ERRATA FOR “RATIONAL POINTS ON VARIETIES”

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This is an errata list for the book

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The name in parentheses is the discoverer of the error.

- Definition 1.3.1: “direct product” should be “finite direct product”.
- Section 1.5.7.4, first sentence: “a another” should be “another”. (Francesc Fité)
- Example 2.3.10, last sentence: $\mathbb{A}^n$ should be $\mathbb{A}^{n-3}$. (Douglas Ulmer)
- Theorem 2.5.1(b): p is the characteristic of k. (Francesc Fité)
- Exercise 2.6: The morphism $\operatorname{Spec} k \to S$ is unnecessary. (Douglas Ulmer)
- Example 3.1.5: Exercise should be Example. (Francesc Fité)
- Section 3.5.15: The sentence “But X'_R need not be smooth.” is correct, but it would be more to the point to say “But X'_R need not be regular.”
- Proof of Proposition 3.5.19: $\frac{\partial g_i}{\partial t_j}(x)$ should be $\frac{\partial g_i}{\partial t_j}$. (Francesc Fité)
- Proof of Lemma 3.5.57: The first period should be a comma. (Francesc Fité)
- Remark 3.5.62: “Theorem 3.5.59” should be “Definition 3.5.59”.
- Line before Example 4.1.3: $\mathscr{F}_{S_{ij}}$ should be $\mathscr{F}|_{S_{ij}}$. (Francesc Fité)
- Proof of Theorem 5.3.1: The actions were not specified clearly. The *left* translation action on G induces a right G-action on A, which can be turned into a left G-action on A (in which g acts as right action of g^{-1}). It is this left G-action on A and the induced contragredient left G-action on A^* that are used in Step 2.
- Remark 5.6.24: One should assume that G is a connected algebraic group, and that $\operatorname{char} k = 0$ or G is reductive. (Alex Youcis)
- Section 5.6.6: The definition of simple algebraic group is too restrictive: no positive-dimensional algebraic group in characteristic p would be simple by this definition, because the kernel of Frobenius would be a normal subgroup scheme.
- The references to [**Wit10**] at the beginning of Section 5.7.2 and in the proof of Theorem 5.7.13 should be to [**Wit08**]. (Borys Kadets.)
- Section 5.11: The terminology needs to be corrected to reflect standard usage, which is as follows. Let $\operatorname{Inn} G_{k_s}$ be the group of inner automorphisms of $G(k_s)$. The homomorphisms $G(k_s) \to \operatorname{Inn} G_{k_s} \to \operatorname{Aut} G_{k_s}$ induce maps $\mathrm{H}^1(k, G) \to \mathrm{H}^1(k, \operatorname{Inn} G_{k_s}) \to \mathrm{H}^1(k, \operatorname{Aut} G_{k_s})$. The algebraic groups corresponding to elements in the image of the second map (resp. the composition) are called inner forms (resp. pure inner forms).

  Researchers working on the Langlands program equip these with rigidifying data. They define an inner twist to be a pair (H, ξ), where H is an algebraic group over k and $\xi \colon G_{k_s} \to H_{k_s}$ is an isomorphism such that $\xi^{-1}({}^\sigma\xi) \in \operatorname{Inn} G_{k_s}$ for all $\sigma \in \mathfrak{G}_k$;

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then $\sigma \mapsto \xi^{-1}({}^\sigma\xi)$ is a 1-cocycle representing an element of $\mathrm{H}^1(k, \operatorname{Inn} G_{k_s})$. They then define a pure inner twist (pure rational form in [Vog93, Definition 2.6]) to be (H, ξ, z), where (H, ξ) is as above and $z\colon \mathfrak{G}_k \to G(k_s))$ is a 1-cocycle such that $\xi^{-1}({}^\sigma\xi)$ equals the inner automorphism $\operatorname{inn}_{z_\sigma}$ for all $\sigma \in \mathfrak{G}_k$; then z represents an element of $\mathrm{H}^1(k, G)$. See also [Kal16, §2.3]. (Alex Youcis)

- Remark 5.12.7: In the second paragraph, one should assume that X is reduced. (Alex Youcis)
- Theorem 5.12.24(c): $A(k)$ should be $A^t(k)$, where A^t is the dual abelian variety (Olivier de Gaay Fortman). Also, Tate's proof is for finite extensions of $\mathbb{Q}_p$; the case $k = \mathbb{R}$ is related to older results of Witt [Wit34] (see [Sch94, p. 221]), and the characteristic p case is proved in [Mil70].
- Warning 5.12.26: In the second sentence, k should be nonarchimedean. (Olivier de Gaay Fortman)
- Warning 5.12.27: In the last sentence, delete "the group".
- Definition 6.2.7(1): "collection" should be "the collection". (Francesc Fité)
- Definition 6.3.21: "equipped" should be "equipped with". (Francesc Fité)
- Proof of Proposition 6.3.22: "such" should be "such that". (Francesc Fité)
- Definition 6.5.1: "An" should be "an". (Francesc Fité)
- Proposition 6.5.9: "a isomorphism" should be "an isomorphism". (Francesc Fité)
- Section 6.5.6.4: The second sentence should say "Let $\tau \in \mathrm{H}^1(S, G)$ be the class of T." (Francesc Fité)
- Two lines before Proposition 6.7.1, it should say "$E_\infty^{n,0}$ is a subobject of L^n." (Anthony Várilly-Alvarado)
- Paragraph after Proposition 6.8.1: Proposition 1.3.15(iii) should be Proposition 1.5.13(iii). (Anthony Várilly-Alvarado)
- Thanks to the proof of the purity conjecture [Česk19], some simplifications are possible:
  - In Theorem 6.8.3, the "caveat" can be simplified to "the caveat that one must exclude the p-primary part of all the groups if there exists $x \in X^{(1)}$ such that $\mathbf{k}(x)$ is imperfect of characteristic p".
  - In Corollaries 6.8.5 and 6.8.7, the caveats are unnecessary.
  - In the proof of Proposition 6.9.10, the $\operatorname{char} k = 0$ proof then works in arbitrary characteristic.
- Warning 6.8.4: $\operatorname{Br} k(X)$ should be $\operatorname{Br} \mathbf{k}(X)$.
- Proof of Lemma 6.9.8: In the first sentence, the claims are true, but the logic is not presented correctly. The commutative square involving s^* initially involves the *downward* homomorphism on the left induced by the restriction of s to $\operatorname{Spec} \mathbf{k}(B)$. It is only after knowing that the two vertical homomorphisms on the left are inverses that one can use s^* to deduce $\operatorname{Br} X \subset \operatorname{Br} B$. (Anthony Várilly-Alvarado and Ken Zheng)
- Proof of Lemma 6.9.8: Where Theorem 6.9.7 is invoked, Corollary 6.7.8 should be mentioned too.
- Section 7.3.3, first sentence: Change "over a" to "of". (Francesc Fité)
- Proposition 7.5.17: $1 \otimes \sigma$ should be $1 \times \sigma$. (Francesc Fité)
- Remark 7.5.19: The Grothendieck–Lefschetz trace formula is not true in the generality suggested (it fails for the morphism $\mathbb{A}^1 \to \mathbb{A}^1$ sending x to $x+1$; see Milne, *Lectures*

on étale cohomology, 2013–03–22, Example 29.1), but Grothendieck did prove it for the Frobenius morphism of a variety over a finite field. See this preprint of Yakov Varshavsky for a generalization. Also, the reference to Deligne is for the Poincaré duality statement. (Kaloyan Slavov)

- Proof of Proposition 7.6.1(b): "subscheme" should be "subschemes". (Francesc Fité)
- Theorem 8.4.10 and Corollary 8.4.11: It is necessary to add the hypothesis "If $\operatorname{char} k = p$, assume that X is proper." In the proof of Theorem 8.4.10, change the sentence starting "For any nonarchimedean $v \in S$" to "For any nonarchimedean $v \in S$, there are only finitely many possibilities for the k_v-scheme $F^{-1}(x_v)$ as x_v ranges over $X(k_v)$: when $\operatorname{char} k = 0$, this follows since k_v has only finitely many extensions of each degree; when $\operatorname{char} k = p$, use Krasner's lemma (Proposition 3.5.74) and compactness of $X(k_v)$." (Fei Xu)
- Remark 8.4.12: Change "irrational" to "rational", and "dominant morphism $\mathbb{P}^1 \twoheadrightarrow X$" to "morphism $\mathbb{P}^1 \to X$ inducing a surjection $\mathbb{P}^1(\mathbf{A}^S) \to X(\mathbf{A}^S)$".
- Paragraph after Definition 9.4.1: [Kol96, Corollary III.2.3.5.2] should be [Kol96, Corollary III.3.2.5.2]. (Osami Yasukura)
- Section A.4: The set $\{x, \{y\}\}$ does not necessarily determine x and y. It should be changed to Kuratowski's definition $\{\{x\}, \{x, y\}\}$. (Juan Climent Vidal)

Acknowledgments

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