A BRIEF SUMMARY OF THE STATEMENTS OF CLASS FIELD THEORY

BJORN POONEN

0. Profinite completions of topological groups

Let G be a topological group. The profinite completion of G is

$$\widehat{G} := \varprojlim_{U} \frac{G}{U},$$

where U ranges over the finite-index open normal subgroups of G. There is a natural continuous homomorphism $G \to \widehat{G}$ through which every other continuous homomorphism from G to a profinite group factors uniquely. If G is profinite already, then $G \to \widehat{G}$ is an isomorphism.

In general, $G \to \widehat{G}$ need not be injective or surjective. Nevertheless, we think of G as being almost isomorphic to $\widehat{G}$: The finite-index open subgroups of G are in bijection with those of $\widehat{G}$. And finite-index open subgroups of certain Galois groups are what we are interested in. . .

1. Local class field theory

1.1. Notation associated to a discrete valuation ring.

$\mathcal{O}$: a complete discrete valuation ring

$K := \operatorname{Frac}(\mathcal{O})$

v: the valuation $K^\times \twoheadrightarrow \mathbb{Z}$

$\mathfrak{p}$: the maximal ideal of $\mathcal{O}$

k: the residue field $\mathcal{O}/\mathfrak{p}$

K^{s}: a fixed separable closure of K

K^{ab}: the maximal abelian extension of K in K^{s}

K^{unr}: the maximal unramified extension of K in K^{s}

k^{s}: the residue field of K^{unr}, so k^{s} is a separable closure of k.

Equip K and its subsets with the topology coming from the absolute value $|x| := \exp(-v(x))$.

Date: February 17, 2012.

I thank David Zywina for comments. The writing of this article was supported by NSF grant DMS-0301280.

1.2. **Local fields.**

Definition 1.1. A nonarchimedean local field is a complete discrete-valued field K as in Section 1.1 such that the residue field k is finite. An archimedean local field is $\mathbb{R}$ or $\mathbb{C}$.

Facts:

- A nonarchimedean local field of characteristic 0 is isomorphic to a finite extension of $\mathbb{Q}_p$.
- A (nonarchimedean) local field of characteristic $p > 0$ is isomorphic to $\mathbb{F}_q((t))$ for some power q of p.

1.3. **The local Artin homomorphism.** Let K be a local field. Local class field theory says that there is a homomorphism

$$\theta\colon K^\times \to \operatorname{Gal}(K^{\mathrm{ab}}/K)$$

that is almost an isomorphism. The homomorphism θ is called the local Artin homomorphism. It cannot be literally an isomorphism, because $\operatorname{Gal}(K^{\mathrm{ab}}/K)$ is a profinite group, hence compact, while $K^\times$ is not. What is true is that θ induces an isomorphism of topological groups $\widehat{K^\times} \to \operatorname{Gal}(K^{\mathrm{ab}}/K)$.

If K is archimedean, then $\theta\colon K^\times \to \operatorname{Gal}(K^{\mathrm{ab}}/K)$ is surjective and its kernel is the connected component of the identity in $K^\times$.

For the rest of Section 1.3, we assume that K is nonarchimedean. Then θ is injective: The choice of a uniformizer $\pi \in \mathcal{O}$ lets us write $K^\times = \mathcal{O}^\times \pi^{\mathbb{Z}} \simeq \mathcal{O}^\times \times \mathbb{Z}$, and $\mathcal{O}^\times$ is already profinite, so $\widehat{K^\times} \simeq \mathcal{O}^\times \times \hat{\mathbb{Z}}$. Thus local class field theory says that there is an isomorphism

$$\mathcal{O}^\times \times \hat{\mathbb{Z}} \to \operatorname{Gal}(K^{\mathrm{ab}}/K).$$

More canonically, without choosing π, the two horizontal exact sequences below are almost isomorphic:

(1)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}^\times & \longrightarrow & K^\times & \xrightarrow{v} & \mathbb{Z} & \longrightarrow & 0 \\
 & & \| & & \big\downarrow \theta & & \big\downarrow & & \\
0 & \longrightarrow & \operatorname{Gal}(K^{\mathrm{ab}}/K^{\mathrm{unr}}) & \longrightarrow & \operatorname{Gal}(K^{\mathrm{ab}}/K) & \xrightarrow{\mathrm{res}} & \operatorname{Gal}(K^{\mathrm{unr}}/K) & \longrightarrow & 0
\end{array}$$

With the identification of the group at lower right

$$\operatorname{Gal}(K^{\mathrm{unr}}/K) \simeq \operatorname{Gal}(k^{\mathrm{s}}/k) \simeq \hat{\mathbb{Z}}$$

mapping the Frobenius automorphism to $1 \in \hat{\mathbb{Z}}$, the right vertical map in (1) becomes the natural inclusion $\mathbb{Z} \hookrightarrow \hat{\mathbb{Z}}$. In other words, θ maps $K^\times$ isomorphically to the set of $\sigma \in \operatorname{Gal}(K^{\mathrm{ab}}/K)$ inducing an *integer* power of Frobenius on the residue field (as opposed to a $\hat{\mathbb{Z}}$-power). The bottom row of (1) is simply the profinite completion of the top row.

Also from (1), one sees that $\theta(\mathcal{O}^\times)$ is the inertia subgroup $\operatorname{Gal}(K^{\mathrm{ab}}/K^{\mathrm{unr}})$ of $\operatorname{Gal}(K^{\mathrm{ab}}/K)$, and that θ maps any uniformizer to a Frobenius automorphism in $\operatorname{Gal}(K^{\mathrm{ab}}/K)$. Moreover, the descending chain

$$\mathcal{O}^\times \supset 1 + \mathfrak{p} \supset 1 + \mathfrak{p}^2 \supset \cdots$$

is mapped isomorphically by θ to the descending chain of ramification subgroups of $\operatorname{Gal}(K^{\mathrm{ab}}/K)$ in the upper numbering.

1.4. **Functoriality.** Let L be a finite extension of K. Let $N_{L/K}\colon L^\times \to K^\times$ be the norm map. Let θ_L, θ_K be the local Artin homomorphisms associated to L, K, respectively. Let $\operatorname{res}\colon \operatorname{Gal}(L^{\mathrm{ab}}/L) \to \operatorname{Gal}(K^{\mathrm{ab}}/K)$ be the homomorphism mapping an automorphism σ of L^{ab} to its restriction $\sigma|_{K^{\mathrm{ab}}}$. Then the square

$$\begin{array}{ccc} L^\times & \xrightarrow{\theta_L} & \operatorname{Gal}(L^{\mathrm{ab}}/L) \\ {\scriptstyle N_{L/K}}\downarrow & & \downarrow{\scriptstyle \mathrm{res}} \\ K^\times & \xrightarrow{\theta_K} & \operatorname{Gal}(K^{\mathrm{ab}}/K) \end{array}$$

commutes.

1.5. **Finite abelian extensions.** Because θ is almost an isomorphism, and because of Galois theory, the following sets are in bijection:

- The finite-index open subgroups of $K^\times$.
- The (finite-index) open subgroups of $\operatorname{Gal}(K^{\mathrm{ab}}/K)$.
- The finite abelian extensions of K contained in K^{s}.

Going backwards, if L is a finite abelian extension of K in K^{s}, the corresponding subgroup of $K^\times$ is $N_{L/K}L^\times$. (This could be viewed as a consequence of the functoriality above.)

The composition

$$K^\times \to \operatorname{Gal}(K^{\mathrm{ab}}/K) \stackrel{\mathrm{res}}{\twoheadrightarrow} \operatorname{Gal}(L/K)$$

is surjective with kernel $N_{L/K}L^\times$, and $\mathcal{O}^\times$ maps to the inertia subgroup $I_{L/K} \trianglelefteq \operatorname{Gal}(L/K)$, and any uniformizer π maps to a Frobenius element of $\operatorname{Gal}(L/K)$.

2. Global class field theory (via ideles)

2.1. Global fields.

Definition 2.1. A number field is a finite extension of $\mathbb{Q}$. A global function field is a finite extension of $\mathbb{F}_p(t)$ for some prime p, or equivalently is the function field of a geometrically integral curve over a finite field $\mathbb{F}_q$ (called the constant field), where q is a power of some prime p. A global field is a number field or a global function field.

Throughout Sections 2 and 3, K is a global field. If v is a nontrivial place of K (given by an absolute value on K), then the completion K_v is a local field. If v is nonarchimedean, let $\mathcal{O}_v$ be the valuation subring of K_v; if v is archimedean, let $\mathcal{O}_v = K_v$.

2.2. **The adele ring.** The adele ring of K is the restricted direct product

$$\mathbf{A}_K := \prod_v{}' (K_v, \mathcal{O}_v) := \left\{ (a_v) \in \prod_v K_v : a_v \in \mathcal{O}_v \text{ for all but finitely many } v \right\}.$$

It is a topological ring: the topology is uniquely characterized by the condition that $\prod_v \mathcal{O}_v$ is open in $\mathbf{A}_K$ and has the product topology. The diagonal map $K \to \mathbf{A}_K$ is like $\mathbb{Z} \to \mathbb{R}$: it embeds K as a discrete co-compact subgroup of $\mathbf{A}_K$.

2.3. **The idele group and idele class group.** The idele group of K is

$$\mathbf{A}_K^\times = \prod_v{}' (K_v^\times, \mathcal{O}_v^\times) := \left\{ (a_v) \in \prod_v K_v^\times : a_v \in \mathcal{O}_v^\times \text{ for all but finitely many } v \right\}.$$

It is a topological group: the topology is uniquely characterized by the condition that $\prod_v \mathcal{O}_v^\times$ is open in $\mathbf{A}_K^\times$ and has the product topology.[1] The diagonal map $K^\times \to \mathbf{A}_K^\times$ is like $\mathbb{Z}^\times \to \mathbb{R}^\times$: it embeds $K^\times$ as a discrete subgroup of $\mathbf{A}_K^\times$, but the quotient $C_K := \mathbf{A}_K^\times / K^\times$ is not compact. The topological group C_K is called the idele class group.

2.4. **The global Artin homomorphism.** Let K^{s} be a fixed separable closure of K. Let K^{ab} be the maximal abelian extension of K contained in K^{s}.

The group C_K plays the role in global class field theory played by $K^\times$ in local class field theory. Namely, if K is a global field, there is a global Artin homomorphism

$$\theta \colon C_K \to \operatorname{Gal}(K^{\mathrm{ab}}/K)$$

that induces an isomorphism $\widehat{C_K} \xrightarrow{\sim} \operatorname{Gal}(K^{\mathrm{ab}}/K)$.

If K is a number field, then θ is surjective and its kernel is the connected component of the identity in C_K.

If K is a global function field with constant field k, then θ is injective and $\theta(C_K)$ equals the set of $\sigma \in \operatorname{Gal}(K^{\mathrm{ab}}/K)$ whose restriction in $\operatorname{Gal}(k^{\mathrm{s}}/k)$ is an *integer* power of the Frobenius generator.

2.5. **Functoriality.** Let L be a finite extension of K of degree n. Then $\mathbf{A}_L \simeq \mathbf{A}_K \otimes_K L$ is free of rank n over $\mathbf{A}_K$, so there is a norm map $N_{L/K} \colon \mathbf{A}_L \to \mathbf{A}_K$. We write $N_{L/K}$ also for the induced homomorphism $N_{L/K} \colon C_L \to C_K$. Then

$$\begin{array}{ccc} C_L & \xrightarrow{\theta_L} & \operatorname{Gal}(L^{\mathrm{ab}}/L) \\ \downarrow{\scriptstyle N_{L/K}} & & \downarrow{\scriptstyle \mathrm{res}} \\ C_K & \xrightarrow{\theta_K} & \operatorname{Gal}(K^{\mathrm{ab}}/K) \end{array}$$

commutes.

2.6. **Finite abelian extensions.** The following sets are in bijection:

- The finite-index open subgroups of C_K.
- The finite-index open subgroups of $\operatorname{Gal}(K^{\mathrm{ab}}/K)$.
- The finite abelian extensions of K contained in K^{s}.

Going backwards, if L is a finite abelian extension of K in K^{s}, the corresponding subgroup of C_K is $N_{L/K} C_L$. The composition

$$C_K \to \operatorname{Gal}(K^{\mathrm{ab}}/K) \overset{\mathrm{res}}{\twoheadrightarrow} \operatorname{Gal}(L/K)$$

is surjective with kernel $N_{L/K} C_L$.

[1] Alternatively, one can use the general recipe for getting the topology on the units of a topological ring R: not the subspace topology on $R^\times$ as a subset of R (this may fail to make the inverse map $R^\times \to R^\times$ continuous), but the subspace topology on the set of solutions to $xy = 1$ in $R \times R$ (this is what one gets if one expresses the multiplicative group scheme $\mathbb{G}_m$ as an affine variety).

2.7. **Connection between the global and local Artin homomorphisms.** Let v be a place of K. Identify $K_v^\times$ with a subgroup of $\mathbf{A}_K^\times$ by mapping $\alpha \in K_v^\times$ to the idele with α in the v-th position and 1 in every other position. The composition $K_v^\times \hookrightarrow \mathbf{A}_K^\times \twoheadrightarrow C_K$ is injective. Let θ_v be the local Artin homomorphism for K_v. Then the diagram

$$\begin{array}{ccc} K_v^\times & \xrightarrow{\theta_v} & \operatorname{Gal}(K_v^{\mathrm{ab}}/K_v) \\ \big\downarrow & & \big\downarrow \mathrm{res} \\ C_K & \xrightarrow{\theta} & \operatorname{Gal}(K^{\mathrm{ab}}/K) \end{array}$$

commutes. Thus θ determines θ_v.

Conversely, if one knows θ_v for all v, one can construct θ as follows. Let L be a finite abelian extension of K contained in K^{s}. Define

$$\begin{aligned} \mathbf{A}_K^\times &\to \operatorname{Gal}(L/K) \\ (a_v) &\mapsto \prod_v \theta_v(a_v); \end{aligned}$$

if v is unramified in L/K, and $a_v \in \mathcal{O}_v^\times$, then $\theta_v(a_v) = 1$, so all but finitely many terms in the infinite product are 1, and the product makes sense. Take the inverse limit over all possible L to get

$$\mathbf{A}_K^\times \to \operatorname{Gal}(K^{\mathrm{ab}}/K).$$

The idelic version of the Artin reciprocity law says that $K^\times$ is in the kernel, so we get a homomorphism

$$C_K \to \operatorname{Gal}(K^{\mathrm{ab}}/K),$$

which is θ.

2.8. **Moduli.**

Definition 2.2. A modulus is a formal product $\mathfrak{m} = \prod_v v^{e_v}$ where $e_v \in \mathbb{Z}_{\geq 0}$, all but finitely many e_v equal 0, and $e_v \in \{0, 1\}$ for real v, and $e_v = 0$ for complex v. The support $\operatorname{supp} \mathfrak{m}$ is the (finite) set of *nonarchimedean* places v such that $e_v \neq 0$.

If K is a number field, then a modulus can be viewed as a pair consisting of

(1) an integral ideal of the ring of integers $\mathcal{O}_K$, and
(2) a subset of the real places.

If K is the function field of a smooth projective curve X over a finite field, then a modulus is the same thing as an effective divisor on X.

2.9. **Ray class groups and ray class fields.** In this section we assume that K is a number field. Fix a modulus $\mathfrak{m} = \prod_v v^{e_v}$. We will define a finite-index open subgroup $U_{\mathfrak{m},v} \subseteq \mathcal{O}_v^\times$ for each v. If $e_v = 0$, define $U_{\mathfrak{m},v} := \mathcal{O}_v^\times$. If $e_v > 0$ and v is nonarchimedean, define $U_{\mathfrak{m},v} := 1 + \mathfrak{p}_v^{e_v}$, where $\mathfrak{p}_v$ is the maximal ideal of $\mathcal{O}_v$. If $e_v > 0$ and v is real, define $U_{\mathfrak{m},v}$ as $\mathbb{R}_{>0} \subseteq \mathbb{R}^\times \simeq K_v^\times$. Define $U_{\mathfrak{m}} := \prod_v U_{\mathfrak{m},v} \subseteq \mathbf{A}_K^\times$. The image of $U_{\mathfrak{m}}$ under $\mathbf{A}_K^\times \twoheadrightarrow C_K$ is a finite-index open subgroup $U'_{\mathfrak{m}}$ of C_K (this is equivalent to finiteness of the class number of K, as we will see in Section 3.4). The corresponding finite abelian extension $R_{\mathfrak{m}}$ of K is called the ray class field of modulus $\mathfrak{m}$, and $R_\mathfrak{m}$ over K is unramified at all v with $e_v = 0$. The ray class group of modulus $\mathfrak{m}$ is

$$\frac{C_K}{U'_\mathfrak{m}} = \frac{\mathbf{A}_K^\times}{U_\mathfrak{m} K^\times},$$

which is isomorphic to $\operatorname{Gal}(R_\mathfrak{m}/K)$ via the global Artin homomorphism.

Every finite-index open subgroup of $\mathbf{A}_K^\times$ contains $U_\mathfrak{m}$ for some $\mathfrak{m}$, so every finite abelian extension of K is contained in $R_\mathfrak{m}$ for some $\mathfrak{m}$.

3. Global class field theory (via ideals)

In this section we assume that K is a number field.

3.1. **Classical ray class groups.** Let I be the group of fractional ideals of K, or equivalently, the free abelian group on the nonarchimedean places of K. Let P be the subgroup of principal ideals. The class group is $\operatorname{Cl}\mathcal{O}_K := I/P$.

We now generalize to an arbitrary modulus $\mathfrak{m} = \prod_v v^{e_v}$. Let $I_\mathfrak{m}$ be the subgroup of fractional ideals that do not involve the primes dividing $\mathfrak{m}$; i.e., $I_\mathfrak{m}$ is the free abelian group on the nonarchimedean places v satisfying $e_v = 0$. For $a \in K^\times$, the notation $a \equiv 1 \pmod{^\times \mathfrak{m}}$ means that $a \in U_{\mathfrak{m},v}$ for every v satisfying $e_v > 0$. The group $P_\mathfrak{m} \subseteq I_\mathfrak{m}$ is the group of principal ideals generated by some $a \in K^\times$ with $a \equiv 1 \pmod{^\times \mathfrak{m}}$. The classical ray class group of modulus $\mathfrak{m}$ is $\operatorname{Cl}_\mathfrak{m}\mathcal{O}_K := I_\mathfrak{m}/P_\mathfrak{m}$. Section 3.4 will prove that this is isomorphic to the ray class group $C_K/U'_\mathfrak{m}$ defined in Section 2.9.

3.2. **The classical Artin homomorphism.** Let L/K be a finite abelian extension of number fields. Let S be a finite set of finite primes of K such that S contains every prime that ramifies in L. Let I_S be the group of fractional ideals that do not involve the primes in S. The classical Artin homomorphism is the map

$$\Theta\colon I_S \to \operatorname{Gal}(L/K)$$

sending each prime ideal $\mathfrak{p} \notin S$ to the Frobenius element $\operatorname{Frob}_\mathfrak{p} \in \operatorname{Gal}(L/K)$.

3.3. **The main theorems.** The Artin reciprocity law states that there exists a modulus $\mathfrak{m}$ (depending on L/K) with $\operatorname{supp}\mathfrak{m} = S$ such that the subgroup $P_\mathfrak{m} \subseteq I_\mathfrak{m} = I_S$ is contained in $\ker\Theta$. The existence theorem states that given a modulus $\mathfrak{m}$ and group H with $P_\mathfrak{m} \subseteq H \subseteq I_\mathfrak{m}$ there exists an abelian extension L of K unramified outside $\operatorname{supp}\mathfrak{m}$ such that the kernel of Θ for L/K equals H.

3.4. **Comparison of ideal groups and idele groups.** Consider the trivial modulus $\mathfrak{m} = 1$ (with $e_v = 0$ for all v). Taking the restricted direct product of the valuation maps $v\colon K_v^\times \twoheadrightarrow \mathbb{Z}$ gives a surjective homomorphism

$$\mathbf{A}_K^\times \to I$$

that discards the archimedean components of its input, and its kernel is $U_1 = \prod_v \mathcal{O}_v^\times$. Thus $\frac{\mathbf{A}_K^\times}{U_1} \simeq I$. If we take the quotient by the image of $K^\times$ on both sides, we find that the ray class group $\frac{\mathbf{A}_K^\times}{U_1 K^\times}$ of modulus 1 is isomorphic to the class group $I/P = \operatorname{Cl}\mathcal{O}_K$. The ray class field R_1 of modulus 1 is called the Hilbert class field, which can be characterized also as the maximal abelian extension of K in K^s that is unramified at all places of K (including the archimedean ones). We get

$$\frac{C_K}{U'_1} = \frac{\mathbf{A}_K^\times}{U_1 K^\times} \quad \simeq \quad \frac{I}{P} = \operatorname{Cl} \mathcal{O}_K \quad \simeq \quad \operatorname{Gal}(R_1/K).$$

This can generalized to an arbitrary modulus $\mathfrak{m} = \prod v^{e_v}$ as follows. Let $\mathbf{A}_K^{\mathfrak{m}} \subseteq \mathbf{A}_K^\times$ be the subgroup consisting of (a_v) with (a_v) with $a_v \in U_{\mathfrak{m},v}$ for every v with $e_v > 0$. Let $K^{\mathfrak{m}} = \mathbf{A}_K^{\mathfrak{m}} \cap K^\times$. We have an isomorphism

$$\frac{\mathbf{A}_K^{\mathfrak{m}}}{U_{\mathfrak{m}}} \xrightarrow{\sim} I_{\mathfrak{m}}.$$

Dividing by the image of $K^{\mathfrak{m}}$ on both sides gives

$$\frac{\mathbf{A}_K^{\mathfrak{m}}}{U_{\mathfrak{m}} K^{\mathfrak{m}}} \xrightarrow{\sim} \frac{I_{\mathfrak{m}}}{P_{\mathfrak{m}}}. \tag{2}$$

On the other hand, $\mathbf{A}_K^\times = \mathbf{A}_K^{\mathfrak{m}} K^\times$, so there is an isomorphism

$$\frac{\mathbf{A}_K^{\mathfrak{m}}}{K^{\mathfrak{m}}} \xrightarrow{\sim} \frac{\mathbf{A}_K^\times}{K^\times} = C_K.$$

Dividing by the image of $U_{\mathfrak{m}}$ on both sides, and combining with (2), we get isomorphisms

$$\frac{C_K}{U'_{\mathfrak{m}}} = \frac{\mathbf{A}_K^\times}{U_{\mathfrak{m}} K^\times} \quad \simeq \quad \frac{I_{\mathfrak{m}}}{P_{\mathfrak{m}}} = \operatorname{Cl}_{\mathfrak{m}} \mathcal{O}_K \quad \simeq \quad \operatorname{Gal}(R_{\mathfrak{m}}/K).$$

4. An introduction to an introduction to the Langlands program

Let K be a local or global field. Every 1-dimensional character (continuous homomorphism)

$$\operatorname{Gal}(K^{\mathrm{s}}/K) \to \mathbb{C}^\times$$

factors through $\operatorname{Gal}(K^{\mathrm{ab}}/K)$ and has finite image. These characters form a discrete abelian group, the Pontryagin dual of the profinite group $\operatorname{Gal}(K^{\mathrm{ab}}/K)$. It follows that the problem of classifying finite abelian extensions of K is more or less the same as the problem of describing all these characters.

The Langlands program is an attempt to understand $\operatorname{Gal}(K^{\mathrm{s}}/K)$ more completely by describing its higher-dimensional representations: the group $\mathbb{C}^\times = \operatorname{GL}_1(\mathbb{C})$ is replaced by $\operatorname{GL}_n(\mathbb{C})$, or even $G(\mathbb{C})$ for other linear algebraic groups G. The continuous homomorphisms

$$\operatorname{Gal}(K^{\mathrm{s}}/K) \to G(\mathbb{C})$$

are conjectured to correspond to certain "automorphic" objects defined intrinsically in terms of K, just as class field theory gives a description of the group $\operatorname{Gal}(K^{\mathrm{ab}}/K)$ (which is defined in terms of extrinsic objects such as finite abelian extensions, which are initially mysterious) in terms of intrinsic objects ($K^\times$ or C_K) obtained directly from K.

Ultimately, the program would give information about nonabelian algebraic extensions of K.

5. Further reading

For basics on profinite groups, see [Ser02, I.§1] and [Gru86]. The latter discusses infinite Galois theory as well.

For local class field theory, see [Ser86]. For the approach to global class field theory via cohomology of ideles, see [Tat86]. For a treatment of global class field theory via ideals, see [Jan96]. All these topics are covered also in [Neu99].

For an introduction to the Langlands program, see [BG03].

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Department of Mathematics, University of California, Berkeley, CA 94720-3840, USA

E-mail address: `poonen@math.berkeley.edu`

URL: `http://math.berkeley.edu/~poonen`