THE EXCEPTIONAL LOCUS IN THE BERTINI IRREDUCIBILITY THEOREM FOR A MORPHISM

BJORN POONEN AND KALOYAN SLAVOV

Abstract. We introduce a novel approach to Bertini irreducibility theorems over an arbitrary field, based on random hyperplane slicing over a finite field. Extending a result of Benoist, we prove that for a morphism $\phi\colon X \to \mathbb{P}^n$ such that X is geometrically irreducible and the nonempty fibers of ϕ all have the same dimension, the locus of hyperplanes H such that $\phi^{-1}H$ is not geometrically irreducible has dimension at most $\operatorname{codim}\phi(X)+1$. We give an application to monodromy groups above hyperplane sections.

1. Introduction

Most Bertini theorems state that a moduli space of hyperplanes contains a dense open subset whose points correspond to hyperplanes with some good property, so if the moduli space has dimension n, the locus of bad hyperplanes has dimension at most $n-1$. In contrast, we exhibit Bertini theorems in which the bad locus is often much smaller.

1.1. **Bertini irreducibility theorems.** We work over an arbitrary ground field k. By variety, we mean a separated scheme of finite type over k; subvarieties need only be locally closed. Given $\mathbb{P}^n$, let $\check{\mathbb{P}}^n$ be the dual projective space, so $H \in \check{\mathbb{P}}^n$ means that H is a hyperplane in $\mathbb{P}^n$ (over the residue field of the corresponding point). The following is part of a theorem of Olivier Benoist:

Theorem 1.1 (cf. [Ben11, Théorème 1.4]). *Let X be a geometrically irreducible subvariety of $\mathbb{P}^n$ for some $n \geq 0$. Let $\mathcal{M}_{\mathrm{bad}} \subseteq \check{\mathbb{P}}^n$ be the constructible locus parametrizing hyperplanes H such that $H \cap X$ is not geometrically irreducible. Then* $\dim \mathcal{M}_{\mathrm{bad}} \leq \operatorname{codim} X + 1$.

Example 1.2. For a hypersurface $X \subset \mathbb{P}^n$, Theorem 1.1 says $\dim \mathcal{M}_{\mathrm{bad}} \leq 2$.

Remark 1.3. The bound $\operatorname{codim} X + 1$ is best possible: see [Ben11, Section 3.1].

Remark 1.4. Benoist assumes that X is closed and geometrically integral, but these additional hypotheses can easily be removed. In fact, he bounds a larger set $\mathcal{M}_{\mathrm{bad}}$ that includes also the H such that $H \cap X$ is not generically reduced. Additionally, he proves a best possible analogue in which hyperplanes are replaced by hypersurfaces of a fixed degree e. Benoist's proof uses a degeneration to a union of hyperplanes.

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Our key new observation is that the statistics of random hyperplane slices over a finite field give another way to bound the bad locus in Bertini irreducibility theorems. We were inspired by [Tao12], which used random slicing to give a proof of the Lang–Weil theorem, and by [Sla17], which used random slicing to refine the Lang–Weil bounds for hypersurfaces. Using this approach, we give a new proof of Theorem 1.1, and generalize it to a setting analogous to that in [Jou83, Théorème 6.3(4)]:

Theorem 1.5. *Let X be a geometrically irreducible variety. Let $\phi\colon X \to \mathbb{P}^n$ be a morphism. Let $\mathcal{M}_{\text{bad}}$ be the constructible locus parametrizing hyperplanes $H \subset \mathbb{P}^n$ such that $\phi^{-1}H$ is not geometrically irreducible. If the nonempty fibers of ϕ all have the same dimension, then* $\dim \mathcal{M}_{\text{bad}} \leq \operatorname{codim} \phi(X) + 1$.

Example 1.6. In the setting of Theorem 1.5, if ϕ is dominant, then the conclusion states that $\dim \mathcal{M}_{\text{bad}} \leq 1$.

Theorem 1.5 can fail if the nonempty fibers of ϕ have differing dimensions:

Example 1.7. If $X \to \mathbb{P}^n$ is the blow-up of a linear subspace $L \subset \mathbb{P}^n$ with $\operatorname{codim} L \geq 2$, then $\mathcal{M}_{\text{bad}}$ parametrizes the hyperplanes containing L, so $\dim \mathcal{M}_{\text{bad}} = \operatorname{codim} L - 1$, but $\operatorname{codim} \phi(X) + 1 = 1$ (so the conclusion of Theorem 1.5 fails if $\operatorname{codim} L \geq 3$).

Nevertheless, Theorem 1.5 admits the following generalization. Fix an algebraic closure $\overline{k} \supset k$.

Theorem 1.8. *Let X be a geometrically irreducible variety. Let $\phi\colon X \to \mathbb{P}^n$ be a morphism. Let W be the closed subset of $x \in X$ at which the relative dimension* $\dim_x \phi$ *is greater than at the generic point. Let $W_1, \ldots, W_r$ be the irreducible components of $W_{\overline{k}}$ of dimension* $\dim X - 1$. *Let $\mathcal{M}_{\text{bad}}$ be the locus of hyperplanes H such that $\phi^{-1}H$ is not geometrically irreducible. Let $\mathcal{N}$ be the locus of hyperplanes over $\overline{k}$ that contain $\phi(W_i)$ for some i; since $\{W_1, \ldots, W_r\}$ is Galois-stable, $\mathcal{N}$ is definable over k. Then $\mathcal{M}_{\text{bad}}$ differs from $\mathcal{N}$ in a constructible set of dimension at most* $\operatorname{codim} \phi(X) + 1$.

1.2. **Monodromy.** A generically étale morphism $\phi\colon X \to Y$ between integral varieties has a monodromy group $\operatorname{Mon}(\phi)$, defined as the Galois group of the Galois closure of the function field extension $k(X)/k(Y)$; see Section 8 for a more general definition requiring only Y to be integral. Now suppose $Y \subset \mathbb{P}^n$. For a hyperplane $H \subset \mathbb{P}^n$, let ϕ_H be the restriction $\phi^{-1}(H \cap Y) \to H \cap Y$. The following theorem states that for all $H \in \check{\mathbb{P}}^n$ outside a low-dimensional locus, $\operatorname{Mon}(\phi_H) \simeq \operatorname{Mon}(\phi)$.

Theorem 1.9. *Let $\phi\colon X \to Y$ be a generically étale morphism with Y an integral subvariety of $\mathbb{P}^n$ over an algebraically closed field. Let $\mathcal{M}_{\text{good}} \subset \check{\mathbb{P}}^n$ be the locus parametrizing hyperplanes H such that*

(i) *$H \cap Y$ is irreducible;*
(ii) *the generic point of $H \cap Y$ has a neighborhood U in Y such that U is normal and $\phi^{-1}U \to U$ is finite étale; and*
(iii) *the inclusion* $\operatorname{Mon}(\phi_H) \hookrightarrow \operatorname{Mon}(\phi)$ *is an isomorphism.*

Then the locus $\mathcal{M}_{\text{bad}} := \check{\mathbb{P}}^n - \mathcal{M}_{\text{good}}$ is a constructible set of dimension at most $\operatorname{codim} Y + 1$.

Remark 1.10. Conditions (i) and (ii) are needed to define the inclusion in (iii): see Section 8.

Example 1.11. Let k be an algebraically closed field of characteristic not 2, let $X = \operatorname{Spec} k[x_1, \dots, x_n, y]/(y^2 - x_1)$, and let $\phi\colon X \to \mathbb{A}^n_k \subset \mathbb{P}^n$ be the projection to $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \dots, x_n]$. Then $\mathcal{M}_{\text{bad}}$ is the 1-dimensional locus consisting of the hyperplanes $x_1 = a$ for $a \in k$.

1.3. **Structure of the article.** After a brief notation section, Theorem 1.5 is proved in Sections 3 to 6; see especially Lemma 6.1. We apply it to prove Theorems 1.8 and 1.9 in Sections 7 and 8, respectively. The heart of our paper is the random slicing in Section 4 and its application towards irreducibility in Section 6.

2. Notation

The empty scheme is not irreducible. For a noetherian scheme X, let $\operatorname{Irr} X$ be the set of irreducible components of X. If X is an irreducible variety, let $k(X)$ be the function field of the associated reduced subscheme X_{red}. The dimension of a constructible subset C of a variety V (viewed as a topological subspace) equals the maximal dimension of a subvariety of V contained in C; then $\operatorname{codim} C := \dim V - \dim C$.

Let S be a scheme, and let X be an S-scheme. Given a morphism of schemes $T \to S$, let X_T denote $X \times_S T$; in this context, if $T = \operatorname{Spec} A$, we may write A instead of $\operatorname{Spec} A$. If $s \in S$, let X_s be the fiber of $X \to S$ above s. If moreover C is a constructible subset of X, then C_T denotes the inverse image of C under $X_T \to X$, and C_s is defined similarly.

For a finite-type morphism $\phi\colon X \to Y$ of noetherian schemes with Y irreducible, ϕ is étale over the generic point of Y if and only if ϕ is étale over some dense open $V \subset Y$; in this case, call ϕ generically étale.

3. Reduction to finite fields

We begin the proof of Theorem 1.5 by reducing to the case of a finite field. There exists a finitely generated $\mathbb{Z}$-algebra $R \subset k$ such that $\phi\colon X \to \mathbb{P}^n_k$ is the base change of a separated finite-type morphism (denoted using the same letters) $\phi\colon X \to \mathbb{P}^n_R$. In the new notation, the original morphism is $\phi_k\colon X_k \to \mathbb{P}^n_k$. By shrinking $\operatorname{Spec} R$ if necessary, we may assume that for each $\mathfrak{p} \in \operatorname{Spec} R$, the fiber $X_{\mathfrak{p}}$ is geometrically irreducible [EGA IV$_3$, 9.7.7(i)], the nonempty fibers of $\phi_{\mathfrak{p}}$ all have the same dimension [EGA IV$_3$, 9.2.6(iv)], and $\dim \phi_{\mathfrak{p}}(X_{\mathfrak{p}}) = \dim \phi_k(X_k)$.

Let $\mathcal{M}_{\text{bad}} \subset \check{\mathbb{P}}^n_R$ be the subset parametrizing hyperplanes H such that $\phi^{-1}H$ is not geometrically irreducible; since the $\phi^{-1}H$ are the fibers of a family, $\mathcal{M}_{\text{bad}}$ is constructible [EGA IV$_3$, 9.7.7(i)]. If we prove Theorem 1.5 for a finite ground field, so that $\dim (\mathcal{M}_{\text{bad}})_{\mathfrak{p}} \leq \operatorname{codim} \phi_{\mathfrak{p}}(X_{\mathfrak{p}})+1$ for every closed point $\mathfrak{p}$, then $\dim (\mathcal{M}_{\text{bad}})_k \leq \operatorname{codim} \phi_k(X_k)+1$ too. Therefore from now on we assume that k is finite.

4. Random hyperplane slicing

The following lemma is purely set-theoretic; for the time being, X is just a set.

Lemma 4.1. *Let $\phi\colon X \to \mathbb{P}^n(\mathbb{F}_q)$ be a map of sets for some $n \geq 1$. Let s be an upper bound on the size of its fibers. For a (set-theoretic) hyperplane $H \subset \mathbb{P}^n(\mathbb{F}_q)$ chosen uniformly at random, define the random variable $Z := \#(\phi^{-1}H)$. Then its mean μ and variance σ^2 satisfy*

$$\mu = \#X\,(q^{-1} + O(q^{-2}))$$
$$\sigma^2 = O(\#(\phi(X))\,s^2 q^{-1}).$$

Proof. For any $y \in \mathbb{P}^n(\mathbb{F}_q)$, define

$$p_1 := \operatorname{Prob}(y \in H) = \frac{q^n - 1}{q^{n+1} - 1} = q^{-1} + O(q^{-2}).$$

Similarly, for any *distinct* $y, z \in \mathbb{P}^n(\mathbb{F}_q)$, define

$$p_2 := \operatorname{Prob}(y, z \in H) = \frac{q^{n-1} - 1}{q^{n+1} - 1} = q^{-2} + O(q^{-3}).$$

The mean of Z is

$$\mu = \mathbb{E}Z = \sum_{x \in X} \operatorname{Prob}(\phi(x) \in H) = (\#X)\, p_1 = \#X\,(q^{-1} + O(q^{-2})),$$

and the variance is

$$\begin{aligned}
\sigma^2 &= \mathbb{E}(Z^2) - (\mathbb{E}Z)^2 \\
&= \sum_{u,v \in X} \Big(\operatorname{Prob}(\phi(u), \phi(v) \in H) - \operatorname{Prob}(\phi(u) \in H)\ \operatorname{Prob}(\phi(v) \in H) \Big) \\
&= \sum_{\phi(u)=\phi(v)} (p_1 - p_1^2) + \sum_{\phi(u)\neq\phi(v)} (p_2 - p_1^2) \\
&\le \sum_{\phi(u)=\phi(v)} p_1 + \sum_{\phi(u)\neq\phi(v)} 0 \quad \text{(we have } p_2 < p_1^2 \text{ since } (q^{n+1}-1)^2(p_1^2 - p_2) = q^{n-1}(q-1)^2\text{)} \\
&\le \sum_{y \in \phi(X)} \#(\phi^{-1}y)^2\, p_1 \\
&= O(\#(\phi(X))\, s^2 q^{-1}).
\end{aligned}$$

□

5. The Lang–Weil bound

We will apply Lemma 4.1 when ϕ comes from a morphism of varieties over $\mathbb{F}_q$, so we need bounds on the number of $\mathbb{F}_q$-points of a variety. Throughout the rest of this paper, the implied constant in a big-O depends on the geometric complexity[1] but not on q.

Theorem 5.1 ([LW54]). *Let X be a variety over $\mathbb{F}_q$. Let $r = \dim X$.*

(a) *We have $\#X(\mathbb{F}_q) = O(q^r)$.*
(b) *If X is geometrically irreducible, then $\#X(\mathbb{F}_q) = q^r + O(q^{r-1/2})$.*
(c) *More generally, if a is the number of irreducible components of X that are geometrically irreducible of dimension r, then $\#X(\mathbb{F}_q) = aq^r + O(q^{r-1/2})$.*

Proof. Parts (a) and (b) are Lemma 1 and Theorem 1 in [LW54]. As is well-known, (c) follows from (a) and (b), since if Z is an irreducible component that is not geometrically irreducible, then $Z(\mathbb{F}_q)$ is contained in the intersection of the geometric components of Z, which is of lower dimension. □

[1] Say that a variety is of complexity $\le M$ if it is a union of $\le M$ open subschemes, each cut out by $\le M$ polynomials of degree $\le M$ in $\mathbb{A}^n$ for some $n \le M$, with each pair of subschemes glued along $\le M$ open sets $D(f)$ with each f given by a polynomial of degree $\le M$, with each gluing isomorphism defined by polynomials of degree $\le M$.

6. Counting very bad hyperplanes

Consider a finite field $\mathbb{F}_q \supset k$. Call a hyperplane $H \in \check{\mathbb{P}}^n(\mathbb{F}_q)$ very bad if the number of $\mathbb{F}_q$-irreducible components of $\phi^{-1}H$ that are geometrically irreducible is not 1. The bound on the variance in Lemma 4.1 will bound the number of very bad hyperplanes, because each such hyperplane contributes noticeably to the variance.

Lemma 6.1. *Let X be a geometrically irreducible variety over a finite field $\mathbb{F}_q$ with a morphism $\phi\colon X \to \mathbb{P}^n$ whose nonempty fibers are all of the same dimension. Then the number of very bad hyperplanes in $\check{\mathbb{P}}^n(\mathbb{F}_q)$ is $O(q^{\operatorname{codim}\phi(X)+1})$.*

Proof. Let $Y = \overline{\phi(X)}$. Let $r = \dim X$ and $m = \dim Y$, so the nonempty fibers of ϕ have dimension $r - m$. Consider the random variable $\#(\phi^{-1}H)(\mathbb{F}_q)$ for H chosen uniformly at random in $\check{\mathbb{P}}^n(\mathbb{F}_q)$. Let μ and σ^2 denote its mean and variance. By Lemma 4.1 and Theorem 5.1(a,b) applied to X, Y, and the fibers of ϕ,

$$\mu = (q^r + O(q^{r-1/2}))(q^{-1} + O(q^{-2})) = q^{r-1} + O(q^{r-3/2}) \tag{1}$$

$$\sigma^2 = O(q^m q^{2(r-m)} q^{-1}) = O(q^{2r-m-1}). \tag{2}$$

If H is very bad, then $\phi^{-1}H \neq X$, so each irreducible component of $\phi^{-1}H$ has dimension $r-1$, and Theorem 5.1(c) implies that $\#(\phi^{-1}H)(\mathbb{F}_q)$ is either $O(q^{r-3/2})$ or at least $2q^{r-1} - O(q^{r-3/2})$, so by (1),

$$|\#(\phi^{-1}H)(\mathbb{F}_q) - \mu| \geq q^{r-1} - O(q^{r-3/2}) \geq \frac{1}{2}q^{r-1} \quad \text{for large } q.$$

Define t so that $\frac{1}{2}q^{r-1} = t\sigma$. Then

$$\begin{aligned}
\operatorname{Prob}(H \text{ is very bad}) &\leq \operatorname{Prob}\left(|\#(\phi^{-1}H)(\mathbb{F}_q) - \mu| \geq t\sigma\right) \\
&\leq \frac{1}{t^2} \quad \text{(by Chebyshev's inequality)} \\
&= \frac{4\sigma^2}{q^{2r-2}} \\
&= O(q^{1-m}) \quad \text{(by (2))}.
\end{aligned}$$

Multiplying by the total number of hyperplanes over $\mathbb{F}_q$, which is $O(q^n)$, gives $O(q^{n-m+1})$. □

Lemma 6.2. *Let $\psi\colon V \to B$ be a morphism of varieties over a finite field k. Suppose that B is irreducible, and the generic fiber of ψ is not geometrically irreducible. Call a point $b \in B(\mathbb{F}_q)$ very bad if the number of $\mathbb{F}_q$-irreducible components of $\psi^{-1}b$ that are geometrically irreducible is not 1. Then there exists $c > 0$ such that there exist arbitrarily large finite fields $\mathbb{F}_q \supset k$ such that $B(\mathbb{F}_q)$ contains at least $cq^{\dim B}$ very bad points.*

Proof. If $B' \to B$ is a quasi-finite dominant morphism of irreducible varieties, and the result holds for the base change $V' \to B'$ of $V \to B$, then the result holds for $V \to B$, because the image under $B' \to B$ of a set of $c'q^{\dim B}$ points of $B'(\mathbb{F}_q)$ has size at least $cq^{\dim B}$ for a possibly smaller c.

Let η be the generic point of B. All geometric components of $\psi^{-1}\eta$ are defined over a finite extension K' of $k(B)$. By choosing $B' \to B$ as above with $k(B') = K'$, we may reduce to the case that all irreducible components of $\psi^{-1}\eta$ are *geometrically* irreducible. By passing to a finite extension of k and replacing B by an irreducible component of the base extension, we may assume also that B is geometrically irreducible.

If $\psi^{-1}\eta$ is empty, then there is a dense open subset U of B above which the fibers are empty, and $\#U(\mathbb{F}_q) = q^{\dim B} + O(q^{\dim B - 1/2})$ by Theorem 5.1(b), so the conclusion holds with $c = 1/2$.

Otherwise $\psi^{-1}\eta$ has ≥ 2 irreducible components. Let W_1 and W_2 be their closures in V. The locus of $b \in B$ such that the fibers of $W_1 \to B$ and $W_2 \to B$ above b are distinct geometrically irreducible components of $\psi^{-1}b$ is constructible, so by replacing B by a dense open subvariety we may assume that the locus is all of B. Now for any $\mathbb{F}_q \supset k$, all $b \in B(\mathbb{F}_q)$ are very bad, and their number is $q^{\dim B} + O(q^{\dim B - 1/2})$ by Theorem 5.1(b). □

Proof of Theorem 1.5. By Section 3, we may assume that the ground field is finite. Let B be an irreducible variety contained in $\mathcal{M}_{\text{bad}}$. Let $V \to B$ be the morphism whose fiber over a point corresponding to a hyperplane H is $\phi^{-1}H$. By Lemma 6.2, for arbitrarily large q there are at least $cq^{\dim B}$ very bad hyperplanes $H \in B(\mathbb{F}_q)$. On the other hand, by Lemma 6.1 there are at most $O(q^{\operatorname{codim} Y + 1})$ very bad hyperplanes. Thus $\dim B \leq \operatorname{codim} Y + 1$. Since this holds for all irreducible $B \subset \mathcal{M}_{\text{bad}}$, we obtain $\dim \mathcal{M}_{\text{bad}} \leq \operatorname{codim} Y + 1$. □

7. Proof of the most general version

Lemma 7.1. *For a constructible set $Y \subset \mathbb{P}^n$, the locus of hyperplanes containing Y is a variety of dimension at most* $\operatorname{codim} Y - 1$.

Proof. Let L be the linear span of Y in $\mathbb{P}^n$. The hyperplanes containing Y are those containing L, which form a projective space of dimension $\operatorname{codim} L - 1 \leq \operatorname{codim} Y - 1$. □

Proof of Theorem 1.8. We may assume that k is algebraically closed. By Lemma 7.1, we may ignore hyperplanes H containing $\phi(X)$. Now every irreducible component of $\phi^{-1}H$ is of dimension $\dim X - 1$. Let $X' = X - W$. By Theorem 1.5 applied to $X' \to \mathbb{P}^n$, it suffices to consider H such that $\phi^{-1}H \cap X'$ is geometrically irreducible. For such H, the following are equivalent:

- $H \in \mathcal{M}_{\text{bad}}$;
- $\phi^{-1}H$ is not irreducible;
- $\phi^{-1}H$ contains a closed subset of W of dimension $\dim X - 1$;
- $\phi^{-1}H$ contains W_i for some i;
- H contains $\phi(W_i)$ for some i. □

8. Application to monodromy

Let K be a field. Fix a separable closure K_s of K, and let $G_K = \operatorname{Gal}(K_s/K)$. If $f\colon X \to \operatorname{Spec} K$ is finite étale, let $\operatorname{Mon}(f)$ be the image of $G_K \to \operatorname{Aut}(X(K_s))$. More generally, if $f\colon X \to Y$ is generically étale with Y irreducible, let $f_K\colon X_K \to \operatorname{Spec} K$ be the generic fiber, and define the monodromy group $\operatorname{Mon}(X/Y) = \operatorname{Mon}(f) := \operatorname{Mon}(f_K)$.

Let $f\colon X \to Y$ be a degree d finite étale morphism of schemes. As in [Vak06, Section 3.5], define the Galois scheme of f as the following open and closed Y-subscheme of the dth fibered power $X \times_Y \cdots \times_Y X$:

$$\operatorname{GS}(f) := \{(x_1, ..., x_d) \in X \times_Y \cdots \times_Y X \mid x_i \neq x_j \text{ for } i \neq j\}.$$

If L/K is a finite separable extension, with Galois closure $\widetilde{L}$, and f is $\operatorname{Spec} L \to \operatorname{Spec} K$, then $\operatorname{Mon}(f) \simeq \operatorname{Gal}(\widetilde{L}/K)$ and any connected component Z of $\operatorname{GS}(f)$ is isomorphic to $\operatorname{Spec} \widetilde{L}$.

Lemma 8.1. *Let $f\colon X \to Y$ be a finite étale morphism with Y irreducible. Let Z be a nonempty open and closed subscheme of $\operatorname{GS}(f)$. Then $\operatorname{Mon}(X/Y) \simeq \operatorname{Mon}(Z/Y)$.*

Proof. Let $K = k(Y)$. Then $Z(K_s)$ is a union of one or more G_K-orbits in the set of bijections $\{1, \ldots, d\} \to X(K_s)$. Thus $G_K \to \operatorname{Aut}(X(K_s))$ and $G_K \to \operatorname{Aut}(Z(K_s))$ have the same kernel, and hence canonically isomorphic images. □

Lemma 8.2. *Let $f\colon X \to Y$ be an open morphism of noetherian schemes. Suppose that Y is irreducible, with generic point η. Then there is a bijection $\operatorname{Irr} X \to \operatorname{Irr} X_\eta$ sending Z to Z_η.*

Proof. If $Z \in \operatorname{Irr} X$, then the set $Z' := X - \bigcup_{W \in \operatorname{Irr} X,\, W \neq Z} W \subset Z$ is nonempty and open in X, so $f(Z')$ is nonempty and open in Y, so $\eta \in f(Z') \subset f(Z)$, so Z meets X_η. By [EGA I, 0, 2.1.13], $\{Z \in \operatorname{Irr} X : Z \text{ meets } X_\eta\}$ is in bijection with $\operatorname{Irr} X_\eta$. □

Lemma 8.3. *Let $Z \to Y$ be a right G-torsor for a finite group G, with Y irreducible and Z connected.*

(a) *Let $y \in Y$. Let T be a connected component of Z_y. Let $G_T \subset G$ be the decomposition group of T. Then $\operatorname{Mon}(T/y) \simeq G_T \subset G$.*
(b) *The injection $\operatorname{Mon}(T/y) \hookrightarrow G$ in* (a) *is an isomorphism if and only if Z_y is connected.*
(c) *If Z is irreducible, then $\operatorname{Mon}(Z/Y) \xrightarrow{\sim} G$.*

Proof. Part (a) is just the usual theory of the decomposition and inertia groups specialized to the Galois étale case: use [SGA 1, V.1.3 and V.2.4]; the residue field extension is Galois. Part (b) follows since G acts transitively on Z_y. Part (c) follows by applying (b) to the generic point η of Y and noting that Z_η is irreducible by Lemma 8.2. □

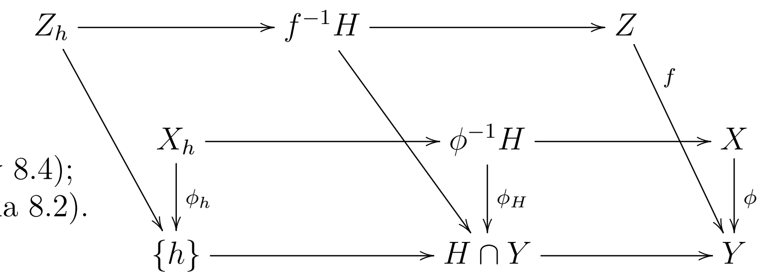
Corollary 8.4. *Let $f\colon X \to Y$ be a finite étale morphism, where Y is irreducible and normal. Let Z be a connected component of $\operatorname{GS}(f)$. Let $y \in Y$. Then there is an injection $\operatorname{Mon}(X_y/y) \hookrightarrow \operatorname{Mon}(X/Y)$, well-defined up to an inner automorphism of $\operatorname{Mon}(X/Y)$, and it is an isomorphism if and only if Z_y is connected.*

Proof. Since Y is normal, Z is irreducible. The proof of [Fu15, Proposition 3.2.10] implies that $Z \to Y$ is a G-torsor. Choose a connected component T of Z_y. By Lemma 8.3, we have an injection $\operatorname{Mon}(T/y) \hookrightarrow G \simeq \operatorname{Mon}(Z/Y)$ which is an isomorphism if and only if Z_y is connected. By Lemma 8.1 applied to $X_y \to \{y\}$ and $X \to Y$, this injection identifies with $\operatorname{Mon}(X_y/y) \hookrightarrow \operatorname{Mon}(X/Y)$. □

Proof of Theorem 1.9. Let $V \subset Y$ be the largest normal open subscheme above which ϕ is finite étale. If H satisfies (i), then (ii) holds if and only if $H \cap V$ is irreducible. By Theorem 1.1 applied to $Y \subset \mathbb{P}^n$ and to $V \subset \mathbb{P}^n$, we may discard all H that fail (i) or (ii). Replace $X \to Y$ by $\phi^{-1}V \to V$.

Let Z be a connected component of $\mathrm{GS}(\phi)$, and let f be the morphism $Z \to Y$. For H such that $H \cap Y$ is irreducible, let h denote the generic point of $H \cap Y$; then the following are equivalent:

- $H \in \mathcal{M}_{\text{good}}$;
- $\mathrm{Mon}(\phi_H) \xrightarrow{\sim} \mathrm{Mon}(\phi)$;
- $\mathrm{Mon}(\phi_h) \xrightarrow{\sim} \mathrm{Mon}(\phi)$;
- Z_h is irreducible (by Corollary 8.4);
- $f^{-1}H$ is irreducible (by Lemma 8.2).



Since Y is normal, Z is irreducible, so by Theorem 1.5 applied to f, the last condition above fails on a constructible locus of dimension at most $\operatorname{codim} Y + 1$. □

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Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139-4307, USA
Email address: poonen@math.mit.edu
URL: http://math.mit.edu/~poonen/

Department of Mathematics, ETH Zürich, Rämistrasse 101, 8006 Zürich, Switzerland
Email address: kaloyan.slavov@math.ethz.ch