ABELIAN VARIETIES OF PRESCRIBED ORDER OVER FINITE FIELDS

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Abstract. Given a prime power q and $n \gg 1$, we prove that every integer in a large subinterval of the Hasse–Weil interval $[(\sqrt{q}-1)^{2n}, (\sqrt{q}+1)^{2n}]$ is $\#A(\mathbb{F}_q)$ for some geometrically simple ordinary principally polarized abelian variety A of dimension n over $\mathbb{F}_q$. As a consequence, we generalize a result of Howe and Kedlaya for $\mathbb{F}_2$ to show that for each prime power q, every sufficiently large positive integer is realizable, i.e., $\#A(\mathbb{F}_q)$ for some abelian variety A over $\mathbb{F}_q$. Our result also improves upon the best known constructions of sequences of simple abelian varieties with point counts towards the extremes of the Hasse–Weil interval. A separate argument determines, for fixed n, the largest subinterval of the Hasse–Weil interval consisting of realizable integers, asymptotically as $q \to \infty$; this gives an asymptotically optimal improvement of a 1998 theorem of DiPippo and Howe. Our methods are effective: We prove that if $q \le 5$, then every positive integer is realizable, and for arbitrary q, every positive integer $\ge q^{3\sqrt{q}\log q}$ is realizable.

1. Introduction

1.1. **Orders of abelian varieties over a finite field.** By work of Weil (a consequence of [Wei48a, pp. 70–71] and [Wei48b, pp. 137–138], generalizing [Has36, p. 206]), if A is an abelian variety of dimension n over a finite field $\mathbb{F}_q$, then $\#A(\mathbb{F}_q)$ lies in the interval

$$\left[\ \left(q - 2q^{1/2} + 1\right)^n\ ,\ \left(q + 2q^{1/2} + 1\right)^n\ \right]. \tag{1}$$

We prove an almost-converse (compare (1) and (3)):

Theorem 1.1. *Fix a prime power q. Let $\tau(x) = x + \sqrt{x^2-1}$. Let I be a closed interval contained in*

$$I_{\text{attained}} := \left(\ \tau(q/2 - q^{1/2} + 3/2)\ ,\ \tau(q/2 + q^{1/2} - 1/2)\ \right). \tag{2}$$

For n sufficiently large, if m is a positive integer with $m^{1/n} \in I$, then there exists an n-dimensional abelian variety A with $\#A(\mathbb{F}_q) = m$. Moreover, A can be chosen to be ordinary, geometrically simple, and principally polarized.

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Corollary 1.2. *Fix a prime power q. Then, for n sufficiently large, every integer in the interval*

$$\left[\ \left(q-2q^{1/2}+3-q^{-1}\right)^n\ ,\ \left(q+2q^{1/2}-1-q^{-1}\right)^n\ \right] \tag{3}$$

is $\#A(\mathbb{F}_q)$ for some geometrically simple ordinary principally polarized abelian variety A of dimension n over $\mathbb{F}_q$.

The interval (3) in Corollary 1.2 contains $[q^n, q^{n+1}]$ if n is large enough, so Corollary 1.2 implies the following:

Corollary 1.3. *Fix a prime power q. Every sufficiently large positive integer is $\#A(\mathbb{F}_q)$ for some geometrically simple ordinary principally polarized abelian variety A over $\mathbb{F}_q$.*

Corollary 1.3 answers a question of Howe and Kedlaya, who proved that every positive integer is the order of an ordinary abelian variety over $\mathbb{F}_2$ [HK21, Theorem 1]. For effective versions, see Section 1.5.

Remark 1.4. Marseglia and Springer refined [HK21] to prove that every finite abelian group is isomorphic to $A(\mathbb{F}_2)$ for some ordinary abelian variety A over $\mathbb{F}_2$ [MS21]. Our Corollary 1.3 combined with [MS21, Theorem 4.2] implies that for any fixed q, every cyclic group of sufficiently large order is isomorphic to $A(\mathbb{F}_q)$ for some ordinary abelian variety A over $\mathbb{F}_q$.

Throughout, p denotes the characteristic of $\mathbb{F}_q$.

Remark 1.5. Theorem 1.1 can be extended to produce non-ordinary abelian varieties. First, define the p-rank of an n-dimensional abelian variety A over $\mathbb{F}_q$ to be the integer $\dim_{\mathbb{F}_p} A[p](\overline{\mathbb{F}}_q)$ in $[0,n]$. For example, A is ordinary if and only if the p-rank is n. Then Theorem 1.1 holds with "ordinary" replaced by "of prescribed p-rank r" for any $r \in [0,n]$, provided that when $r=0$, we assume $m \equiv 1 \pmod{p}$; see Remark 5.9.

Remark 1.6. It may be that Theorem 1.1 holds for an interval larger than I_{attained}. There is a largest open interval I_{true} containing q for which Theorem 1.1 holds.

1.2. **Extreme point counts for simple abelian varieties.** Other authors have studied the extreme values of $\#A(\mathbb{F}_q)^{1/\dim A}$ without trying to realize every order in between. Following [Kad21], let $\mathcal{A}_q$ be the set of simple abelian varieties over $\mathbb{F}_q$ up to isogeny and consider

$$I_{\text{simple}} := \left[\ \liminf_{A\in\mathcal{A}_q} \#A(\mathbb{F}_q)^{1/\dim A}\ ,\ \limsup_{A\in\mathcal{A}_q} \#A(\mathbb{F}_q)^{1/\dim A}\ \right].$$

(If one did not require simplicity and take lim sup and lim inf, then for square q the minimum and maximum would be achieved by elliptic curves of order $q \pm 2q^{1/2}+1$ and their powers.) Then

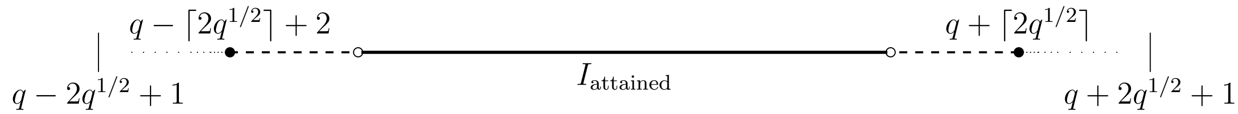
$$I_{\text{attained}} \subseteq I_{\text{true}} \subseteq I_{\text{simple}} \subseteq I_{\text{Weil}} := [\ q-2q^{1/2}+1\ ,\ q+2q^{1/2}+1\ ].$$

Aubry, Haloui and Lachaud [AHL13, Corollaries 2.2 and 2.14] and Kadets [Kad21, Theorem 1.8] found inner and outer bounds $I_{\text{inner}}, I_{\text{outer}}$ for I_{simple}:

$$\left[\ q-\lfloor 2q^{1/2}\rfloor+3\ ,\ q+\lfloor 2q^{1/2}\rfloor-1-q^{-1}\ \right]\ \subseteq\ I_{\text{simple}}\ \subseteq\ \left[\ q-\lceil 2q^{1/2}\rceil+2\ ,\ q+\lceil 2q^{1/2}\rceil\ \right]. \tag{4}$$

Our inner bound I_{attained} for I_{simple} improves upon I_{inner}, but careful consideration shows that Kadets's argument yields a better result than he claimed, an inner bound matching our I_{attained} when q is a square.

The following diagram shows $I_{\text{attained}} \subset I_{\text{outer}} \subset I_{\text{Weil}}$, bounded by open dots, solid dots, and vertical bars, respectively. The endpoints of I_{true} and I_{simple} are unknown, but they lie somewhere in the (closed) dashed intervals.



1.3. **Strategy of proof.** Given an abelian variety A over the finite field $\mathbb{F}_q$, let $f_A(x) \in \mathbb{Z}[x]$ be the characteristic polynomial of the q-power Frobenius acting on a Tate module $T_\ell A$. Then $\#A(\mathbb{F}_q) = f_A(1)$. Honda–Tate theory implies that for $f \in \mathbb{Z}[x]$, we have $f = f_A$ for some ordinary n-dimensional abelian variety A over $\mathbb{F}_q$ if and only if f is monic of degree $2n$ with complex roots $\alpha_1, \bar{\alpha}_1, \ldots, \alpha_n, \bar{\alpha}_n$ satisfying $|\alpha_i| = q^{1/2}$, and p does not divide the coefficient of x^n. Therefore, as in [HK21], we need to find a polynomial f satisfying these conditions with a prescribed value of $f(1)$.

One ingredient that lets us go beyond [HK21] is a lemma more general than [DH98, Lemma 3.3.1] for constructing polynomials whose roots lie on the circle $|z| = q^{1/2}$ (Lemma 3.1). Using this lemma alone, we can give a quick proof of Corollary 1.3, if we omit "geometrically simple" and "principally polarized": see Section 4.

To force A to be geometrically simple and principally polarized, we prove that it suffices to impose certain congruence conditions on the coefficients of f (Proposition 5.8); unlike [DH98, Lemma 3.3.1], our Lemma 3.1 is robust enough to permit a wide enough range of values $f(1)$ even when such congruence conditions are imposed. To prove Theorem 1.1, we start with rescaled Chebyshev polynomials similar to those in [Kad21] (Proposition 6.2), but we improve on [Kad21] by temporarily allowing *non-integral* real coefficients, and later making adjustments to make the coefficients integral while preserving $f(1)$ and the bounds needed to apply Lemma 3.1. To obtain the widest interval of realizable values, we must adjust differently in three different ranges of exponents, and the adjustments do something more elaborate than changing one coefficient at a time; see Section 7.

Although we do not know if the bounds in Theorem 1.1 are sharp, Appendix A shows that the rescaled Chebyshev polynomials are asymptotically optimal for our *method*.

1.4. **Large q limit.** So far we have discussed the possibilities for $\#A(\mathbb{F}_q)$ for an n-dimensional abelian variety over a fixed finite field $\mathbb{F}_q$, as $n \to \infty$. We also obtain a sharp asymptotic for the possibilities for fixed n as $q \to \infty$:

Theorem 1.7. *Fix $n \geq 3$. Let $\lambda_1 = 2n - \sqrt{\frac{2n}{n-1}}$. Then the largest interval in which every integer is $\#A(\mathbb{F}_q)$ for some n-dimensional abelian variety A over $\mathbb{F}_q$ has the form*

$$\left[\, q^n - \lambda_1 q^{n-1/2} + o(q^{n-1/2}) \,,\; q^n + \lambda_1 q^{n-1/2} + o(q^{n-1/2}) \,\right] \tag{5}$$

as $q \to \infty$ through prime powers.

Remark 1.8. The interval (5) is a fraction $\lambda_1/(2n)$ of the Hasse–Weil interval, approximately.

Remark 1.9. For $n = 1$, if q is prime, then every integer in $[q - 2q^{1/2} + 1, q + 2q^{1/2} + 1]$ is $\#A(\mathbb{F}_q)$ for some elliptic curve A over $\mathbb{F}_q$. This fails for $q = p^e$ with $e \geq 2$ because of Remark 2.3 below.

Remark 1.10. For $n = 2$, Theorem 1.7 holds if q tends to ∞ through primes only. If instead q tends to ∞ through *non-prime* prime powers, then the constant $\lambda_1 = 2$ (asymptotically 50% of the Hasse–Weil interval) must be replaced by $\lambda_2 := 4 - 2\sqrt{2}$ (about 29% of the Hasse–Weil interval); see Remark 8.5.

Remark 1.11. If we allow only *ordinary* abelian varieties, then Theorem 1.7 remains true for $n \geq 3$, as the proof will show, but for $n = 2$ one must use λ_2 in place of λ_1, even if q is prime.

Remark 1.12. DiPippo and Howe proved a result implying that for any $n \geq 2$, all integers in an interval of the form (5) with λ_1 replaced by $1/2$ are realized by ordinary abelian varieties [DH98, Theorem 1.4]. Thus Theorem 1.7 and Remark 1.11 give an asymptotically optimal improvement of their result.

Theorem 1.7 will be proved in Section 8.

1.5. **Effective bounds.** The polynomial constructions we used to prove Theorems 1.1 and 1.7 are difficult to analyze explicitly for specific values of q and n, even when $q = 3$. In Section 9, we give *another* construction, and this one, combined with some computations with rigorous error bounds, will allow us to prove the following.

Theorem 1.13. *Let q be a prime power.*

(a) *For each $q \leq 5$, every positive integer is $\#A(\mathbb{F}_q)$ for some abelian variety A over $\mathbb{F}_q$.*
(b) *For arbitrary q, every integer $\geq q^{3\sqrt{q}\log q}$ is $\#A(\mathbb{F}_q)$ for some abelian variety A over $\mathbb{F}_q$.*

Remark 1.14. Theorem 1.13(a) is best possible: As remarked in [HK21], if $q \geq 7$, then 2 lies outside the union of the Hasse–Weil intervals (1).

Remark 1.15. Theorem 1.13(b) is best possible too, except for the constant 3, which we have not attempted to optimize. It becomes false for large q if 3 is replaced by any number $\delta < 1/4$, because if $n = (\delta + o(1))\sqrt{q}\log q$, then

$$\begin{aligned}
\log \frac{(\sqrt{q}-1)^{2(n+1)}}{(\sqrt{q}+1)^{2n}} &= \log q + o(1) + 2n \log \frac{\sqrt{q}-1}{\sqrt{q}+1} \\
&= \log q + o(1) + 2(\delta + o(1))(q^{1/2}\log q)(-2q^{-1/2} + o(q^{-1})) \\
&= (1 - 4\delta + o(1))\log q,
\end{aligned}$$

which means that there is a large gap between the nth Hasse–Weil interval and the $(n+1)$st.

Remark 1.16. Suppose that we require A to be ordinary. Both statements in Theorem 1.13 remain true, except that when $q = 4$ one must exclude order 3 (that 3 over $\mathbb{F}_4$ must be excluded follows from [Kad21, Theorem 3.2]).

Remark 1.17. For $q = 7$, the only positive integers not of the form $\#A(\mathbb{F}_7)$ are 2, 14, and 17. If we require A to be ordinary, then 8 and 73 are the only additional integers that must be excluded.

Remark 1.18. Suppose that we require f_A to be squarefree. Then all the claims in this section remain true except that for $q = 7$, the integer 16 is no longer realized.

2. Honda–Tate theory

Throughout the paper, if f is a polynomial, then $f^{[i]}$ denotes the coefficient of its degree i term. All the results of this section are restatements of results in [Wat69, Chapter 2].

Theorem 2.1 (Honda–Tate)**.** *A polynomial $f \in \mathbb{Z}[x]$ is the characteristic polynomial of an ordinary abelian variety A of dimension n over $\mathbb{F}_q$ if and only if*

(a) *f is monic of degree $2n$;*
(b) *f is q-symmetric, by which we mean $f^{[i]} = q^{n-i} f^{[2n-i]}$ for $i = 0, \dots, n-1$;*
(c) *all complex roots of f have absolute value $q^{1/2}$; and*
(d) *$p \nmid f^{[n]}$.*

Remark 2.2. Condition (c) implies (b) if $x + q^{1/2}$ and $x - q^{1/2}$ each appear to an even power in the factorization of $f(x)$ over $\mathbb{C}$.

Remark 2.3. Let $v \colon \mathbb{Q}_p \to \mathbb{Z} \cup \{\infty\}$ be the p-adic valuation. If in Theorem 2.1 we replace (d) by the weaker condition

(d′) the multiplicity μ of each $\mathbb{Q}_p$-irreducible factor g in f is such that $\mu\, v(g(0))/v(q) \in \mathbb{Z}$,

then we obtain the criterion for f to be the characteristic polynomial of a not-necessarily-ordinary abelian variety A of dimension n over $\mathbb{F}_q$. If q is prime, then (d′) holds automatically.

3. Roots on a circle

For $r > 0$, let $\mathbb{C}_{\le r}$ be the closed disk $\{z \in \mathbb{C} : |z| \le r\}$. Let $D = \mathbb{C}_{\le q^{-1/2}}$. For

$$h(z) = a_0 + a_1 z + \cdots + a_s z^s \ \in \mathbb{R}[z]$$

with $s < 2n$, define

$$\begin{aligned}\widehat{h}(x) &= x^{2n} h(1/x) + q^n h(x/q) \\ &= a_0 x^{2n} + a_1 x^{2n-1} + \cdots + a_s x^{2n-s} \\ &\qquad\qquad + q^{n-s} a_s x^s + \cdots + q^{n-1} a_1 x + q^n a_0,\end{aligned}$$

which is a q-symmetric polynomial of degree $\le 2n$ (the notation implicitly depends on a choice of n). To prove Theorem 1.1, we will eventually need $\widehat{h}$ for some polynomials h of degree $s > n$, in which case the two ranges of exponents of x overlap.

Lemma 3.1. *Let $h(z) \in \mathbb{R}[z]$ be a polynomial of degree $< 2n$ such that h is nonvanishing on D. Then all complex roots of $\widehat{h}(x)$ have absolute value $q^{1/2}$.*

Proof. As z goes around the circle $|z| = q^{-1/2}$, the winding number of $h(z)$ around 0 is 0, so the winding number of $x^n h(1/x)$ as x goes around the circle $|x| = q^{1/2}$ is n. Thus the real-valued function $2\operatorname{Re}(x^n h(1/x)) = x^n h(1/x) + q^n x^{-n} h(x/q)$ on the circle $|x| = q^{1/2}$ crosses 0 at least $2n$ times. Multiplying by x^n shows that $\widehat{h}(x)$ has at least $2n$ roots on the circle $|x| = q^{1/2}$. It cannot have more than $2n$ roots, since $\deg \widehat{h} = 2n$. $\square$

Remark 3.2. If $h(z) = 1 + a_1 z + \cdots + a_n z^n$ with $\sum_{i=1}^n |a_i| q^{-i/2} < 1$, then $h(D) \subset \{z \in \mathbb{C} : |z - 1| < 1\}$, so $0 \notin h(D)$. Thus Lemma 3.1 subsumes [DH98, Lemma 3.3.1], which appears also (with a different proof) as [HK21, Lemma 2]. The feature of Lemma 3.1 that allows us to obtain stronger results is that $\{h : 0 \notin h(D)\}$ is closed under multiplication, a natural property given that one can take products of abelian varieties.

Remark 3.3. The polynomials $\widehat{h}(x)$ produced by Lemma 3.1 are squarefree.

Remark 3.4. Applying Lemma 3.1 to $h(rx)$ as $r \to 1^-$ shows that the hypothesis could be weakened to assume only that h is nonvanishing on the *interior* of D.

For use in the proof of Lemma 7.1, we record the following result.

Lemma 3.5. *Let $R \in \mathbb{C}[z]$ be a polynomial with no zeros inside D. Then*

$$|R(1)| \leq q^{(\deg R)/2} |R(1/q)|. \tag{6}$$

Proof. By multiplicativity in R, we may assume that $R(z) = z - w$ for some $w \in \mathbb{C}$ with $|w| \geq q^{-1/2}$. We must prove $|(1-w)/(1/q - w)| \leq q^{1/2}$. The Möbius transformation $M(z) := (1-z)/(1/q - z)$ maps the circle $|z| = q^{-1/2}$ to a complex-conjugation-invariant circle passing through $M(\pm q^{-1/2}) = \pm q^{1/2}$, and it maps the exterior to the interior since $M(\infty) = 1$. □

4. Abelian varieties of all sufficiently large orders

Theorem 4.1. *Fix a prime power q and a closed interval $I \subset \mathbb{R}_{>0}$. For $n \gg 1$, each integer $m \in q^n I$ is $\#A(\mathbb{F}_q)$ for some ordinary abelian variety A of dimension n over $\mathbb{F}_q$.*

Proof. For $k \geq 1$, let $\mathcal{J}_k$ be the set of power series of the form $1 + a_k z^k + a_{k+1} z^{k+1} + \cdots$ with integer coefficients in $[-q/2, q/2]$. Choose k such that $1 - \sum_{r \geq k} \lfloor q/2 \rfloor q^{-r/2} \geq 1/2$; then $|j(w)| \geq 1/2$ for all $j \in \mathcal{J}_k$ and $w \in D$. Choose $\epsilon > 0$ such that $[1-\epsilon, 1+\epsilon] \subset \{j(1/q) : j \in \mathcal{J}_k\}$. Choose N such that $[(1-\epsilon)^N, (1+\epsilon)^N] \supset I$. Then, given $m \in q^n I$, we may choose $j \in \mathcal{J}_k$ with $j(1/q)^N = m/q^n$. Write $j^N = h_0 + h_1$ such that $h_0 \in 1 + z^k \mathbb{Z}[z]$ is of degree $\leq n$, and $h_1 \in z^{n+1} \mathbb{Z}[[z]]$. Let $E = m - \widehat{h}_0(1)$. Let

$$h = h_0 + (E/2) z^n + s(z^{n-1} - ((q+1)/2) z^n),$$

where $s \in \{0, 1\}$ is chosen so that p does not divide the coefficient of x^n in

$$\widehat{h} = \widehat{h}_0 + E x^n + s(x^{n+1} - (q+1) x^n + q x^{n-1}).$$

Then $\widehat{h}$ is a monic polynomial of degree $2n$ in $\mathbb{Z}[x]$ and $\widehat{h}(1) = \widehat{h}_0(1) + E = m$. The conclusion follows from Lemma 3.1 and Theorem 2.1 if we can show that h is nonvanishing on D. We will do so by estimating the error in the approximations $h \approx h_0 \approx j^N$.

Since j has bounded coefficients, induction on N shows that $|(j^N)^{[r]}| = O(r^{N-1})$ as $r \to \infty$, uniformly for $j \in \mathcal{J}_k$. Thus

$$|h_0(1)| = \left| \sum_{r=0}^{n} (j^N)^{[r]} \right| \leq \sum_{r=0}^{n} O(r^{N-1}) = O(n^N),$$

$$|h_1(1/q)| = \left| \sum_{r=n+1}^{\infty} (j^N)^{[r]} q^{-r} \right| \leq \sum_{r=n+1}^{\infty} O(r^{N-1}) q^{-r} = O(n^{N-1} q^{-n-1}),$$

$$|E| = |m - \widehat{h}_0(1)| = |q^n j(1/q)^N - (q^n h_0(1/q) + h_0(1))| \leq |q^n h_1(1/q)| + |h_0(1)| = O(n^N).$$

Now

$$h(z) = j(z)^N - h_1(z) + (E/2) z^n + s(z^{n-1} - ((q+1)/2) z^n),$$

so for $w \in D$,

$$|h(w)| \geq 2^{-N} - O(n^{N-1}) q^{-n/2} - O(n^N q^{-n/2}) - O(q \cdot q^{-n/2}) > 0$$

if n is large enough. $\square$

Corollary 4.2. *Fix a prime power q. Every sufficiently large positive integer is $\#A(\mathbb{F}_q)$ for some ordinary abelian variety A over $\mathbb{F}_q$.*

Proof. Apply Theorem 4.1 with $I = [1, q]$. $\square$

5. A congruence condition forcing geometric simplicity and the existence of principal polarizations

The goal of this section is Proposition 5.8, which provides a congruence condition on the characteristic polynomial of an abelian variety A over $\mathbb{F}_q$ which guarantees that A is geometrically simple and isogenous to a principally polarized abelian variety. Moreover, the congruence condition will be compatible with prescribing $\#A(\mathbb{F}_q)$. The lemmas in this section are used only to prove Proposition 5.8.

Lemma 5.1. *For every prime power q, prime $\ell \geq 7$ not dividing q, and integer $n \geq 1$, there exists $j(x) \in \mathbb{F}_\ell[x]$ such that $j(x)$ and $x^n j(q/x)$ are relatively prime irreducible polynomials of degree n not vanishing at 1.*

Proof. If $n = 1$, choose $j(x) = x - a$ where $a \in \mathbb{F}_\ell - \{0, 1, q, \pm\sqrt{q}\}$. If $n = 2$, let $j(x)$ be the minimal polynomial of an element $\alpha \in \mathbb{F}_{\ell^2}^\times - \mathbb{F}_\ell^\times$ such that $\alpha \neq q/\alpha$ and $\alpha^\ell \neq q/\alpha$; there are at least $(\ell^2 - \ell) - 2 - (\ell + 1) > 0$ such elements α.

Now suppose that $n \geq 3$. Let α be a generator of the multiplicative group $\mathbb{F}_{\ell^n}^\times$. Let $j(x)$ be the minimal polynomial of α over $\mathbb{F}_\ell$. If $j(x)$ and $x^n j(q/x)$ are not relatively prime, then $\alpha^{\ell^a} = q/\alpha$ for some $a \in \{0, 1, \ldots, n-1\}$. Then $\alpha^{(\ell-1)(\ell^a+1)} = q^{\ell-1} = 1$ in $\mathbb{F}_{\ell^n}$, so $\ell^n - 1$ divides $(\ell - 1)(\ell^a + 1)$, contradicting $0 < (\ell - 1)(\ell^a + 1) < \ell^n - 1$. $\square$

Lemma 5.2. *Let q be a prime power, let $\ell \geq 7$ be a prime not dividing q, let $n \in \mathbb{Z}_{\geq 1}$, and let $m \in \mathbb{Z}$. Suppose that $d_1, \ldots, d_r$ are positive integers summing to n such that 1 appears exactly once or twice among $d_1, \ldots, d_r$ and every other positive integer appears at most once. Then there exists a monic q-symmetric polynomial $g(x) \in \mathbb{F}_\ell[x]$ such that*

- $g(1) = m \bmod \ell$,
- *the roots of g form n* distinct *multiset pairs $\{\alpha, q/\alpha\}$; and*
- *the Frobenius element of $\operatorname{Gal}(\overline{\mathbb{F}}_\ell/\mathbb{F}_\ell)$ acts on these n pairs as a permutation consisting of cycles of lengths $d_1, \ldots, d_r$.*

Proof. For each i with $d_i \geq 2$, let $j_i(x)$ be the polynomial of degree d_i provided by Lemma 5.1, and let $g_i(x) = j_i(x) \cdot x^{d_i} j_i(q/x)$. For each i with $d_i = 1$, let $g_i(x) = x^2 - a_i x + q$ for some $a_i \in \mathbb{F}_\ell$ to be determined. Then $g(x) = \prod_{i=1}^r g_i(x)$ gives the correct cycle type, and its irreducible factors are distinct, except possibly for the factors of the g_i for which $d_i = 1$.

If exactly one d_i equals 1, then there is a unique choice of a_i in $\mathbb{F}_\ell$ that makes $g(1) = m \bmod \ell$. If d_i and d_j both equal 1 (with $i \neq j$), then there are at least $\ell - 1$ choices for (a_i, a_j) that make $g(1) = m \bmod \ell$ and at most two of these satisfy $a_i = a_j$; thus we can ensure $g(1) = m \bmod \ell$ while making g separable. $\square$

Lemma 5.3. *For every prime power q, integer m, prime $\ell > q + 2\sqrt{q} + 1$, and integer $n \geq 8\sqrt{q} + 5$, there exists a monic q-symmetric polynomial $g(x) \in \mathbb{F}_\ell[x]$ of degree $2n$ such that $g(1) = m \bmod \ell$ and $g(x)$ has no factor of the form $x^2 - \overline{a}x + q$ with $a \in \mathbb{Z}$ and $|a| \leq 2\sqrt{q}$.*

Proof. Since $\ell > q + 2\sqrt{q} + 1$, none of the polynomials $x^2 - \overline{a}x + q$ vanish at 1 mod ℓ. Lagrange interpolation provides a monic degree n polynomial $j(x) \in \mathbb{F}_\ell[x]$ such that $j(0) = 1$, $j(1) = m$, $j(q) = 1$, and $j(\alpha) = 1$ for every root $\alpha \in \overline{\mathbb{F}}_\ell$ of the quadratic polynomials $x^2 - \overline{a}x + q$ (the number of values to specify is at most $3 + 2(4\sqrt{q} + 1) \le n$). Take $g(x) := j(x) \cdot x^n j(q/x)$. □

Lemma 5.4. *Let $n \ge 3$. A subgroup G of S_n containing an $(n-1)$-cycle, an $(n-2)$-cycle, and a 2-cycle is either S_n or the stabilizer S_{n-1} of the fixed point of the $(n-1)$-cycle.*

Proof. Without loss of generality, the fixed point of the $(n-1)$-cycle is n. If $G \le S_{n-1}$, then G acts on $\{1, 2, \ldots, n-1\}$ transitively (because of the $(n-1)$-cycle) and primitively (because of the $(n-2)$-cycle); a primitive subgroup of S_{n-1} containing a 2-cycle is the whole group S_{n-1} [Isa08, Theorem 8.17]. Otherwise G acts on $\{1, \ldots, n\}$ transitively (because the orbit of 1 is larger than $\{1, 2, \ldots, n-1\}$) and primitively (because of the $(n-1)$-cycle), and then the 2-cycle forces $G = S_n$. □

Lemma 5.5. *Let $n \ge 5$. Let A be an n-dimensional abelian variety over $\mathbb{F}_q$. Write $f_A(x) = x^n R(x + q/x)$ for some monic degree n polynomial $R(x) \in \mathbb{Z}[x]$. If the Galois group of R is S_n or the stabilizer S_{n-1} of a point, then A is either geometrically simple or a product of geometrically simple abelian varieties over $\mathbb{F}_q$ of dimensions $n-1$ and 1.*

Proof. If A is isogenous to $A_1 \times A_2$ over $\mathbb{F}_q$, then R factors correspondingly into $R_1 R_2$. Since R is either irreducible or a product of irreducible polynomials of degrees 1 and $n-1$, the abelian variety A is either simple or a product of simple abelian varieties of dimensions 1 and $n-1$. Let A' be the simple factor of dimension $d \in \{n, n-1\}$, and define R' accordingly.

Suppose that A' is not geometrically simple. Let $r > 1$ be such that $A'_{\mathbb{F}_{q^r}}$ is not simple. Then $f_{A'}$ has roots $\alpha, \beta \in \overline{\mathbb{Q}}$ giving rise to distinct roots $\alpha + q/\alpha \neq \beta + q/\beta$ of R' such that $\alpha^r = \beta^r$. Now $\beta = \zeta\alpha$ for some root of unity ζ. Thus the extension $\mathbb{Q}(\alpha, \zeta) \supset \mathbb{Q}(\alpha + q/\alpha)$, being the compositum of two abelian extensions, is abelian, so its subfield $\mathbb{Q}(\alpha + q/\alpha, \beta + q/\beta)$ is Galois over $\mathbb{Q}(\alpha + q/\alpha)$, contradicting the fact that S_{d-2} is not normal in S_{d-1}. □

Lemma 5.6. *For every prime power $q = p^e$, prime $\lambda \ge 7$ such that q is a nonzero square modulo λ, and integers $n \ge 5$ and m, there exists a monic q-symmetric degree $2n$ polynomial $g(x) \in (\mathbb{Z}/\lambda^2\mathbb{Z})[x]$ with $g(1) = m \bmod \lambda^2$ such that if A is a simple abelian variety over $\mathbb{F}_q$ with $f_A \bmod \lambda^2$ equal to g, then the isogeny class of A contains a principally polarized abelian variety.*

Proof. By Hensel's lemma, we can choose $s \in \mathbb{Z}$ such that the discriminant of $x^2 - sx + q$ is 0 mod λ but nonzero mod λ^2. Replace s by $-s$, if necessary, to make $q + 1 - s \not\equiv 0 \pmod{\lambda}$. Choose a monic irreducible polynomial $S(x) \in \mathbb{F}_\lambda[x]$ of degree $n-3$. Choose $a, b \in \mathbb{F}_\lambda$ such that the polynomial $\bar{R} := (x-s)(x-a)(x-b)S(x) \in \mathbb{F}_\lambda[x]$ is separable and $\bar{R}(q+1) = m \bmod \lambda$; this amounts to choosing two elements of $\mathbb{F}_\lambda$ (namely, $q+1-a$ and $q+1-b$) with prescribed product, not equal to $q+1-s$ or each other, which is possible because $\lambda - 1 > 4$. Let $R \in (\mathbb{Z}/\lambda^2\mathbb{Z})[x]$ be a lift of $\bar{R}$ such that $R(s) = 0$ and $R(q+1) = m$ in $\mathbb{Z}/\lambda^2\mathbb{Z}$. Let $g(x) = x^n R(x + q/x) \in (\mathbb{Z}/\lambda^2\mathbb{Z})[x]$.

Suppose that A is a simple abelian variety over $\mathbb{F}_q$ such that $f_A \bmod \lambda^2$ is g. Since A is simple, f_A is a power of an irreducible polynomial [Wat69, Chapter 2], but its reduction $g \bmod \lambda$ has some simple roots (for example, the roots of $x^{n-3}S(x + q/x)$), so f_A must be irreducible, of degree $2n$. Let $\pi \in \overline{\mathbb{Q}}$ be a root of f_A. Let $K = \mathbb{Q}(\pi)$ and $K^+ = \mathbb{Q}(\pi + q/\pi)$, so K is a CM field and K^+ is its totally real subfield. Since the minimal polynomial of

$\pi + q/\pi$ reduces to $\bar{R}$, the extension $K^+/\mathbb{Q}$ is unramified above λ. On the other hand, K/K^+ is ramified at the prime above λ corresponding to the root s of g, because the discriminant of $x^2 - sx + q$ has odd valuation 1. By [How96, Theorem 1.1], the isogeny class of A contains a principally polarized abelian variety. ☐

Lemma 5.7. *For any prime power q, there exists a prime λ such that $7 \le \lambda < q^3$ and q is a nonzero square mod λ.*

Proof. We will choose λ to be a prime factor of $u^2 - q$ for some integer u in $[\sqrt{q} - 30, \sqrt{q} + 30]$ chosen so that $u^2 - q \neq \pm 1$ and $u^2 - q$ is not divisible by 2, 3, or 5. There are at least six integers u in $[\sqrt{q} - 30, \sqrt{q} + 30]$ such that $u^2 - q$ is not divisible by 2, 3, or 5. At most two of them satisfy $u^2 - q = \pm 1$; among the other four are two differing by 30, and one of them is prime to p. Thus u can be found. Then $\lambda \neq 2, 3, 5, p$, and $\lambda \le (\sqrt{q} + 30)^2 - q$, which is less than q^3, except for some small q for which we instead compute an explicit λ. ☐

Proposition 5.8. *Given a prime power q, there exists a positive integer L such that for any integers $n \gg 1$ and m, there exists a monic q-symmetric polynomial $g(x) \in (\mathbb{Z}/L\mathbb{Z})[x]$ of degree $2n$ with $g(1) = m \bmod L$ such that any n-dimensional abelian variety A over $\mathbb{F}_q$ whose characteristic polynomial reduces modulo L to $g(x)$ is ordinary, geometrically simple, and isogenous to a principally polarized abelian variety. Moreover, we may choose $L < q^{23}$.*

Proof. Let λ be as in Lemma 5.7. Let $L = p\lambda^2\ell_0\ell_1\ell_2\ell_3$, where p is the characteristic, and $p, \lambda, \ell_0, \dots, \ell_3$ are distinct primes such that $\ell_0 > q + 2\sqrt{q} + 1$ and $\ell_i \ge 7$ for $i = 1, \dots, 3$. Suppose that $n \ge 8\sqrt{q} + 5$. Let $\gamma(x) \in \mathbb{F}_p[x]$ be a monic q-symmetric polynomial of degree $2n$ such that $\gamma(1) = m \bmod p$; add $x^{n+1} - x^n$, if necessary, to make $\gamma^{[n]} \neq 0 \bmod p$ (here q-symmetry means only that $\gamma^{[i]} = 0$ for $i < n$). Let $g_\lambda(x) \in (\mathbb{Z}/\lambda^2\mathbb{Z})[x]$ be as in Lemma 5.6. Apply Lemma 5.3 to produce a polynomial $g_0(x) \in \mathbb{F}_{\ell_0}[x]$. Apply Lemma 5.2 to produce polynomials $g_i(x) \in \mathbb{F}_{\ell_i}[x]$ for $i = 1, 2, 3$ corresponding to the partitions

- $(n - 1, 1)$
- $(n - 2, 1, 1)$
- $(n - 3, 2, 1)$ if n is even; and $(n - 4, 2, 1, 1)$ n is odd,

respectively. Let $g \in (\mathbb{Z}/L\mathbb{Z})[x]$ be the monic q-symmetric polynomial of degree $2n$ reducing to γ, the g_i, and g_λ.

Suppose that A is an n-dimensional abelian variety over $\mathbb{F}_q$ such that $f_A(x) \bmod L = g(x)$. Write $f_A(x) = x^n R(x + q/x)$. Let $G \le S_n$ be the Galois group of R, which encodes the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the pairs $\{\alpha, q/\alpha\}$ of roots of F. By choice of g_1, g_2, g_3, the group G contains permutations $\sigma_1, \sigma_2, \sigma_3$ whose cycle types are given by the partitions above. Raising σ_3 to an odd power produces a 2-cycle. By Lemma 5.4, G is S_n or S_{n-1}. By Lemma 5.5, A is either geometrically simple or a product of geometrically simple abelian varieties over $\mathbb{F}_q$ of dimensions $n - 1$ and 1. In the second case, $f_A(x)$ would have a factor $x^2 - ax + q$ for some integer a with $|a| \le 2\sqrt{q}$, which is ruled out by the choice of g_0. Thus A is geometrically simple. Since $\gamma^{[n]} \neq 0 \bmod p$, A is ordinary. By Lemma 5.6, A is isogenous to a principally polarized abelian variety.

In proving $L < q^{23}$, the worst case is $q = 2$, in which case we take $L = 2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 < 2^{23}$. ☐

Remark 5.9. It is not hard to adapt Proposition 5.8 for the purpose of constructing abelian varieties of prescribed order that have *prescribed p-rank*. Namely, one can prove that it suffices to impose congruences modulo pq^2 on the coefficients of a q-symmetric monic degree $2n$ polynomial f to guarantee that its Newton polygon is the lowest Newton polygon corresponding to p-rank r and that its segments of slope in $[-1/2, 0]$ correspond to $\mathbb{Q}_p$-irreducible factors, in which case the other segments do too by q-symmetry, so that the condition in Remark 2.3 is satisfied; moreover one can make these congruences compatible with $f(1) \equiv m \pmod{pq^2}$, provided that, in the case $r = 0$, one has $m \equiv 1 \pmod{p}$. This last hypothesis is necessary: if A has p-rank 0, then all roots of f_A have positive p-adic valuation, so $\#A(\mathbb{F}_q) \equiv 1 \pmod{p}$.

6. Chebyshev polynomials

Choose the branch of $\sqrt{z^2-1}$ on $\mathbb{C} - [-1,1]$ that is $z + o(1)$ as $z \to \infty$. Let $\tau(z) = z + \sqrt{z^2-1}$. Define the dth Chebyshev polynomial

$$T_d(z) = \frac{1}{2}\left(\left(z+\sqrt{z^2-1}\right)^d + \left(z-\sqrt{z^2-1}\right)^d\right) = (\tau(z)^d + \tau(z)^{-d})/2. \tag{7}$$

Lemma 6.1. *For a suitable choice of dth root, the functions $T_d(z)^{1/d}/z$ and $\tau(z)/z$ extend to holomorphic functions on $\mathbb{P}^1(\mathbb{C}) \setminus [-1,1]$, and $T_d(z)^{1/d}/z \to \tau(z)/z$ uniformly on any compact subset of that domain as $d \to \infty$.*

Proof. Since τ is nonvanishing with a simple pole at ∞, the maximum modulus principle applied to $1/\tau$ shows that $|\tau(z)|$ is minimized as z approaches $[-1,1]$, in which case $|\tau(z)| \to 1$. Thus $|\tau(z)| > 1$ on $\mathbb{P}^1(\mathbb{C}) \setminus [-1,1]$. The uniform convergence claim now follows from $T_d(z)/z^d = \frac{1}{2}z^{-d}(\tau(z)^d + \tau(z)^{-d})$. □

Proposition 6.2. *Let I be a closed interval contained in I_{attained} (see (2)). Then for even $d \gg 1$, there exists a degree d polynomial $P(z) \in \mathbb{R}[z]$ such that*

(a) *$P(0) = 1$;*
(b) *P is positive on $\mathbb{R}$;*
(c) *$|P(w)|^{1/d} \geq q^{-1/4}$ for all $w \in D := \mathbb{C}_{\leq q^{-1/2}}$; and*
(d) *$(qP(1/q)^{2/d}, qP(-1/q)^{2/d})$ contains I.*

Remark 6.3. In Appendix A, we use potential theory to prove that Proposition 6.2 is optimal in the sense that it fails if I_{attained} is enlarged.

Proof. For $\epsilon > 0$ to be specified later, let

$$\begin{aligned} \ell(z) &= (q^{1/2}/2)z - (q^{1/2}-1), \\ f_d(z) &= 2q^{-d/4}z^{d/2}\,T_{d/2}(\ell(z+1/z)), \\ P(z) &= f_d((1-\epsilon)q^{1/2}z). \end{aligned}$$

(a) The leading coefficient of $T_{d/2}$ is $2^{d/2-1}$, so $f_d(0) = 2q^{-d/4}2^{d/2-1}(q^{1/2}/2)^{d/2} = 1$ and $P(0) = f_d(0) = 1$.
(b) The roots of $T_{d/2}$ are in $[-1,1)$, and $\ell^{-1}([-1,1)) \subset (-2,2)$, so all the roots of $f_d(z)$ are on the unit circle and not at ± 1. Thus f_d does not change sign on $\mathbb{R}$. Since $f_d(0) > 0$, the sign is positive. Thus P is positive on $\mathbb{R}$.

(c) The function $(1-\epsilon)q^{1/2}z$ maps D to $\mathbb{C}_{\le 1-\epsilon}$, so we need to prove that $|f_d|^{1/d} \ge q^{-1/4}$ on $\mathbb{C}_{\le 1-\epsilon}$. First, $zT_{d/2}(\ell(z+1/z))^{2/d}$ is the product of the polynomial $z\,\ell(z+1/z)$ and holomorphic function $T_{d/2}(\ell(z+1/z))^{2/d}/\ell(z+1/z)$ on $\mathbb{C}_{\le 1-\epsilon}$, so Lemma 6.1 implies that

$$|f_d(z)|^{1/d} \to q^{-1/4}|z|^{1/2}\,|\tau(\ell(z+1/z))|^{1/2} \tag{8}$$

uniformly for $z \in \mathbb{C}_{\le 1-\epsilon}$. The function $z\,\tau(\ell(z+1/z))$ is holomorphic, nonconstant, and nonvanishing on $\mathbb{C}_{<1}$, and it extends to a continuous function on $\mathbb{C}_{\le 1}$ having absolute value ≥ 1 on the boundary, so the maximum modulus principle applied to its inverse shows that there exists $M > 1$ such that $|z\,\tau(\ell(z+1/z))| > M$ for all $z \in \mathbb{C}_{\le 1-\epsilon}$. The lower bound on $|f_d|$ follows for $d \gg 1$.

(d) It suffices to prove that $\lim_{\epsilon\to 0^+}\lim_{d\to\infty} qP(1/q)^{2/d}$ equals the left endpoint of I_{attained}, and likewise at the other end. In fact, (8) implies that $\lim_{d\to\infty} qP(1/q)^{2/d}$ is a continuous function of $\epsilon \in [0,1]$, so we may simply *substitute* $\epsilon = 0$. Then

$$\begin{aligned}\lim_{d\to\infty} qP(1/q)^{2/d} &= \lim_{d\to\infty} qf_d(q^{-1/2})^{2/d} \\ &= q\cdot q^{-1/2}q^{-1/2}\,|\tau(\ell(q^{-1/2}+q^{1/2}))| \\ &= \tau(q/2 - q^{1/2} + 3/2).\end{aligned}$$

Similarly, $\lim_{\epsilon\to 0^+}\lim_{d\to\infty} qP(-1/q)^{2/d} = |\tau(-q/2-q^{1/2}+1/2)| = \tau(q/2+q^{1/2}-1/2)$. □

7. Construction of polynomials

We now begin the proof of Theorem 1.1. Let I be a closed interval in I_{attained}. Let $P(z)$ be as in Proposition 6.2 and let $d = \deg P$; we may assume that $d \ge 53$.

The polynomial P was optimized to have a small value at $1/q$ and large value at $-1/q$. Lemma 7.1 below shows that this makes $\widehat{P^b}(1)$ small and $\widehat{P(-z)^b}(1)$ large, where b is chosen to make P^b of degree close to $2n$. The polynomial Q in Lemma 7.2 interpolates between $P(z)$ and $P(-z)$ to make $\widehat{Q^b}(1)$ equal a prescribed intermediate value.

Lemma 7.1. *Let $b = b(n)$ and $\ell = \ell(n)$ be functions of n tending to ∞ such that* $\deg P^b = 2n - 2\ell$ *and $\ell = o(n)$. Then*

$$\widehat{P^b}(1)^{1/n} \longrightarrow qP(1/q)^{2/d} \qquad \text{and} \qquad \widehat{P(-z)^b}(1)^{1/n} \longrightarrow qP(-1/q)^{2/d}$$

as $n \to \infty$. (Recall that $\widehat{P^b}(1) := q^nP^b(1/q) + P^b(1)$, which depends on n.)

Proof. We have

$$\widehat{P^b}(1) = q^nP^b(1/q) + P^b(1) = (q^n + O(q^{n-\ell}))P^b(1/q)$$

by Lemma 3.5 applied to P^b. Taking nth roots yields the left endpoint limit, since $\ell \to \infty$ and $b/n = (2n-2\ell)/(dn) \to 2/d$. The right endpoint limit follows similarly. □

Choose integers $\ell = \ell(n)$ and $b = b(n)$ such that $\ell = 4\log_q n + O(1)$ and $bd = 2n - 2\ell$. The statements in the rest of this section will hold if n is sufficiently large. Given $m \in \mathbb{Z}$ such that $m^{1/n} \in I$, we want to construct an n-dimensional, ordinary, geometrically simple, principally polarized abelian variety A with $\#A(\mathbb{F}_q) = m$.

Lemma 7.2. *There exists $Q(z) \in 1 + z\mathbb{R}[z]$ of degree $\le d$ such that Q is positive on $\mathbb{R}$, $\widehat{Q^b}(1) = m$, and $|Q(w)|^{1/d} \ge q^{-1/4}$ for all $w \in D$.*

Proof. Because n is sufficiently large, Proposition 6.2(d) and Lemma 7.1 show that

$$(\widehat{P^b}(1)^{1/n}, \widehat{P(-z)^b}(1)^{1/n}) \supset I \ni m^{1/n}. \tag{9}$$

By the intermediate value theorem, there exists $s \in [-1, 1]$ such that the polynomial

$$Q(z) := P(sz) \in 1 + z\mathbb{R}[z]$$

satisfies $\widehat{Q^b}(1)^{1/n} = m^{1/n}$. Thus $\widehat{Q^b}(1) = m$. Moreover, Q is positive on $\mathbb{R}$, and $|Q(w)|^{1/d} = |P(sw)|^{1/d} \ge q^{-1/4}$ for all $w \in D$ by Proposition 6.2(b,c). □

In the rest of this section, the implied constant in big-O notation may depend on q, L, d, P, and Q, but not on n.

The polynomial Q^b has real coefficients. We could round them to the nearest integer to produce a polynomial $h \in \mathbb{Z}[x]$ and adjust the middle coefficients to make $\widehat{h}(1) = m$, as in Section 4, but it turns out that we cannot guarantee that such an h is nonvanishing on D, as required for Lemma 3.1. So instead we adjust the coefficients of Q (inside the bth power) by only $O(1/n)$ each to make the first d coefficients of $\widehat{Q^b}$ integral (and to make them satisfy the congruences in Proposition 5.8), and then, to correct the later coefficients, we add correction polynomials designed to be small on D, because as we go along, we need to bound the difference between Q^b and the final h to ensure that h is still nonvanishing on D.

Let us outline the entire construction; then in a series of lemmas, we will prove that the steps make sense.

Construction 7.3.

1. Let $Q \in 1 + z\mathbb{R}[z]$ be as in Lemma 7.2.
2. Let $g \in (\mathbb{Z}/L\mathbb{Z})[x]$ be as in Proposition 5.8.
3. Let $Q_0 = Q$.
4. For $i = 1, \ldots, d-1$ in turn, let $a_i \in [0, L/b)$ and $Q_i := Q_{i-1} + a_i z^i$ and $h_i := Q_i^b$ be such that $\widehat{h}_i^{[2n-i]} \in \mathbb{Z}$ and $\widehat{h}_i^{[2n-i]} \equiv g^{[2n-i]} \pmod{L}$.
5. Let $\tilde{Q} = Q_{d-1} - cz^d$ and $h_d = \tilde{Q}^b$, where $c \in \mathbb{R}$ is chosen so that $\widehat{h}_d(1) = m$.
6. Define "correction polynomials" as follows:
   - For $i = d, \ldots, \ell - 1$, let $k_i = z^i\, \tilde{Q}(z)^b$.
   - For $i = \ell, \ldots, n-1$, let $k_i = z^i\, \tilde{Q}(z)^a$, where $a \in \mathbb{Z}_{\ge 0}$ is chosen as large as possible such that $\deg k_i < 2n - i$.
   - Define $k_n = z^n/2$.

   The definitions are so that $\widehat{k}_i$ is monic of degree $2n - i$ for all integers $i \in [d, n]$.
7. For $i = d, \ldots, n-1$, let $r_i \in [0, L)$ and $s_{i+1} \in \mathbb{R}_{\ge 0}$ and $h_{i+1} := h_i + r_i k_i - s_{i+1} k_{i+1}$, where r_i is such that $\widehat{h}_{i+1}^{[2n-i]} \in \mathbb{Z}$ and $\widehat{h}_{i+1}^{[2n-i]} \equiv g^{[2n-i]} \pmod{L}$, and s_{i+1} is such that $\widehat{h}_{i+1}(1) = m$.
8. Let A be an abelian variety over $\mathbb{F}_q$ with $f_A = \widehat{h}_n$.

Lemma 7.4. *The a_i can be chosen as specified in Step 4, and they are $O(1/n)$.*

Proof. In Step 4, once $a_1, \ldots, a_{i-1}$ have been fixed, $\widehat{h}_i^{[2n-i]}$ as a function of a_i is a linear polynomial with leading coefficient b, so $a_i \in [0, L/b)$ can be found. Then $a_i = O(L/b) = O(1/n)$. □

Lemma 7.5.

(a) *The real number c can be chosen as specified in Step 5, and c is $O(1/n)$.*

(b) *We have $\tilde{Q}(1) > 0$ and $\tilde{Q}(1/q) > 0$.*

(c) *The values $\tilde{Q}(1)$ and $\tilde{Q}(1/q)$ are $O(1)$.*

Proof.

(a) Since $a_i \geq 0$, we have $Q_{d-1} \geq \cdots \geq Q_0 = Q > 0$ on $\mathbb{R}_{\geq 0}$, so

$$\widehat{Q_{d-1}^b}(1) \geq \widehat{Q^b}(1) = m. \tag{10}$$

Let

$$c' := q^{d-1}a_1 + q^{d-2}a_2 + \ldots + qa_{d-1}.$$

Let $R = Q_{d-1} - c'z^d$. Then

$$R(1) = Q_{d-1}(1) - c' = Q(1) - (q^{d-1} - 1)a_1 - \cdots - (1-1)a_{d-1} \ \in \ (0, Q(1)],$$

for large n, by Lemma 7.4, and

$$\begin{aligned} R(1/q) &= Q_{d-1}(1/q) - c'/q^d \\ &= (Q(1/q) + a_1 q^{-1} + \cdots + a_{d-1}q^{-(d-1)}) - (a_1 q^{-1} + \cdots + a_{d-1}q^{-(d-1)}) \\ &= Q(1/q) > 0, \end{aligned}$$

so

$$\widehat{R^b}(1) \leq \widehat{Q^b}(1) = m. \tag{11}$$

By (10) and (11) and the intermediate value theorem, there exists $c \in [0, c']$ such that $\widehat{(Q_{d-1} - cz^d)^b}(1) = m$. Moreover, $c = O(c') = O((d-1)q^{d-1}(1/n)) = O(1/n)$.

(b) We have $\tilde{Q}(1) \geq R(1) > 0$ and $\tilde{Q}(1/q) \geq R(1/q) > 0$.

(c) For $w \in \{1, 1/q\}$, we have $\tilde{Q}(w) = Q(w) + O(1/n)$, and $Q(w) \in P([-1,1])$, an interval independent of n. □

Lemmas 7.6 through 7.9 show that $\tilde{Q}^b$ is large enough on D and the corrections are small enough that h_n is nonvanishing on D.

Lemma 7.6. *We have $|\tilde{Q}(w)| \geq q^{-d/4} - O(1/n)$ for every $w \in D$.*

Proof. By Lemma 7.2, $|Q(w)| \geq q^{-d/4}$, and $|\tilde{Q}(w)|$ differs from $|Q(w)|$ by at most $|a_1 w + \cdots + a_{d-1}w^{d-1} - cw^d| = O(1/n)$, by Lemmas 7.4 and 7.5. □

Lemma 7.7. *We have $k_i(1) > 0$ and $k_i(1/q) > 0$.*

Proof. These follow from Lemma 7.5(b). □

Lemma 7.8. *The $r_i \in [0, L)$ and s_{i+1} can be chosen as specified in Step 7, and s_{i+1} is $O(1)$. For $i = d, \ldots, \ell - 2$, we have the more precise bound $s_{i+1} \in [0, qL]$.*

Proof. This is similar to the proof of Lemma 7.5. The value $\widehat{h}_{i+1}^{[2n-i]}$ is r_i plus terms that have already been fixed, so there is a unique choice $r_i \in [0, L)$ such that $\widehat{h}_{i+1}^{[2n-i]} \in \mathbb{Z}$ and $\widehat{h}_{i+1}^{[2n-i]} \equiv g^{[2n-i]} \pmod{L}$.

We seek s_{i+1} making the value $\widehat{h}_{i+1}(1) = m + r_i \widehat{k}_i(1) - s_{i+1}\widehat{k}_{i+1}(1)$ equal to m. By Lemma 7.7,

$$m + r_i \widehat{k}_i(1) \geq m. \tag{12}$$

Let $V = k_i/k_{i+1}$ and $v = \max\{V(1), V(1/q)\}$. By Lemma 7.7, $\widehat{k}_i(1) \le v\,\widehat{k}_{i+1}(1)$, so

$$m + r_i\,\widehat{k}_i(1) - vr_i\,\widehat{k}_{i+1}(1) \le m. \tag{13}$$

Now (12), (13), and the intermediate value theorem yield $s_{i+1} \in [0, vr_i] \subseteq [0, vL]$ making $\widehat{h}_{i+1}(1) = m$.

To bound s_{i+1}, we need to bound v. The function V is $1/z$, $\tilde{Q}(z)/z$, or $2/z$; accordingly, v is q, $O(1)$, or $2q$, with the middle case following from Lemma 7.5(b,c). In every case, $v = O(1)$, so $s_{i+1} = O(1)$. If $i \in [d, \ell - 1)$, then $V = 1/z$, so $v = q$, so $s_{i+1} \in [0, qL]$. □

Lemma 7.9. *The polynomial h_n is nonvanishing on D.*

Proof. By construction,

$$h_n = \tilde{Q}^b + \sum_{i=d}^{n-1}(r_i k_i - s_{i+1}k_{i+1}),$$

so it suffices to prove that

$$\sum_{i=d}^{n-1}\left|\frac{r_i k_i}{\tilde{Q}^b}\right| + \sum_{i=d+1}^{n}\left|\frac{s_i k_i}{\tilde{Q}^b}\right| < 1 \tag{14}$$

on D. We claim that

$$\left|\frac{k_i}{\tilde{Q}^b}\right| \le \begin{cases} q^{-i/2} & \text{if } i \in [d, \ell), \\ O(n^{-2}) & \text{if } i \in [\ell, n], \end{cases} \tag{15}$$

on D. The case $i \in [d, \ell)$ follows since $k_i/\tilde{Q}^b = z^i$. In particular, for $i \in [\ell - d/2, \ell)$, we have $|k_i/\tilde{Q}^b| \le q^{-(\ell-d/2)/2} = O(q^{-\ell/2}) = O(n^{-2})$. From then on, changing i to $i + d/2$ multiplies $|k_i/\tilde{Q}^b|$ by $|z^{d/2}/\tilde{Q}| \le q^{-d/4}/(q^{-d/4} - O(1/n)) = 1 + O(1/n)$ by Lemma 7.6 (or, at the last step with $i + d/2 = n$, by $|(z^n/2)/z^i| = |z^{d/2}/2| \le 1$); this happens fewer than n times, and $(1 + O(1/n))^n = O(1)$, so (15) for $i \in [\ell, n]$ follows.

By Lemma 7.8 and (15), the left hand side of (14) is at most

$$\sum_{i=d}^{\ell-1} Lq^{-i/2} + \sum_{i=\ell}^{n-1} L\,O(n^{-2}) + \sum_{i=d+1}^{\ell-1} qL\,q^{-i/2} + \sum_{i=\ell}^{n} O(1)\,O(n^{-2}) \le \frac{2Lq^{-(d-1)/2}}{1 - q^{-1/2}} + O(1/n) < 1$$

if n is large, since $L < q^{23}$ and $d \ge 53$. □

Lemma 7.10. *The polynomial $\widehat{h}_n$ is monic of degree $2n$. Also, $\widehat{h}_n \in \mathbb{Z}[x]$ and $\widehat{h}_n \equiv g \pmod{L}$.*

Proof. In Steps 4 and 7, adjusting h_i to produce h_{i+1} does not change the coefficients of z^{2n}, z^{2n-1}, …, z^{2n-i} in $\widehat{h}_i$, which are integers congruent modulo L to the corresponding coefficients of g; by q-symmetry, the same holds for the coefficients of 1, z, …, z^i. Thus $\widehat{h}_n$ is monic and has integer coefficients congruent to the coefficients of g, except perhaps the coefficient of z^n; actually it holds for this coefficient too since $\widehat{h}_n(1)$ is an integer (namely, m) and $\widehat{h}_n(1) = m \equiv g(1) \pmod{L}$. □

End of proof of Theorem 1.1.

- The polynomial $\widehat{h}_n$ is monic of degree $2n$, with integer coefficients, by Lemma 7.10.
- The polynomial $\widehat{h}_n$ is q-symmetric, by definition of the hat.

- All complex roots of $\widehat{h}_n$ have absolute value $q^{1/2}$, by Lemmas 7.9 and 3.1.
- The characteristic p does not divide $\widehat{h}_n^{[n]}$, because by Lemma 7.10, $\widehat{h}_n^{[n]}$ is congruent modulo L to $g^{[n]}$, which is nonzero modulo p, and $p \mid L$, by construction of g.

By Theorem 2.1, there exists an ordinary n-dimensional abelian variety A over $\mathbb{F}_q$ with $f_A = \widehat{h}_n$. Then $\#A(\mathbb{F}_q) = f_A(1) = \widehat{h}_n(1) = m$. By Proposition 5.8, A is geometrically simple, and principally polarized after replacing A by an isogenous abelian variety. □

8. Large q limit

In this section, we prove Theorem 1.7, which for fixed n and large q determines the largest subinterval of the Hasse–Weil interval in which all integers are realizable as $\#A(\mathbb{F}_q)$ for an n-dimensional abelian variety A over $\mathbb{F}_q$. First let us explain the idea. For any n-dimensional abelian variety A over $\mathbb{F}_q$, we have $f_A(x) = x^n\, G(x + q/x)$ for some polynomial

$$G(x) = x^n + c_1 x^{n-1} + c_2 x^{n-2} + \cdots + c_n \in \mathbb{Z}[x] \tag{16}$$

all of whose roots lie in $[-2q^{1/2}, 2q^{1/2}]$. Then $c_i = O(q^{i/2})$, and

$$\#A(\mathbb{F}_q) = f_A(1) = G(q+1) = (q+1)^n + c_1(q+1)^{n-1} + c_2(q+1)^{n-2} + \cdots + c_n.$$

For each integer c_1 in the possible range $[-2nq^{1/2}, 2nq^{1/2}]$, let I_{c_1} be the smallest interval containing the possible values of $c_2(q+1)^{n-2} + \cdots + c_n$; then we prove that the ranges for $c_2, \ldots, c_n$ are large enough that all integers in I_{c_1} are realized, possibly ignoring a negligible fraction of the interval at the ends. The interval I_{c_1} has width $O(q^{n-1})$ and does not change much when c_1 is incremented by 1 — its endpoints move by $o(q^{n-1})$. The big-O constant matters: for c_1 close to the extremes of its range (with $|c_1|$ greater than about $\left(2n - \sqrt{\frac{2n}{n-1}}\right) q^{1/2}$), it turns out that I_{c_1} has length significantly less than q^{n-1}, so that there is a gap between the intervals $(q+1)^n + c_1(q+1)^{n-1} + I_{c_1}$ and $(q+1)^n + (c_1+1)(q+1)^{n-1} + I_{c_1+1}$, a gap in which $\#A(\mathbb{F}_q)$ cannot lie; see Lemma 8.2. On the other hand, for the c_1 towards the middle of the range, I_{c_1} has width significantly greater than q^{n-1}, so the intervals $(q+1)^n + c_1(q+1)^{n-1} + I_{c_1}$ overlap to cover a large interval in the middle of the Hasse–Weil interval. Figure 1 shows these overlapping intervals when $n = 2$ and $q \in \{11, 9\}$; for the non-prime 9, there is an additional phenomenon explained in Remark 8.5.

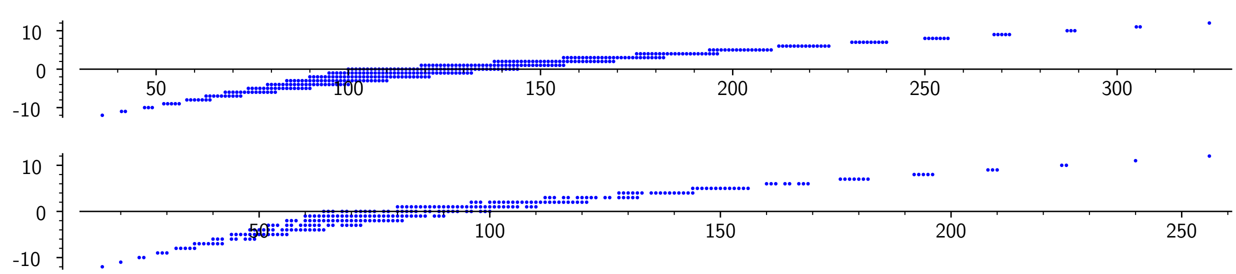


Figure 1. For $q = 11$ and $q = 9$, respectively, the graph shows all the points $(\#A(\mathbb{F}_q), c_1)$, where A ranges over abelian surfaces over $\mathbb{F}_q$, and $c_1 = G^{[n-1]} = f_A^{[2n-1]}$ with $n = 2$; see (16).

As the previous paragraph indicates, the coefficients of x^{n-1} and x^{n-2} are what matter most. After using the normalization $g(x) := q^{-n/2}G(q^{1/2}x)$, we are led to study

$$\mathcal{G} := \{\, g \in \mathbb{R}[x] \colon\ g \text{ is monic of degree } n \text{ with all roots in } [-2,2] \,\}$$
$$\mathcal{S} := \{\, (g^{[n-1]}, g^{[n-2]}) \in \mathbb{R}^2 : g \in \mathcal{G} \,\}.$$

Let $\lambda_1 = 2n - \sqrt{\frac{2n}{n-1}}$ and $\lambda_2 = 2n - \sqrt{\frac{4n}{n-1}}$.

Lemma 8.1. *If $n \ge 2$, then there exist continuous functions $B_{\min}, B_{\max} \colon [-2n, 2n] \to \mathbb{R}$ such that*

(a) *We have $\mathcal{S} = \{\, (a,b) \in [-2n,2n] \times \mathbb{R} : B_{\min}(a) \le b \le B_{\max}(a) \,\}$.*
(b) *The difference $B_{\mathrm{diff}}(a) := B_{\max}(a) - B_{\min}(a)$ is*
  - *nonnegative on $[-2n, 2n]$, positive on $(-2n, 2n)$,*
  - *less than 1 if $\lambda_1 < |a| \le 2n$, greater than 1 if $|a| < \lambda_1$,*
  - *less than 2 if $\lambda_2 < |a| \le 2n$, and greater than 2 if $|a| < \lambda_2$.*

(c) *There exists a compact subset $\mathcal{G}_0 \subset \mathcal{G}$ surjecting onto $\mathcal{S}$ such that any $g \in \mathcal{G}_0$ mapping into the interior of $\mathcal{S}$ has distinct roots in $(-2, 2)$.*

Proof. If $g = \prod_{i=1}^{n}(x - r_i)$, then $(g^{[n-1]}, g^{[n-2]}) = (-\sum r_i, \sum_{i<j} r_i r_j)$. Given $a \in [-2n, 2n]$, let

$$\mathcal{C}_a = \{(r_1, \ldots, r_n) \in [-2,2]^n : \textstyle\sum r_i = -a\}.$$

Since $\mathcal{C}_a$ is compact and connected, (a) holds with $B_{\min}$ and $B_{\max}$ being the minimum and maximum of $\sum_{i<j} r_i r_j$ on $\mathcal{C}_a$. If any two of the r_i are different, then $\sum_{i<j} r_i r_j$ can be increased by bringing them closer together; thus the maximum occurs when the r_i are all equal, so $B_{\max}(a) = \binom{n}{2}(a/n)^2$. If there are two r_i in $(-2, 2)$, then $\sum_{i<j} r_i r_j$ can be decreased by moving them slightly apart; thus the minimum occurs when all but one r_i are at ± 2. Given a, there is at most one such $(r_1, \ldots, r_n)$ with $\sum r_i = -a$ up to permuting coordinates — as a increases, the roots move linearly from 2 to -2 one at a time, so $B_{\min}$ is the piecewise-linear continuous function such that for each $k \in \{0, \ldots, n-1\}$,

$$B_{\min}(a) = (4k - 2n + 2)a - 8k^2 + 8k(n-1) - 2(n-1)n \quad \text{for } a \in [4k - 2n, 4k - 2n + 4].$$

The minimum value of B_{diff} on $[4k - 2n, 4k - 2n + 4]$ is

$$B_{\mathrm{diff}}(4k - 2n + 4k/(n-1)) = 8k(n-1-k)/(n-1),$$

which for $k \in \{1, \ldots, n-2\}$ is at least $8(n-2)/(n-1) \ge 4$. On the other hand, for $t \in [0, 4]$, we have $B_{\mathrm{diff}}(2n - t) = B_{\mathrm{diff}}(-2n + t) = \frac{n-1}{2n} t^2$. The claims in (b) follow.

Given a, let $\prod_{i=1}^{n}(x - r_i)$ and $\prod_{i=1}^{n}(x - r_i'')$ be the polynomials realizing $B_{\min}(a)$ and $B_{\max}(a)$, each with roots listed in increasing order. (So all but one r_i are ± 2, and $r_i'' = -a/n$ for all i.) Let $\epsilon \ge 0$ be the distance from $-a/n$ to the boundary of $[-2, 2]$, and let $r_1', \ldots, r_n'$ be an arithmetic progression with $r_1' = -a/n - \epsilon/2$ and $r_n' = -a/n + \epsilon/2$. For each $s \in [0, 1]$, consider the monic degree n polynomial whose roots are $(1-s)r_i + s r_i'$ for $i = 1, \ldots, n$ and the analogous polynomial with roots $(1-s)r_i' + s r_i''$. These depend continuously on $(a, s) \in [-2n, 2n] \times [0, 1]$, so the set of all such polynomials is a compact subset $\mathcal{G}_0$ of $\mathcal{G}$. For fixed a, the coefficients of x^{n-2} in these polynomials vary continuously from $B_{\min}(a)$ to $B_{\max}(a)$, so $\mathcal{G}_0 \to \mathcal{S}$ is surjective. Finally, by construction, all polynomials in $\mathcal{G}_0$ except for the ones realizing $B_{\min}(a)$ and $B_{\max}(a)$ have distinct roots in $(-2, 2)$. □

Lemma 8.2. *Suppose $n \geq 2$. For $\lambda \in \mathbb{R}$ satisfying $\lambda_1 < |\lambda| < 2n$, there exists $\epsilon > 0$ such that if q is sufficiently large and $r = \lfloor \lambda q^{1/2} \rfloor$, then the interval*

$$\left[\ (q{+}1)^n{+}r(q{+}1)^{n-1}{+}(B_{\max}(\lambda){+}\epsilon)q^{n-1}\ ,\ (q{+}1)^n{+}(r{+}1)(q{+}1)^{n-1}{+}(B_{\min}(\lambda){-}\epsilon)q^{n-1}\ \right] \tag{17}$$

has width > 1 and does not contain $\#A(\mathbb{F}_q)$ for any n-dimensional abelian variety A over $\mathbb{F}_q$.

Proof. By Lemma 8.1(b), $B_{\text{diff}}(\lambda) < 1$. Choose $\epsilon > 0$ such that $B_{\text{diff}}(\lambda) < 1 - 2\epsilon$. Then the width of the interval (17) is $(q+1)^{n-1} - (B_{\text{diff}}(\lambda) + 2\epsilon)q^{n-1} > 1$.

Let A be an n-dimensional abelian variety over $\mathbb{F}_q$. Then $f_A(x) = x^n\, G(x + q/x)$ for some $G(x) = x^n + c_1 x^{n-1} + \cdots + c_n \in \mathbb{Z}[x]$ with all roots in $[-2q^{1/2}, 2q^{1/2}]$. We have $c_i = O(q^{i/2})$ and $(a, b) := (q^{-1/2}c_1, q^{-1}c_2) \in \mathcal{S}$. Now

$$\#A(\mathbb{F}_q) = f_A(1) = G(q+1) = (q+1)^n + c_1(q+1)^{n-1} + bq^{n-1} + O(q^{n-3/2}). \tag{18}$$

Since $b = O(1)$, if $\#A(\mathbb{F}_q)$ lies in the interval (17), then $c_1 = r + O(1)$, so $a = q^{-1/2}c_1 = \lambda + O(q^{-1/2})$. Then

$$b \in [B_{\min}(a), B_{\max}(a)] \subset [B_{\min}(\lambda) - \epsilon/2, B_{\max}(\lambda) + \epsilon/2]$$

by continuity, if q is large enough. If $c_1 \leq r$, then the right side of (18) is too small to lie in (17). If $c_1 \geq r + 1$, then it is too large. □

Lemma 8.3. *Suppose that $n \geq 3$ and $\lambda \in \mathbb{R}$ satisfies $0 < \lambda < \lambda_1$. Then for sufficiently large q, every integer in*

$$\left[\ q^n - \lambda q^{n-1/2}\ ,\ q^n + \lambda q^{n-1/2}\ \right] \tag{19}$$

is $\#A(\mathbb{F}_q)$ for some n-dimensional abelian variety A over $\mathbb{F}_q$.

Proof. By Lemma 8.1(b), $B_{\text{diff}} > 1$ on $[-\lambda, \lambda]$. Choose $\epsilon > 0$ so that $B_{\text{diff}} > 1 + 2\epsilon$ on $[-\lambda, \lambda]$. Let

$$\mathcal{S}_\epsilon = \{\, (a, b) \in [-2n, 2n] \times \mathbb{R} : B_{\min}(a) + \epsilon \leq b \leq B_{\max}(a) - \epsilon \,\}.$$

Then $\mathcal{S}_\epsilon$ is a compact subset of the interior of $\mathcal{S}$. Let $\mathcal{G}_\epsilon$ be the inverse image of $\mathcal{S}_\epsilon$ under $\mathcal{G}_0 \twoheadrightarrow \mathcal{S}$. By Lemma 8.1(c), $\mathcal{G}_\epsilon$ is compact and consists of polynomials with distinct real roots in $(-2, 2)$, so we can choose $\delta > 0$ such that any polynomial whose coefficients are within δ of some $g \in \mathcal{G}_\epsilon$ again has distinct real roots in $(-2, 2)$.

Suppose that m is an integer in $[\ q^n - \lambda q^{n-1/2}\ ,\ q^n + \lambda q^{n-1/2}\ ]$. The rest of the proof relies on the following construction.

Construction 8.4.

1. Let $a \in [-\lambda, \lambda]$ be such that $m = q^n + aq^{n-1/2}$.
2. Write $m = (q+1)^n + (c_1 + b)(q+1)^{n-1}$ with $c_1 \in \mathbb{Z}$ and $b \in [B_{\min}(a) + \epsilon, B_{\max}(a) - \epsilon]$ (possible since $[B_{\min}(a) + \epsilon, B_{\max}(a) - \epsilon]$ has length > 1). Then $(a, b) \in \mathcal{S}_\epsilon$.
3. Choose $g \in \mathcal{G}_\epsilon$ mapping to (a, b).
4. Let $G(x) = q^{n/2}\, g(q^{-1/2}x) = x^n + q^{1/2}ax^{n-1} + qbx^{n-2} + \cdots \in \mathbb{R}[x]$.
5. Let G_1 be the same as G except with the coefficient of x^{n-1} changed to c_1.
6. For $i = 2, \ldots, n$, let G_i be the same as G_{i-1} except with the coefficient of x^{n-i} changed to the integer c_i that makes $G_i(q+1) - m \in [0, (q+1)^{n-i})$.
7. Let $G_{\text{final}} = G_n + s(x - (q+1))$, where $s \in \{0, 1\}$ is chosen so that $p \nmid G_{\text{final}}^{[0]}$.

Continuation of proof of Lemma 8.3. We now bound the coefficients of $G_{\mathrm{final}} - G$ in order to prove that for q large enough, the roots of G_{final} are still distinct and all in $[-2q^{1/2}, 2q^{1/2}]$. Since $\mathcal{S}_\epsilon$ is compact, b is $O(1)$. By Steps 1 and 2,

$$\begin{aligned} q^n + aq^{n-1/2} = m = (q+1)^n + (c_1+b)(q+1)^{n-1} &= q^n + c_1 q^{n-1} + O(q^{n-1}), \\ c_1 &= q^{1/2}a + O(1). \end{aligned} \tag{20}$$

Now

$$\begin{aligned} G_1(q+1) &= (q+1)^n + c_1(q+1)^{n-1} + qb(q+1)^{n-2} + O(q^{3/2})(q+1)^{n-3} + \cdots + O(q^{n/2})1 \\ &= (q+1)^n + (c_1+b)(q+1)^{n-1} + O(q^{n-3/2}) \\ &= m + O(q^{n-3/2}), \end{aligned}$$

so

$$c_2 - G^{[n-2]} = O(q^{n-3/2})/(q+1)^{n-2} = O(q^{1/2}). \tag{21}$$

Similarly, for $i = 3, \ldots, n$, we have

$$c_i - G^{[n-i]} = O((q+1)^{n-(i-1)})/(q+1)^{n-i} = O(q). \tag{22}$$

Equations (20), (21), and (22) imply that

$$G_n^{[n-i]} - G^{[n-i]} = O(q^{(i-1)/2})$$

for all $i \geq 1$. Since $n \geq 3$, the same holds with G_n replaced by G_{final}. Thus the coefficients of $g_{\mathrm{final}}(x) = q^{-n/2}\, G_{\mathrm{final}}(q^{1/2}x)$ are within $O(q^{-1/2}) < \delta$ of the corresponding coefficients of g if q is sufficiently large, so g_{final} has all its roots in $[-2, 2]$. Thus G_{final} has all its roots in $[-2q^{1/2}, 2q^{1/2}]$. By construction, $G_{\mathrm{final}} \in \mathbb{Z}[x]$. Also $G_{\mathrm{final}}(q+1) - m = G_n(q+1) - m \in [0,1)$, so $G_{\mathrm{final}}(q+1) = m$.

Let $f(x) = x^n\, G_{\mathrm{final}}(x + q/x) \in \mathbb{Z}[x]$. We have $f^{[n]} \equiv G_{\mathrm{final}}^{[0]} \not\equiv 0 \pmod{p}$. By Theorem 2.1, $f = f_A$ for some n-dimensional ordinary abelian variety over $\mathbb{F}_q$. Finally, $\#A(\mathbb{F}_q) = f(1) = G_{\mathrm{final}}(q+1) = m$. ☐

Proof of Theorem 1.7. Lemma 8.3 shows that all integers in $[q^n - \lambda q^{n-1/2}, q^n + \lambda q^{n-1/2}]$ are realizable for λ that can approach λ_1 from below as $q \to \infty$. Lemma 8.2 shows, on the other hand, that for any μ with $|\mu| > \lambda_1$, there are unrealizable integers within $O(q^{n-1})$ of $(q+1)^n + \mu q^{n-1/2}$ if q is sufficiently large. These imply Theorem 1.7. ☐

Remark 8.5. Suppose $n = 2$. Theorem 1.7 holds without change if q tends to ∞ through primes only: the proof of Lemma 8.3 works if we omit Step 7, because of the last sentence of Remark 2.3.

On the other hand, if q tends to ∞ through non-prime prime powers, then Theorem 1.7 holds with λ_1 replaced by the smaller value $\lambda_2 = 4 - 2\sqrt{2}$, as we now explain. In Lemma 8.3, if $0 < \lambda < \lambda_2$, then $B_{\mathrm{diff}} > 2$ on $[-\lambda, \lambda]$, so there are at least *two* consecutive integer possibilities for c_1, and at least one of them will lead to a polynomial f for which (d) in Theorem 2.1 holds. Meanwhile, in Lemma 8.2, if $\lambda_2 < |\mu| < 2n$, so that $B_{\mathrm{diff}}(\mu) < 2$, then there exists $\epsilon > 0$ such that if q is sufficiently large, and r is the multiple of p nearest $\mu q^{1/2}$, then any integer of the form $m = (q+1)^2 + r(q+1) + c_2$ in

$$\left[\ (q+1)^2 + (r-1)(q+1) + (B_{\max}(\mu) + \epsilon)q\ ,\ (q+1)^2 + (r+1)(q+1) + (B_{\min}(\mu) - \epsilon)q\ \right]$$

with $p \mid c_2$ and $p^2 \nmid c_2$ is not $\#A(\mathbb{F}_q)$ for any abelian surface A over $\mathbb{F}_q$, because the only monic quadratic polynomial $G(x) \in \mathbb{Z}[x]$ with roots in $[-2q^{1/2}, 2q^{1/2}]$ such that $G(q+1) = m$ is $x^2 + rx + c_2$, which is Eisenstein at p, which implies that the polynomial $f(x) := x^2\, G(x + q/x)$ fails condition (d′) in Remark 2.3.

9. Effective bounds

Given q and n, we have given three ways to construct polynomials that realize a large interval of integers as $\#A(\mathbb{F}_q)$ for A of dimension n over $\mathbb{F}_q$:

- Section 4 gave a quick construction that realized intervals wide enough to cover all sufficiently large integers as n varies, but not wide enough to be asymptotically close to optimal.
- Section 7 gave a more subtle construction that gave a much wider interval, but it is too complicated to analyze explicitly to make all the big-O constants explicit.
- Section 8 gave a method that again is asymptotically good, but only when q is large compared to n.

In this section, we present a *fourth* construction that, while not asymptotically as good as the construction of Section 7, realizes a wide interval for any q and sufficiently large n, and is still simple enough to analyze fully.

Given q, $n \geq 2$, and an integer m in $[q^{n-1/2}, q^{n+1/2})$, the plan is to find a power series $j(z) \in z\mathbb{R}[[z]]$ such that $j(1/q) = \log(m/q^n)$ and $\exp(j(z)) \in \mathbb{Z}[[z]]$; then we truncate $\exp(j(z))$ to a degree n polynomial and adjust the coefficients of x^{n-1} and x^n to produce a polynomial $h(z)$ such that $\widehat{h}(1) = m$ and $p \nmid \widehat{h}^{[n]}$. This should work well, since $\exp(j(z))$ is automatically nonvanishing on D, and if its coefficients are not too large, then the nonvanishing should persist after truncating and adjusting.

Construction 9.1.

1. For $i = 1, 2, \ldots, n-1$, let c_i be the real number such that
$$\log(m/q^n) - c_1 q^{-1} - \cdots - c_i q^{-i} \in [-q^{-i}/2, q^{-i}/2)$$
and such that the coefficient of z^i in $\exp(c_1 z + \cdots + c_i z^i)$ is an integer; for the existence and uniqueness of c_i, see the proof of Lemma 9.2.
2. Let $c_n \in \mathbb{R}$ be such that $\log(m/q^n) - c_1 q^{-1} - \cdots - c_n q^{-n} = 0$.
3. Let $h_0(z) \in \mathbb{R}[z]$ be the degree n Taylor polynomial of $\exp(c_1 z + \cdots + c_n z^n)$.
4. Let $h_1(z) = h_0(z) + kz^n/2$, where $k \in \mathbb{R}$ is chosen to make $\widehat{h}_1(1) = m$.
5. Let h be h_1 or $h_1 + z^{n-1} - ((q+1)/2)z^n$, whichever makes $p \nmid \widehat{h}^{[n]}$.
6. Let A be an abelian variety with $f_A = \widehat{h}$, if one exists. (If h is nonvanishing on D, then such an A is guaranteed to exist and $\widehat{h}$ is squarefree by Remark 3.3.)

Let $s = \lfloor \frac{1}{2} q \log q + \frac{1}{2} \rfloor$.

Lemma 9.2. *We have $|c_1| \leq s$ and $|c_i| \leq (q+1)/2$ for $i = 2, \ldots, n$.*

Proof. Since $m \in [q^{n-1/2}, q^{n+1/2})$, we have $\log(m/q^n) \in [-\frac{1}{2}\log q, \frac{1}{2}\log q)$, and Step 1 says that c_1 is the integer in the interval $q \log(m/q^n) + [-\frac{1}{2}, \frac{1}{2})$, so $|c_1| \leq s$.

For $i \leq n-1$, let $\epsilon_i = \log(m/q^n) - c_1 q^{-1} - \cdots - c_i q^{-i}$, so $\epsilon_i = \epsilon_{i-1} - c_i q^{-i}$; then $\epsilon_{i-1} \in [-q^{-(i-1)}/2, q^{-(i-1)}/2)$, so the condition $\epsilon_i \in [-q^{-i}/2, q^{-i}/2)$ in Step 1 constrains c_i

to a half-open interval of length 1 contained in $[-(q+1)/2,(q+1)/2]$, while the integer coefficient condition in Step 1 constrains c_i to a coset of $\mathbb{Z}$ in $\mathbb{R}$; thus a unique c_i exists, and $|c_i| \le (q+1)/2$. Finally, $c_n = q^n \epsilon_{n-1} \in q^n[-q^{-(n-1)}/2, q^{-(n-1)}/2) = [-q/2, q/2)$. □

Let $j(z) = c_1 z + \cdots + c_n z^n$. To bound the difference between $\exp(j(z))$ and its degree n Taylor polynomial, we consider the worst case: let

$$J(z) := \exp\left(sz + \frac{q+1}{2}\frac{z^2}{1-z}\right) = J_{\le n}(z) + J_{>n}(z) \quad \in \mathbb{R}_{\ge 0}[[z]],$$

where $J_{\le n}$ is the degree n Taylor polynomial, and $J_{>n}$ is the remainder power series consisting of terms of degree $> n$. By Lemma 9.2, $|(\exp j(z))^{[i]}| \le J^{[i]}$.

Proposition 9.3. *Let q be a prime power. For $n \ge 2$ and $m \in [q^{n-1/2}, q^{n+1/2})$, if*

$$J_{>n}(q^{-1/2}) + \frac{q^{n/2}}{2} J_{>n}(q^{-1}) + \frac{q^{-n/2}}{2} J_{\le n}(1) + \frac{(q^{1/2}+1)^2}{2} q^{-n/2} < \frac{1}{J(q^{-1/2})}, \tag{23}$$

then Construction 9.1 produces an ordinary n-dimensional A over $\mathbb{F}_q$ with $\#A(\mathbb{F}_q) = m$.

Proof. By Step 2, $\exp j(q^{-1}) = m/q^n$, so

$$\begin{aligned} |m - q^n h_0(q^{-1})| \;&=\; q^n |\exp j(q^{-1}) - h_0(q^{-1})| \;\le\; q^n\, J_{>n}(q^{-1}) \\ |h_0(1)| \;&\le\; J_{\le n}(1) \\ |k| \;=\; |\widehat{h}(1) - \widehat{h}_0(1)| \;&\le\; |m - q^n h_0(q^{-1}) - h_0(1)| \;\le\; q^n\, J_{>n}(q^{-1}) + J_{\le n}(1). \end{aligned}$$

On D,

$$\begin{aligned} |\exp j(z)| = \exp(\operatorname{Re} j(z)) &\ge \exp\left(-sq^{-1/2} - \frac{q+1}{2} q^{-2/2} - \cdots - \frac{q+1}{2} q^{-n/2}\right) \ge \frac{1}{J(q^{-1/2})} \\ |h_0(z)| &\ge |\exp j(z)| - J_{>n}(q^{-1/2}) \\ |h_1(z)| &\ge |h_0(z)| - \frac{k}{2} q^{-n/2} \\ |h(z)| &\ge |h_1(z)| - q^{-(n-1)/2} - \frac{q+1}{2} q^{-n/2} = |h_1(z)| - \frac{(q^{1/2}+1)^2}{2} q^{-n/2}. \end{aligned}$$

Combining the previous five inequalities yields

$$|h(z)| \ge \frac{1}{J(q^{-1/2})} - J_{>n}(q^{-1/2}) - \frac{q^{n/2}}{2} J_{>n}(q^{-1}) - \frac{q^{-n/2}}{2} J_{\le n}(1) - q^{-(n-1)/2} - \frac{q+1}{2} q^{-n/2},$$

so (23) implies that h is nonvanishing on D. Theorem 2.1 produces A. □

The following weaker statement has the advantage that if any hypothesis holds for one n, it clearly holds for all larger n:

Corollary 9.4. *Let q be a prime power. For $n \ge 2$ and $m \in [q^{n-1/2}, q^{n+1/2})$, if any of*

$$(1 + q^{-1/2}/2) J_{>n}(q^{-1/2}) + \tfrac{1}{2}\left(\tfrac{4}{3} q^{-1/2}\right)^n J(\tfrac{3}{4}) + \frac{(q^{1/2}+1)^2}{2} q^{-n/2} < \frac{1}{J(q^{-1/2})}, \tag{24}$$

$$q \ge 7 \quad \text{and} \quad 2^{n-1} > J(q^{-1/2}) J(2q^{-1/2}), \quad \text{or} \tag{25}$$

$$q \ge 16 \quad \text{and} \quad n > 3q^{1/2} \log q - 1/2, \tag{26}$$

hold, then Construction 9.1 produces an ordinary n-dimensional A over $\mathbb{F}_q$ with $\#A(\mathbb{F}_q) = m$.

Proof. In (23), $J_{>n}(q^{-1}) \le q^{-(n+1)/2} J_{>n}(q^{-1/2})$ (this holds termwise for any power series J with nonnegative coefficients). Similarly $J_{\le n}(1) \le (\frac{4}{3})^n J_{\le n}(3/4) \le (\frac{4}{3})^n J(3/4)$. Hence the left side of (23) is at most the left side of (24). Thus, if (24) holds, Proposition 9.3 applies.

Now suppose that $q \ge 7$ and $2^{n-1} > J(q^{-1/2})J(2q^{-1/2})$. First,

$$2^{n-1} > J(q^{-1/2})J(2q^{-1/2}) \ge \exp(3sq^{-1/2}) \ge \exp(3q^{-1/2}(q\log q - 1)/2) \ge 2^9,$$

so $n \ge 10$. We use

$$\begin{aligned} J_{>n}(q^{-1/2}) &\le 2^{-(n+1)} J(2q^{-1/2}) \\ J_{>n}(q^{-1}) &\le (2q^{1/2})^{-(n+1)} J(2q^{-1/2}) \\ J_{\le n}(1) &\le 1 + (q^{1/2}/2)^n (J(2q^{-1/2}) - 1) \\ (q^{1/2}+1)^2 &\le (q^{1/2}/2)^n - 1; \end{aligned} \tag{27}$$

the first three are proved termwise, and the last follows from the inequality $(2u+1)^2 \le u^{10} - 1$ for $u := q^{1/2}/2 \ge 7^{1/2}/2$. By (27), the left side of (23) is at most

$$\begin{aligned} &2^{-(n+1)} J(2q^{-1/2}) + \frac{q^{n/2}}{2}(2q^{1/2})^{-(n+1)} J(2q^{-1/2}) \\ &\qquad + \frac{q^{-n/2}}{2}\Big((q^{1/2}/2)^n J(2q^{-1/2}) + 1 - (q^{1/2}/2)^n\Big) + \frac{q^{-n/2}}{2}\Big((q^{1/2}/2)^n - 1\Big) \\ &= 2^{-(n+1)}(2 + q^{-1/2}/2)\, J(2q^{-1/2}) \\ &\le 2^{1-n} J(2q^{-1/2}) \\ &< \frac{1}{J(q^{-1/2})}, \end{aligned}$$

by hypothesis, so Proposition 9.3 applies.

Finally, suppose that $q \ge 16$ and $n > 3q^{1/2}\log q - 1/2$. Then

$$\begin{aligned} s &\le (q\log q + 1)/2 \\ \log\big(J(q^{-1/2})J(2q^{-1/2})\big) &\le 3\Big(\frac{q\log q + 1}{2}\Big) q^{-1/2} + \frac{q+1}{2}\Big(\frac{q^{-1}}{1-q^{-1/2}} + \frac{4q^{-1}}{1-2q^{-1/2}}\Big) \\ &\le (3q^{1/2}\log q - 3/2)\log 2 \\ &< (n-1)\log 2, \end{aligned} \tag{28}$$

so (25) holds; to prove (28), check numerically for $16 \le q \le 100$ and for $q > 100$ use

$$\begin{aligned} &\frac{3}{2}q^{-1/2} + \frac{q+1}{2}\Big(\frac{q^{-1}}{1-q^{-1/2}} + \frac{4q^{-1}}{1-2q^{-1/2}}\Big) + \frac{3}{2}\log 2 \\ &\le \frac{3}{2}(0.1) + q\Big(\frac{q^{-1}}{0.9} + \frac{4q^{-1}}{0.8}\Big) + \frac{3}{2}\log 2 < 8 < (3\log 2 - 3/2)q^{1/2}\log q. \end{aligned}$$ □

Corollary 9.4 proves Theorem 1.13(b) for $q \ge 16$. Also, for each $q < 16$ it provides an n such that all integers $\ge q^{n-1/2}$ are realizable, but too many integers remain to be checked one at a time. Therefore we describe a construction allowing us to realize larger intervals of integers all at once. The plan is to start with h such that $\widehat{h} = f_A$ for some A with $\#A(\mathbb{F}_q) = m$, and then to replace h by $h + \sum_{i=r}^n c_i z^i$ for some r and small c_i (and then adjust to make $p \nmid \widehat{h}^{[n]}$ again); as the c_i vary, we realize all integers in an interval.

Construction 9.5. Suppose that we are given q, n, m, and a polynomial $h \in 1 + x\mathbb{Z}[x]$ of degree $< 2n$ with $\widehat{h}(1) = m$ (given by Construction 9.1 or otherwise).

1. Compute the complex zeros of h and check that none of them are in D. (More precisely: Compute small balls containing the zeros, and check that none of them intersect D.)
2. Compute the complex zeros α of the derivative of $h(z)h(1/(qz))$, evaluate $|h|$ at each α on the boundary ∂D, and let μ be the minimum of these values; see the proof of Lemma 9.6. (More precisely: Compute small balls around these zeros, and let μ be a lower bound for $|h|$ on all these balls that intersect ∂D; if $h = 1$, then let $\mu = 1$.)
3. Let $\mu_{\mathrm{ord}} = \mu - q^{-(n-1)/2} - ((q+1)/2)q^{-n/2}$; abort if $\mu_{\mathrm{ord}} \le 0$.
4. Let r be the smallest positive integer $\le n+1$ such that $\sum_{i=r}^{n} \lfloor q/2 \rfloor q^{-i/2} < \mu_{\mathrm{ord}}$.
5. Let $N = \lfloor q/2 \rfloor \sum_{j=r}^{n} (q^{n-j} + 1) = \lfloor q/2 \rfloor \left(\frac{q^{n-r+1}-1}{q-1} + (n-r+1) \right)$.
6. Return the interval $\left[\widehat{h}(1) - N, \widehat{h}(1) + N \right]$.

Lemma 9.6. *In Construction 9.5, if Steps 1 and 3 succeed, then every integer in the interval of Step 6 is $\#A(\mathbb{F}_q)$ for some ordinary abelian variety of dimension n over $\mathbb{F}_q$.*

Proof. Suppose that h has no zeros in D. Then the minimum of $|h|$ occurs on ∂D, where $|h|^2 = h(z)\,h(1/(qz))$, and this minimum occurs at a point where the derivative of $h(z)\,h(1/(qz))$ is 0. Thus $|h| \ge \mu$ on D.

Suppose that $H = h + \sum_{i=r}^{n} c_i z^i$ where $|c_i| \le q/2$ for all i, and $c_i \in \mathbb{Z}$ for all i except n, and $c_n \in \frac{1}{2}\mathbb{Z}$. The choice of r guarantees that $|H| > 0$ on D, even if we add $z^{n-1} - ((q+1)/2)z^n$ to H if necessary to make $p \nmid \widehat{H}^{[n]}$, so $\widehat{H}(1)$ is realizable. To realize an integer $\widehat{h}(1) + M$ with $|M| \le N$, write M as $\sum_{j=r}^{n} c_j(q^{n-j}+1)$ with $|c_j| \le \lfloor q/2 \rfloor$, $c_j \in \mathbb{Z}$ for all $j \ne n$, and $c_n \in \frac{1}{2}\mathbb{Z}$, by greedily choosing c_r, c_{r+1}, …, one at a time. ☐

Proof of Theorem 1.13 and Remarks 1.16, 1.17, and 1.18. In this proof, given q, a positive integer is called realizable if it equals $\#A(\mathbb{F}_q)$ for some ordinary abelian variety A over $\mathbb{F}_q$ with f_A squarefree. The case $q = 2$ is done by [HK21]. Criterion (26) of Corollary 9.4 proves Theorem 1.13(b) for $q \ge 16$. For each $q < 16$, we numerically find $n \ge 2$ such that (24) holds; then we check smaller values of n to find the smallest n_0 such that (23) holds for all $n \ge n_0$. (It turns out that $n_0 \le 25$ for each $q < 16$.) For $q \in \{11, 13\}$, it turns out that $q^{3\sqrt{q}\log q} > q^{n_0 - 1/2}$, which proves Theorem 1.13(b) for these q.

For $3 \le q \le 9$, we use variants of Construction 9.1 and 9.5 to realize all integers in an interval $[M_q, q^{n_0-1/2}]$. For $q \in \{8, 9\}$, we have $M_q \le q^{3\sqrt{q}\log q}$, which proves Theorem 1.13(b) for these q. For $q \in \{3, 4, 5, 7\}$, we use the algorithm of [Ked08] (implemented at `https://github.com/kedlaya/root-unitary`) to exhaust over the polynomials f_A for abelian varieties A of dimension ≤ 4 to realize all integers $< M_q$ with the exception of those listed in Remarks 1.16, 1.17, and 1.18. Neither are these exceptions realized by abelian varieties of dimension ≥ 5, because they are all less than $(\sqrt{q} - 1)^{10}$. The calculations in this paragraph took 7.19 CPU hours on an Intel Core i7-9750H CPU @ 2.60GHz. See `https://github.com/edgarcosta/abvar-fq-orders` for the code and further details. ☐

Some calculations were checked against the database of isogeny classes of abelian varieties over finite fields in the L-functions and Modular Forms Database [DKRV20; LMFDB].

Appendix A. Optimality of a potential function

A.1. **Polynomials.** The goal of this appendix is to prove the following.

Proposition A.1. *Choose c in the interval $(0,1)$. For $d \geq 1$, let $\mathscr{F}(d,c)$ be the set of complex polynomials f of degree d satisfying $f(0)=1$ and $|f(w)|^{1/d} \geq c$ for all $w \in \mathbb{C}_{\leq 1}$. On $(-\infty, 1]$ define the decreasing continuous function*

$$M(r) := \frac{1-r+\sqrt{(1-r)^2+4rc^2}}{2}.$$

(a) *For any $f \in \mathscr{F}(d,c)$, we have*

$$|f(r)|^{1/d} \geq M(r) \qquad \text{for all } r \in [0,1], \tag{29}$$

$$|f(r)|^{1/d} \leq M(-r) \qquad \text{for all } r \in [0,\infty). \tag{30}$$

(b) *There exist polynomials $f_1, f_2, \ldots$ with $f_d \in \mathscr{F}(d,c)$ such that for every $r \in (-\infty, 1]$,*

$$\lim_{d\to\infty} |f_d(r)|^{1/d} = M(r). \tag{31}$$

(*Thus* (29) *is asymptotically sharp, and* (30) *is too since $f_d(-z) \in \mathscr{F}(d,c)$.*)

Remark A.2. For $r > 1$, the lower bound in (29) is simply 0 since the function $f(z) := 1-(z/r)^d$ is in $\mathscr{F}(d,c)$.

Remark A.3. If f is in $\mathscr{F}(d,c)$, then so is $f(uz)$ for any $u \in \mathbb{C}$ with $|u|=1$. Thus Proposition A.1 implies the same results for $f(w)$ for any complex number w satisfying $|w| = r$.

Outside the trivial case $r=0$ and the case $r=1$, which was handled in detail in [RV86], Proposition A.1 appears to be new.

Remark A.4. Choose a prime power q, and take $c = q^{-1/4}$ and $r = q^{-1/2}$. Let $I_1, I_2, \ldots$ be an increasing sequence of closed intervals with union I_{attained}. For each positive integer k, let P_k be a polynomial constructed as in Proposition 6.2 from the interval I_k. Then the polynomials $P_k(q^{1/2}z) \in \mathscr{F}(\deg P_k, c)$ have limits as in (31). In the other direction, Proposition A.1(a) shows that we could not hope to construct polynomials satisfying the conditions of Proposition 6.2 for intervals larger than I_{attained}.

A.2. **Potential functions.** Given a nonconstant polynomial f, let μ be the uniform probability measure on the set of zeros of f, counted with multiplicity. Then $\log|f(z)|^{1/d}$ equals $\int \log|w-z|\, d\mu(w)$ minus a constant, so Proposition A.1 can be reformulated in terms of μ. This suggests a generalization in which μ is allowed to be any compactly supported probability measure on $\mathbb{C}_{\geq 1}$. In fact, this generalization, formalized as Proposition A.6 below, is *equivalent* to Proposition A.1.

Definition A.5 ([ST97, p. I.1]). Let Σ be a compact subset of $\mathbb{C}$. Let $\mathcal{M}(\Sigma)$ be the set of (Borel) probability measures on $\mathbb{C}$ with support contained in Σ. For $\mu \in \mathcal{M}(\Sigma)$, define the potential function $U^\mu \colon \mathbb{C} \to \mathbb{R} \cup \{\infty\}$ by

$$U^\mu(z) := \int_{\mathbb{C}} -\log|w-z|\, d\mu(w).$$

For a polynomial F with nonzero constant term, the polynomial $f(z) := F(z)/F(0)$ satisfies $f(0) = 1$, as required in the definition of $\mathscr{F}(d,c)$. Analogously, we will consider $U^\mu(z) - U^\mu(0)$.

Proposition A.6. *Choose c in $(0,1)$, and let $\mathscr{M}(c)$ be the set of probability measures μ with compact support contained in $\mathbb{C}_{\geq 1}$ such that*

$$U^\mu(z) - U^\mu(0) \;\leq\; -\log c \qquad \textit{for all } z \in \mathbb{C}_{\leq 1}. \tag{32}$$

(a) *For any $\mu \in \mathscr{M}(c)$,*

$$U^\mu(r) - U^\mu(0) \leq -\log M(r) \qquad \textit{for all } r \in [0,1], \tag{33}$$

$$U^\mu(r) - U^\mu(0) \geq -\log M(-r) \qquad \textit{for all } r \in [0,\infty). \tag{34}$$

(b) *Let Σ_c be the arc $\{z \in \mathbb{C} : |z| = 1 \text{ and } |z-1| \leq 2c\}$. There exists a measure $\mu_c \in \mathscr{M}(c)$ supported on Σ_c such that for every $r \in (-\infty, 1]$,*

$$U^{\mu_c}(r) - U^{\mu_c}(0) = -\log M(r).$$

Proof that Proposition A.6 implies Proposition A.1. If Proposition A.6(a) holds, apply it to the uniform probability measure μ on the zeros of $f \in \mathscr{F}(d,c)$ counted with multiplicity, and apply $x \mapsto e^{-x}$ to (33) and (34) to get Proposition A.1(a).

Now suppose in addition that Proposition A.6(b) holds. Fix $r \in [0,1]$. For $\lambda \in (0,1)$, Proposition A.6(a) (and rotational symmetry) shows that for $z \in \mathbb{C}_{\leq \lambda}$,

$$U^{\mu_c}(z) - U^{\mu_c}(0) \leq -\log M(\lambda), \tag{35}$$

which is *strictly* less than $-\log c$. By approximating μ_c by uniform probability measures supported on finite subsets of $\mathbb{C}_{\geq 1}$, we find a sequence of polynomials $p_1, p_2, \ldots$ such that

- p_d has degree d, has all roots in $\mathbb{C}_{\geq 1}$, and satisfies $p_d(0) = 1$; and
- on each compact subset of $\mathbb{C} \setminus \Sigma_c$, the sequence $-\log |p_d(z)|^{1/d}$ converges uniformly to $U^{\mu_c}(z) - U^{\mu_c}(0)$.

By (35) and uniform convergence, for any $\lambda < 1$, if d is sufficiently large, then $-\log |p_d(z)|^{1/d} \leq -\log c$ on $\mathbb{C}_{\leq \lambda}$, so the polynomial $f_d(z) := p_d(\lambda z)$ lies in $\mathscr{F}(d,c)$. Then $|f_d(r)|^{1/d} \to \exp(-(U^{\mu_c}(\lambda r) - U^{\mu_c}(0))) = M(\lambda r)$ uniformly on each compact subset of $(-\infty, 1]$. By repeating the argument for each $\lambda \in (0,1)$ to obtain $f_{d,\lambda}$, and then letting λ tend to 1 sufficiently slowly with d, we obtain polynomials satisfying Proposition A.1(b). □

If μ is supported on the unit circle, then $U^\mu(0) = 0$ by definition. In proving Proposition A.6(a), the following lets us assume that μ is supported on the unit circle.

Lemma A.7. *Given a compactly supported probability measure μ on $\mathbb{C}_{\geq 1}$, there is a probability measure $\widehat{\mu}$ supported on the unit circle Σ such that*

$$U^\mu(z) - U^\mu(0) = U^{\widehat{\mu}}(z) \quad \textit{whenever} \quad |z| < 1$$

and

$$U^\mu(z) - U^\mu(0) \geq U^{\widehat{\mu}}(z) \quad \textit{whenever} \quad |z| \geq 1.$$

Proof. Write μ as a sum of nonnegative measures $\mu_\Sigma + \mu'$, where μ_Σ is supported on the circle and $\mu'(\Sigma) = 0$. Apply "balayage" ([ST97, Theorem II.4.7]) to μ' to produce $\widehat{\mu'}$ supported on the circle, and let $\widehat{\mu} = \mu_\Sigma + \widehat{\mu'}$. □

A.3. **Equilibrium measures.**

Definition A.8. Suppose that Σ is of positive capacity, as defined in [ST97, Definition 1.5]; this holds if Σ contains a line segment or circular arc of positive length, for example. The energy of $\mu \in \mathcal{M}(\Sigma)$ is $\int_\Sigma U^\mu(z)\, d\mu(z)$. There is a unique energy-minimizing measure $\mu \in \mathcal{M}(\Sigma)$, called the equilibrium measure on Σ. More generally, for any continuous function $Q\colon \Sigma \to \mathbb{R}$, there is a unique measure μ in $\mathcal{M}(\Sigma)$ minimizing the weighted energy

$$E_Q(\mu) := \int_\Sigma \bigl(U^\mu(z) + 2\,Q(z)\bigr)\, d\mu(z),$$

and this μ is called the weighted equilibrium measure for Q on Σ.

From now on, Σ denotes the unit circle, and $\kappa\colon \mathbb{C}^\times \to \mathbb{C}$ denotes the rational function $\kappa(z) := z + z^{-1}$, which maps Σ onto the interval $[-2, 2]$.

Lemma A.9.

(a) *The map $\mu \mapsto \kappa_*\mu$ sending measures to their pushforwards under κ is a bijection from the set of complex-conjugation-invariant probability measures on the unit circle Σ to $\mathcal{M}([-2, 2])$.*

(b) *For μ as in (a), we have $U^{\kappa_*\mu}(\kappa(z)) = 2\,U^\mu(z) + \log|z|$ for all $z \in \mathbb{C}^\times$.*

(c) *Let Σ' be a positive-capacity complex-conjugation-invariant compact subset of Σ. Let $\alpha \in \mathbb{R}$ and $r \in \mathbb{R}^\times \setminus \Sigma'$. Under κ_*, the equilibrium measure on Σ' for weight $Q(z) := \alpha \log|z - r|$ corresponds to the equilibrium measure on $\kappa(\Sigma')$ for weight $R(z) := \alpha \log|z - \kappa(r)|$.*

Proof.

(a) The map κ induces an isomorphism from the σ-algebra of complex-conjugation-invariant Borel subsets of Σ to the σ-algebra of Borel subsets of $[-2, 2]$.

(b) For $w, z \in \mathbb{C}^\times$ we have

$$\kappa(w) - \kappa(z) = -\frac{1}{z}(w - z)(w^{-1} - z). \tag{36}$$

The claim follows by applying $-\log|\;|$ and integrating against $d\mu(w)$.

(c) Renaming variables in (36) yields $R(\kappa(z)) = 2\,Q(z) - \alpha \log|r|$. By symmetry, the only measures on Σ' we need to consider are those that are complex-conjugation-invariant. For such μ,

$$\begin{aligned} E_R(\kappa_*\mu) &= \int_{\Sigma'} \bigl(U^{\kappa_*\mu}(\kappa(z)) + 2\,R(\kappa(z))\bigr)\, d\mu(z) \\ &= \int_{\Sigma'} \bigl(2\,U^\mu(z) + 4\,Q(z) - 2\alpha \log|r|\bigr)\, d\mu(z) \qquad \text{(by (b))} \\ &= 2\,E_Q(\mu) - 2\alpha \log|r|, \end{aligned}$$

so the μ that minimizes $E_R(\kappa_*\mu)$ is the same as the μ that minimizes $E_Q(\mu)$. □

A.4. **The extreme measure.** Let c and Σ_c be as in Proposition A.6. Let μ_c be the equilibrium measure on Σ_c. Lemma A.10 below shows that μ_c satisfies the requirements of Proposition A.6(b).

Lemma A.10.

(a) *We have $U^{\mu_c}(0) = 0$.*

(b) *The function* $U^{\mu_c}(z)$ *is* $\begin{cases} -\log c & \text{for } z \in \Sigma_c, \\ \leq -\log c & \text{for } z \notin \Sigma_c. \end{cases}$

(c) *For all* $r \in (-\infty, 1]$*, we have* $U^{\mu_c}(r) = -\log M(r)$*.*

Proof.

(a) This holds for any measure supported on the unit circle, by definition of the potential.

(b) The energy of the equilibrium measure on the circular arc Σ_c is $-\log c$, as follows from [Ran95, Table 5.1]. The inequality outside Σ_c follows from nonnegativity of the Green function [ST97, p. I.4]. The equality on Σ_c follows from the fact that the points on Σ_c are regular points for the Dirichlet problem on $\mathbb{C} \setminus \Sigma_c$, as can be checked from [ST97, Theorem I.4.6].

(c) By similar right triangles, the real part of either endpoint of Σ_c is at distance $2c^2$ from 1, so $\kappa(\Sigma_c) = 2[1-2c^2, 1] = [2-4c^2, 2]$. Let $\ell(z) := c^2 z + 2 - 2c^2$, so $\ell([-2,2]) = [2-4c^2, 2]$. Let μ_Σ be the uniform probability measure on Σ, so $U^{\mu_\Sigma}(z) = 0$ on $\mathbb{C}_{\leq 1}$ [ST97, Example 0.5.7]. By Lemma A.9(c), $\kappa_* \mu_c$ and $\kappa_* \mu_\Sigma$ are the equilibrium measures on $[2-4c^2, 2]$ and $[-2, 2]$, respectively, so $\kappa_* \mu_c = \ell_* \kappa_* \mu_\Sigma$. Given $r \in (0, 1]$, define $r' \in (0, 1]$ by $\kappa(r) = \ell(\kappa(r'))$. Applying $U^{\kappa_* \mu_c} = U^{\ell_* \kappa_* \mu_\Sigma}$ yields

$$\begin{aligned} U^{\kappa_* \mu_c}(\kappa(r)) &= U^{\kappa_* \mu_\Sigma}(\kappa(r')) - \log c^2 && \text{(since } \ell \text{ scales distances by } c^2\text{)} \\ 2\, U^{\mu_c}(r) + \log r &= (2\, U^{\mu_\Sigma}(r') + \log r') - \log c^2 && \text{(by Lemma A.9(b) twice)} \\ U^{\mu_c}(r) &= \tfrac{1}{2}\log(r'/r) - \log c && \text{(since } U^{\mu_\Sigma}(z) = 0 \text{ on } \mathbb{C}_{\leq 1}\text{)} \\ &= -\log M(r) && \text{(algebraic computation yields } r'\, M(r)^2 = rc^2\text{).} \end{aligned}$$

To extend to $(-\infty, 1]$, observe that $U^{\mu_c}(r)$ and $-\log M(r)$ are real analytic on $(-\infty, 1)$. □

A.5. **Proof of optimality.** The idea for proving inequality (33) is that it should be a nonnegative linear combination (really an integral) of the inequalities (32). The "coefficients" of the linear combination are given by a measure ν_α belonging to a family that we describe now. The $r = 0$ and $r = 1$ cases of (33) follow from $M(0) = 1$ and $M(1) = c$, so we assume $r \in (0, 1)$. For $\alpha \in \mathbb{R}_{\geq 0}$, let ν_α be the equilibrium measure on Σ for weight $Q_\alpha(z) := \alpha \log |z - r|$.

Lemma A.11.

(a) *For every* $\alpha \geq 0$*, there exists* $\theta(\alpha) \in (0, \pi]$ *such that* $\operatorname{supp}(\nu_\alpha)$ *is the arc* $\{e^{it} \colon |t| \leq \theta(\alpha)\}$*.*

(b) *The function* θ *is decreasing and continuous. Also,* $\theta(0) = \pi$ *and* $\lim_{\alpha \to \infty} \theta(\alpha) = 0$*.*

(c) *Let* α *be such that* $\operatorname{supp}(\nu_\alpha) = \Sigma_c$*. Then there is a constant* C *such that*

$$U^{\nu_\alpha}(z) + Q_\alpha(z) \ \text{ is } \ \begin{cases} C & \text{for all } z \in \Sigma_c, \\ \geq C & \text{for all } z \in \Sigma \setminus \Sigma_c. \end{cases} \tag{37}$$

Proof. The literature contains similar results for an interval; to use them, we push forward by κ. By Lemma A.9(c), $\kappa_* \nu_\alpha$ is the equilibrium measure on $[-2, 2]$ for weight $Q_1(z) := \alpha \log |z - \kappa(r)|$.

(a) We need to prove that $\operatorname{supp}(\kappa_* \nu_\alpha) = [2\cos(\theta(\alpha)), 2]$ for some $\theta(\alpha) \in (0, \pi]$. Pushing forward $\kappa_* \nu_\alpha$ by $z \mapsto 2 - z$ gives the equilibrium measure on $[0, 4]$ for weight $Q_2(z) :=$

$\alpha \log|\kappa(r) - 2 + z|$. The function

$$x\, Q_2'(x) = \frac{\alpha x}{\kappa(r) - 2 + x}$$

is increasing on $[0, 4]$, so [ST97, Theorem IV.1.10(c)] implies that the support is an interval. The corresponding interval for $\kappa_* \nu_\alpha$ is contained in $[-2, 2]$, and must contain 2 since otherwise we could translate the measure right to reduce the weighted energy.

(b) By [ST97, Theorem IV.1.6(f)], $\operatorname{supp}(\nu_\alpha)$ is decreasing and continuous,[1] so θ is too. Since ν_0 is the uniform measure on Σ, we have $\theta(0) = \pi$. For fixed $\epsilon > \epsilon' > 0$, if $\kappa_* \nu_\alpha$ has any mass to the left of $2 - \epsilon$, redistributing it according to the equilibrium measure on $[2 - \epsilon', 2]$ increases the energy by $O(1)$ but decreases the contribution from the weight by at least a positive constant times α, so if α is sufficiently large, $\kappa_* \nu_\alpha$ cannot have such mass; in other words, $\operatorname{supp}(\kappa_* \nu_\alpha) \subset [2 - \epsilon, 2]$ for sufficiently large α. This holds for every ϵ, so $\lim_{\alpha \to \infty} \theta(\alpha) = 0$.

(c) The number α exists by (b). Let C be the modified Robin constant for Q_α [ST97, p. 27]. By [ST97, Theorem I.1.3(d, f)], (37) holds outside a zero-capacity subset of Σ. On the other hand, the points in Σ are regular points for the Dirichlet problem in $\overline{\mathbb{C}} \setminus \Sigma$ by Wiener's theorem [ST97, Theorem I.4.6], so [ST97, Theorem I.5.1(iv')] implies that U^{ν_α} is continuous on Σ, as is Q_α. Thus (37) holds on all of Σ. □

Proof of (33). By Lemma A.7, we may assume that μ is supported on the unit circle Σ, so $U^\mu(0) = 0$. Given $r \in (0, 1)$, let α be as in Lemma A.11(c). Then

$$\begin{aligned}
-U^\mu(r) &= \frac{1}{\alpha} \int_\Sigma Q_\alpha(z)\, d\mu(z) && (\text{since } Q_\alpha(z) = \alpha \log|z - r|) \\
&\geq \frac{1}{\alpha} \left(C - \int_\Sigma U^{\nu_\alpha}(z)\, d\mu(z) \right) && (\text{by the inequality in (37)}) \\
&= \frac{C}{\alpha} + \frac{1}{\alpha} \int_{\Sigma_c} \int_\Sigma \log|z - w|\, d\mu(z)\, d\nu_\alpha(w) && (\text{by definition of } U^{\nu_\alpha}) \\
&= \frac{C}{\alpha} - \frac{1}{\alpha} \int_{\Sigma_c} U^\mu(w)\, d\nu_\alpha(w) && (\text{by definition of } U^\mu) \\
&\geq \frac{C}{\alpha} + \frac{1}{\alpha} \log c && (\text{by (32) with } U^\mu(0) = 0).
\end{aligned}$$

When μ is μ_c, Lemmas A.11(c) and A.10(b) show that both inequalities in this sequence are sharp, so

$$-U^{\mu_c}(r) = \frac{C}{\alpha} + \frac{1}{\alpha} \log c \leq -U^\mu(r).$$

Thus $U^\mu(r) - U^\mu(0) = U^\mu(r) \leq U^{\mu_c}(r) = -\log M(r)$, by Lemma A.10(c). □

Proof of (34). For $\beta \in \mathbb{R}_{\geq 0}$, let ν'_β be the equilibrium measure on Σ for weight $R_\beta(z) := -\beta \log|z + r|$. As in Lemma A.11, there exists $\beta > 0$ and a real constant D such that

[1] Although [ST97, Theorem IV.1.6(f)] claims only right continuity, it can be applied with Q replaced by $-Q$ to get left continuity.

$\mathrm{supp}(\nu'_\beta) = \Sigma_c$ and

$$U^{\nu'_\beta}(z) + R_\beta(z) \ \text{ is } \begin{cases} D & \text{for all } z \in \Sigma_c, \\ \geq D & \text{for all } z \in \Sigma \setminus \Sigma_c. \end{cases}$$

We may replace μ by the $\widehat{\mu}$ given by Lemma A.7, which implies that $U^\mu(r) - U^\mu(0) \geq U^{\widehat{\mu}}(r)$ for every $r \in [0, \infty)$. The rest of the proof is entirely analogous to the proof of (33). ☐

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References

[AHL13] Y. Aubry, S. Haloui, and G. Lachaud, "On the number of points on abelian and Jacobian varieties over finite fields," *Acta Arithmetica*, vol. 160, no. 3, pp. 201–241, 2013.

[DH98] S. A. DiPippo and E. W. Howe, "Real polynomials with all roots on the unit circle and abelian varieties over finite fields," *J. Number Theory*, vol. 73, no. 2, pp. 426–450, 1998, Corrigendum in *J. Number Theory* **83** (2000), no. 1, 182.

[DKRV20] T. Dupuy, K. Kedlaya, D. Roe, and C. Vincent, "Isogeny classes of abelian varieties over finite fields in the lmfdb," *preprint*, 2020, arXiv:2003.05380.

[Has36] H. Hasse, "Zur Theorie der abstrakten elliptischen Funktionenkörper III. Die Struktur des Meromorphismenrings. Die Riemannsche Vermutung," *J. Reine Angew. Math.*, vol. 175, pp. 193–208, 1936.

[How96] E. W. Howe, "Kernels of polarizations of abelian varieties over finite fields," *J. Algebraic Geom.*, vol. 5, no. 3, pp. 583–608, 1996.

[HK21] E. W. Howe and K. S. Kedlaya, "Every positive integer is the order of an ordinary abelian variety over $\mathbb{F}_2$," *preprint*, 2021, arXiv:2103.16530v2.

[Isa08] I. M. Isaacs, *Finite group theory*, ser. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2008, vol. 92, pp. xii+350.

[Kad21] B. Kadets, "Estimates for the number of rational points on simple abelian varieties over finite fields," *Math. Z.*, vol. 297, no. 1-2, pp. 465–473, 2021.

[Ked08] K. S. Kedlaya, "Search techniques for root-unitary polynomials," in *Computational arithmetic geometry*, ser. Contemp. Math. Vol. 463, Amer. Math. Soc., Providence, RI, 2008, pp. 71–81.

[LMFDB] The LMFDB Collaboration, *The L-functions and modular forms database*, `https://www.lmfdb.org`, [Online; accessed 25 June 2021], 2021.

[MS21] S. Marseglia and C. Springer, "Every finite abelian group is the group of rational points of an ordinary abelian variety over $\mathbb{F}_2$," *preprint*, 2021, arXiv:2105.08125v1.

[Ran95] T. Ransford, *Potential theory in the complex plane*, ser. London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 1995, vol. 28, pp. x+232.

[RV86] S. Ruscheweyh and R. S. Varga, "On the minimum moduli of normalized polynomials with two prescribed values," *Constr. Approx.*, vol. 2, no. 4, pp. 349–368, 1986.

[ST97] E. B. Saff and V. Totik, *Logarithmic potentials with external fields*, ser. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1997, vol. 316, pp. xvi+505, Appendix B by Thomas Bloom.

[Wat69] W. C. Waterhouse, "Abelian varieties over finite fields," *Ann. Sci. École Norm. Sup. (4)*, vol. 2, pp. 521–560, 1969.

[Wei48a] A. Weil, *Sur les courbes algébriques et les variétés qui s'en déduisent*, ser. Actualités Scientifiques et Industrielles [Current Scientific and Industrial Topics], No. 1041. Hermann et Cie., Paris, 1948, pp. iv+85, Publ. Inst. Math. Univ. Strasbourg **7** (1945).

[Wei48b] ——, *Variétés abéliennes et courbes algébriques*, ser. Actualités Scientifiques et Industrielles [Current Scientific and Industrial Topics], No. 1064. Hermann & Cie., Paris, 1948, p. 165, Publ. Inst. Math. Univ. Strasbourg **8** (1946).

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