trivial object of Example 7.34. $\square$

Theorem 7.33 gives a way to access the topological fundamental group of X^{an}, or at least its representations, in purely algebraic terms! Likewise, even though applying the definition of local system directly to X with its Zariski topology yields only constant sheaves, Theorem 7.33 says that we can secretly talk about the richer category of

$$\text{(topological) local systems on } X^{\mathrm{an}}$$

in algebraic terms by using

(algebraic) vector bundles on X with regular integrable connection

as a proxy.

7.9. **The Gauss–Manin connection.** Let X be a C^∞ manifold. Fix $q \in \mathbb{Z}_{\geq 0}$. Then we have a comparison isomorphism between

- the Betti cohomology group $\mathrm{H}^q(X, \mathbb{C})$, and
- the de Rham cohomology group $\mathrm{H}^q_{\mathrm{dR}}(X, \mathbb{C})$, defined as the cohomology of the C^∞ de Rham complex of global forms with complex coefficients. If X is a complex manifold, one can alternatively define $\mathrm{H}^q_{\mathrm{dR}}(X, \mathbb{C})$ as the hypercohomology of the holomorphic de Rham complex $\mathbb{H}^q(X, \Omega^\bullet)$. (The sheaves Ω^p in the holomorphic de Rham complex are not acyclic, so one needs to use hypercohomology instead of cohomology.)

Now let $\pi \colon X \to B$ be a proper submersion of complex manifolds. Then we have

- the relative Betti cohomology $\mathrm{R}^q\pi_*\mathbb{C}_X$, a local system on B (see Example 4.8), and
- the relative de Rham cohomology $\mathscr{H}^q_{\mathrm{dR}}(X/B) := \mathbb{R}^q\pi_*\Omega^\bullet_{X/B}$, a vector bundle on B (here $\mathbb{R}^q\pi_*$ is the qth hyperderived functor of π_*).

The local system $\mathrm{R}^q\pi_*\mathbb{C}_X$ and vector bundle $\mathscr{H}^q_{\mathrm{dR}}(X/B)$ are related as in Example 5.5, via the relative Betti–de Rham comparison isomorphism

$$\mathscr{O}_B \underset{\mathbb{C}}{\otimes} \mathrm{R}^q\pi_*\mathbb{C}_X \xrightarrow{\sim} \mathscr{H}^q_{\mathrm{dR}}(X/B)$$

of vector bundles on B. Under this isomorphism, the canonical connection $d \otimes 1$ on $\mathscr{O}_B \underset{\mathbb{C}}{\otimes} \mathrm{R}^q\pi_*\mathbb{C}_X$ corresponds to an integrable connection ∇_{GM} on $\mathscr{H}^q_{\mathrm{dR}}(X/B)$, called the Gauss–Manin connection. By Corollary 7.22, ∇ is an *integrable* connection.

Now let $\pi \colon X \to B$ be a smooth proper morphism of smooth $\mathbb{C}$-varieties. Let $\Omega^\bullet_{X/B}$ be the *algebraic* de Rham complex. Define $\mathscr{H}^q_{\mathrm{dR}}(X/B) := \mathbb{R}^q\pi_*\Omega^\bullet_{X/B}$, a vector bundle on B. Differentiation of differential forms with respect to parameters induces a rule for differentiating de Rham classes with respect to parameters, which yields an (algebraic) connection ∇_{GM} on $\mathscr{H}^q_{\mathrm{dR}}(X/B)$ whose analytification is the Gauss–Manin connection of the previous paragraph [KO68]. It too is called the Gauss–Manin connection. It is a *regular* integrable connection, by work of Griffiths, Katz [Kat70, Theorem 14.1], and Deligne [Del70, II.7.9]. Especially when $B \subset \mathbb{A}^1_{\mathbb{C}}$, the differential equation corresponding to ∇_{GM} is classically known as the Picard–Fuchs equation. For the example of $\mathscr{H}^1_{\mathrm{dR}}(X/B)$ for the Legendre family of elliptic curves $X \to B$, see [Cle03, §2.10].

7.10. **Other kinds of connections.** This section mentions generalizations of the notion of connection to other settings. These will not be needed in the rest of this article, so this section may be skipped.

Remark 7.36 (Generalized connections). Let $E \xrightarrow{p} M$ be a fibered manifold, that is, a submersion of C^∞ manifolds (some authors require p to be surjective). Let T_E and T_M be the tangent bundles. The vertical tangent space at each $e \in E$ is the kernel of $T_{E,e} \xrightarrow{dp} T_{M,p(e)}$. To give a connection is to specify a complementary *horizontal tangent space* at each $e \in E$. More precisely, a (generalized) connection on $E \to M$ is a splitting of the surjection $T_E \twoheadrightarrow p^* T_M$ of vector bundles on E. Such a connection gives rise to notions of curvature, horizontal sections, and parallel transport along a C^∞ path in M, but parallel transport might be defined only in a small neighborhood of the starting point, as in Remark 3.4.

Remark 7.37 (Linear connections). When discussing generalized connections, the generalized connection arising from a connection ∇ on a C^∞ vector bundle $E \to M$ in the sense of (the C^∞ analogue of) Section 7.2 would be called a linear connection.

Remark 7.38 (Principal G-connections). Here we mention an important special case of Remark 7.36. Let G be a C^∞ Lie group. Consider a fibered manifold $P \xrightarrow{p} M$ equipped with a right G-action on P respecting p, so G acts on the fibers of p. Call $P \xrightarrow{p} M$ with G-action a trivial G-bundle if it is isomorphic *as fibered manifold with G-action* to the projection $M \times G \to M$ with right G-action on the second factor. Call $P \xrightarrow{p} M$ with G-action a principal G-bundle if there is an open cover (M_i) of M such that $p^{-1} M_i \xrightarrow{p} M_i$ is a trivial G-bundle. A principal G-connection on a principal G-bundle $P \xrightarrow{p} M$ is a connection in the sense of Remark 7.36 that is G-invariant.

Remark 7.39 (Linear connections versus principal $\mathrm{GL}_n(\mathbb{R})$-connections). To any rank n vector bundle $\mathcal{V}$ on M, one can associate a $\mathrm{GL}_n(\mathbb{R})$-torsor $P := \mathrm{Isom}(\mathscr{O}^n, \mathcal{V})$ whose fiber above $m \in M$ consists of all the vector space isomorphisms $\mathbb{R}^n \to \mathcal{V}_m$; another way to understand this is to observe that the transition function data needed to specify $\mathcal{V}$ and P are the same. To give a linear connection on $\mathcal{V}$ is equivalent to giving a principal $\mathrm{GL}_n(\mathbb{R})$-connection on P.

8. CHARACTERISTIC p

8.1. **Solutions to differential equations.**

Let k be a field of characteristic $p > 0$. Let X be a smooth variety over k. Let $(\mathcal{V}, \nabla)$ be a vector bundle with connection on X. Define $\mathcal{V}^\nabla = \ker \nabla$ as before.

In the complex manifold setting, $\mathcal{V}^\nabla$ was a sheaf of finite-dimensional $\mathbb{C}$-vector spaces. But the direct analogue in characteristic p fails even for the simplest differential equation:

Example 8.1. Let $X = \mathbb{A}^1 = \operatorname{Spec} k[z]$. The set of solutions $f \in k[z]$ to $\frac{d}{dz} f = 0$ is $k[z^p]$, which is not a finite-dimensional k-vector space.

What structure does $\mathcal{V}^\nabla$ have in general? Let $X^{(p)} \to \operatorname{Spec} k$ be the base change of $X \to \operatorname{Spec} k$ by the morphism $\operatorname{Spec} k \to \operatorname{Spec} k$ induced by the pth power map $k \to k$. Thus $X^{(p)}$ is the k-variety defined by polynomials whose coefficients are the pth powers of the corresponding coefficients for X. Let $F = F_{X/k} \colon X \to X^{(p)}$ be the relative Frobenius morphism, the k-morphism that raises coordinates to the pth power. Then $\mathcal{V}^{\nabla}$ is a sheaf of $F^{-1}\mathscr{O}_{X^{(p)}}$-modules.

The sheaf $F^{-1}\mathscr{O}_{X^{(p)}}$ plays the role of $\mathbb{C}_X$ because it is the kernel of $d \colon \mathscr{O}_X \to \Omega^1_X$. From now on, we identify $F^{-1}\mathscr{O}_{X^{(p)}}$-modules on X with $\mathscr{O}_{X^{(p)}}$-modules on $X^{(p)}$. The characteristic p analogues of local systems are vector bundles on $X^{(p)}$.

8.2. ***p*-curvature.** Let $D \in \mathscr{D}er$. Iterating D gives the higher Leibniz rule

$$D^n(fg) = \sum_{i=0}^{n} \binom{n}{i} (D^{n-i}f)(D^i g),$$

for any $f, g \in \mathscr{O}$. Taking $n = p$ gives

$$D^p(fg) = (D^p f)\, g + f\, (D^p g).$$

since $\binom{p}{i} = 0$ in k for $i = 1, 2, \ldots, p-1$, so D^p is another derivation!

Now let $(\mathcal{V}, \nabla)$ be a vector bundle with integrable connection. For each $D \in \mathscr{D}er$, we can iterate ∇_D, but it is not always true that $(\nabla_D)^p = \nabla_{D^p}$. The failure is measured by the map

$$\psi(D) := (\nabla_D)^p - \nabla_{D^p}.$$

A priori, $\psi(D) \colon \mathcal{V} \to \mathcal{V}$ is only k-linear, but a calculation shows that $\psi(D)$ is actually $\mathscr{O}$-linear [Kat70, 5.0.5–5.0.9]. Varying D, we get a map

$$\begin{aligned} \psi \colon \mathscr{D}er &\longrightarrow \mathscr{E}nd\,\mathcal{V} \\ D &\longmapsto \psi(D) \end{aligned}$$

called the p-curvature of ∇.

Warning 8.2. Even though $\psi(D) \colon \mathcal{V} \to \mathcal{V}$ is $\mathscr{O}$-linear for each D, the map $\psi \colon \mathscr{D}er \to \mathscr{E}nd\,\mathcal{V}$ is only p-linear in D, meaning that $\psi(uD) = u^p\,\psi(D)$ for each $u \in \mathscr{O}$ and $D \in \mathscr{D}er$. Any p-linear map can, however, be reinterpreted as an $\mathscr{O}$-linear map, by using the absolute Frobenius morphism $F_X \colon X \to X$ (identity on X, pulls back functions to their pth powers): specifically, ψ induces an $\mathscr{O}$-linear map $F_X^* \mathscr{D}er \to \mathscr{E}nd\,\mathcal{V}$, which also may be called the p-curvature.

Example 8.3. Let k be a field of characteristic 2. Let $X = \mathbb{A}^1 - \{0\} = \operatorname{Spec} k[z, z^{-1}]$. Consider $\mathscr{O}$ with connection $\nabla := d + w\,dz$ for some $w = \sum_{n \in \mathbb{Z}} w_n z^n \in k[z, z^{-1}]$. Since $\dim X = 1$, ∇ is integrable. Let $D = d/dz$. The operator $D^2 = D^p$ kills z and is in $\mathscr{D}er = \mathscr{O}D$, so $D^2 = 0$. For $f \in \mathscr{O}$,

$$\begin{aligned} \nabla f &= df + wf\,dz \\ \nabla_D f &= f' + wf \\ (\nabla_D)^2 f &= (f'+wf)' + w(f'+wf) = (w'+w^2)f \qquad \text{(since } f''=0\text{), and} \\ \nabla_{D^2} &= \nabla_0 = 0 \\ \psi(D)\,f &= ((\nabla_D)^2 - \nabla_{D^2})f = (w'+w^2)f \\ \psi(D) &= w' + w^2, \quad \text{as a global section of } \mathscr{O} = \mathscr{E}nd\,\mathscr{O}. \end{aligned}$$

The following are equivalent: $\psi = 0$; $\psi(D) = 0$ (since $\mathscr{D}er = \mathscr{O}D$); $w' + w^2 = 0$; $\sum n w_n z^{n-1} = \sum w_n^2 z^{2n}$ in $k[z, z^{-1}]$; $w_{2n+1} = w_n^2$ for all $n \in \mathbb{Z}$ (and $w_n = 0$ for all but finitely many n); $w_{-1} = w_{-1}^2$ and $w_n = 0$ for all $n \neq -1$; $w = 0$ or $w = z^{-1}$. Also, the following are equivalent: $\mathscr{O}$ has a basis of horizontal sections; $\mathscr{O}^\nabla \neq 0$; $\nabla f = 0$ for some nonzero $f \in k[z, z^{-1}]$; $f' + fw = 0$ for some nonzero f; w has the form f'/f; $w = 0$ or z^{-1} (since 0 and $z^{-1}\,dz$ are the only elements of $k[z, z^{-1}]\,dz$ with simple poles and integer residues, including at ∞). Conclusion:

$$\mathscr{O} \text{ has basis of horizontal sections} \iff \text{the } p\text{-curvature of } \nabla \text{ is zero.} \tag{8.4}$$

In fact, (8.4) holds for any vector bundle with integrable connection in characteristic p; see Corollary 8.6. First, here is the characteristic p analogue of Theorem 7.20:

Theorem 8.5 (Cartier [Kat70, Theorem 5.1]). *Fix a smooth variety X over a field of characteristic p. Then the following are equivalent:*

- *the category of quasi-coherent $\mathscr{O}_{X^{(p)}}$-modules, and*
- *the category of $(\mathcal{V}, \nabla)$, where $\mathcal{V}$ is a quasi-coherent $\mathscr{O}_X$-module and ∇ is an integrable connection on $\mathcal{V}$ whose p-curvature is 0.*

The functors in each direction are

$$\begin{aligned} \mathcal{L} &\longmapsto (F^*\mathcal{L}, \nabla_{\text{can}}) \\ \mathcal{V}^\nabla &\longleftarrow\!\shortmid (\mathcal{V}, \nabla) \end{aligned}$$

(*for a suitable canonical connection ∇_{can} on $F^*\mathcal{L}$*).

Corollary 7.22 stated that for $(\mathcal{V}, \nabla)$ on a *complex* manifold,

$$\mathcal{V}^\nabla \text{ spans } \mathcal{V} \text{ as an } \mathscr{O}\text{-module} \iff \text{the curvature of } \nabla \text{ is } 0.$$

In characteristic p, however, one also needs the p-curvature to be 0:

Corollary 8.6. *Let $(\mathcal{V}, \nabla)$ be a vector bundle with connection on X. Then*

$$\mathcal{V}^\nabla \textit{ spans } \mathcal{V} \textit{ as an } \mathscr{O}\textit{-module} \iff \textit{the curvature and } p\textit{-curvature of } \nabla \textit{ are } 0.$$

8.3. **The p-curvature conjecture.** Let K be a number field. Let $\mathcal{O}_K$ be its ring of integers. For each prime $\mathfrak{p} \subset \mathcal{O}_K$, let $\mathbb{F}_\mathfrak{p} = \mathcal{O}_K/\mathfrak{p}$. Let $K(z)$ be the rational function field. Its algebraic closure $\overline{K(z)}$ is the field of algebraic functions. The derivation $\frac{d}{dz}$ extends uniquely to $\overline{K(z)}$ (in general, a derivation of a field extends uniquely to any separable algebraic extension, via implicit differentiation of the minimal equation satisfied by any element of the extension). Let $A \in \mathrm{M}_n(K(z))$ for some $n \geq 0$. Consider the system of ODEs $f' = Af$ involving an unknown n-tuple f of functions of z. For all but finitely many primes $\mathfrak{p} \subset \mathcal{O}_K$, we may reduce modulo $\mathfrak{p}$ to obtain a differential equation involving functions with coefficients in $\mathbb{F}_\mathfrak{p}$.

Conjecture 8.7. Let K and A be as above. Suppose that for all but finitely many primes $\mathfrak{p}$, the reduced equation has a full set of solutions, meaning that there are n solutions in $\mathbb{F}_\mathfrak{p}(z)^n$ that are linearly independent over $\mathbb{F}_\mathfrak{p}(z)$. Then $f' = Af$ admits a full set of solutions in $\overline{K(z)}^n$ (that is, n solutions that are linearly independent over $\overline{K(z)}$).

Example 8.8. Let $K = \mathbb{Q}$. Let $n = 1$. Consider the differential equation $f' = \frac{1}{3z} f$. For any $p \neq 3$, there exists $a \in \mathbb{Z}$ with $3a \equiv 1 \pmod{p}$, and then $z^a \in \mathbb{F}_p(z)$ is a solution to the differential equation reduced modulo p. And although there is no nonzero solution in $\mathbb{Q}(z)$, there is an *algebraic* solution $z^{1/3} \in \overline{\mathbb{Q}(z)}$, as the conjecture predicts.

Example 8.9. Let $K = \mathbb{Q}(\sqrt{2})$. Consider $f' = \frac{\sqrt{2}}{z} f$. If $\mathfrak{p}$ is a degree 1 prime of $\mathcal{O}_K$, then there exists $a \in \mathbb{Z}$ with the same image as $\sqrt{2}$ in $\mathbb{F}_\mathfrak{p}$, and then $z^a \in \mathbb{F}_\mathfrak{p}(z)$ is a solution. But if $\mathfrak{p}$ is a degree 2 prime, then there are no nonzero solutions in $\mathbb{F}_\mathfrak{p}(z)$. So it is not true that the differential equation modulo $\mathfrak{p}$ has a nonzero solution for all but finitely many $\mathfrak{p}$. On the other hand, there is also no solution in $\overline{K(z)}$: there are holomorphic solutions like $z^{\sqrt{2}}$ that make sense on a disk around 1 in $\mathbb{C}^\times$, but they are not algebraic functions.

Conjecture 8.7 can be generalized, and we can replace the condition on existence of solutions modulo $\mathfrak{p}$ with the condition that the p-curvature vanishes, because of Theorem 8.5:

Conjecture 8.10. Let R be a finitely generated subring of $\mathbb{C}$. Let $X \to \operatorname{Spec} R$ be a smooth R-scheme with geometrically connected fibers. Let $(\mathcal{V}, \nabla)$ be a vector bundle with integrable connection. Suppose that for every maximal ideal $\mathfrak{p} \subset R$, the p-curvature of $(\mathcal{V}/\mathfrak{p}\mathcal{V}, \nabla)$ is 0. Then the base change $(\mathcal{V}, \nabla)_\mathbb{C}$ on $X_\mathbb{C} := X \times_R \mathbb{C}$ has a full set of algebraic solutions (that is, a basis of solutions after pulling back $(\mathcal{V}, \nabla)_\mathbb{C}$ to a finite étale cover $Y \to X_\mathbb{C}$).

Remark 8.11 (Reformulation in terms of monodromy). Under the assumption that the p-curvature vanishes for all $\mathfrak{p}$, [Kat70, Theorem 13.0] shows that $(\mathcal{V}, \nabla)_\mathbb{C}$ has regular singularities; then Corollary 7.35 shows that

$$\begin{aligned} &(\mathcal{V}, \nabla)_\mathbb{C} \text{ has a full set of algebraic solutions} \\ \iff &\text{ the monodromy group of the corresponding local system is } \textit{finite}. \end{aligned}$$

Conjectures 8.7 and 8.10 originally appeared as questions attributed to Grothendieck in a paper of Katz [Kat72, (I) and (I bis)]. By a specialization argument, Conjecture 8.10 can be reduced to Conjecture 8.7 [Kat72, p. 2]. Either may be called the Grothendieck–Katz p-curvature conjecture. The conjectures are still open, but there have been partial results [Kat72], [CC85], [Bos01], [And04], [FK09], [Sha18], [PSW21]. For a non-abelian variant of the conjecture, see [LL25].

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