18.02 LECTURE NOTES, FALL 2021

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These are an approximation of what was covered in lecture. (Please clear your browser's cache before reloading this file to make sure you are getting the current version.)

Thursday, September 9

We do not live in a 1-dimensional world. And 3 dimensions is not enough either, since applications in engineering, machine learning, economics, etc. routinely involve quantities depending on *lots* of parameters. 18.02 provides the background for understanding functions with multivariable input and output. In Fall 2021 we are modernizing the content of 18.02, in particular to incorporate more linear algebra that is essential for many of the applications above. Time to turn on the firehose!

1. Vectors

1.1. **Vector notation.** A vector $\mathbf{v}$ in $\mathbb{R}^3$ is an ordered triple of real numbers. (In print, use bold face letters like $\mathbf{v}$; in handwriting, use $\vec{v}$.) In 18.02, vectors are often[1] written as column vectors like $\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$; Edwards & Penney write the same vector as $\langle 2, 3, 5 \rangle$. The 3-dimensional space $\mathbb{R}^3$ is the set of all such triples. (You can have vectors in $\mathbb{R}^2$ or $\mathbb{R}^{95786}$ too, if you want.) Geometrically, a vector is an arrow with a length and a direction; its position does not matter.

The standard basis vectors for $\mathbb{R}^3$ are

$$\mathbf{e}_1 := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_3 := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(The symbol $:=$ means "is defined to be".) (Alternative notations: $\mathbf{i}, \mathbf{j}, \mathbf{k}$, or $\hat{x}, \hat{y}, \hat{z}$ to indicate that these are in the directions of the x-, y-, and z-axes.[2])

Define $\mathbf{0} := \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

[1] One reason is to facilitate matrix-vector multiplication later on.

[2] One advantage of the notation $\mathbf{e}_1, \mathbf{e}_2, \ldots$ is that it generalizes to arbitrary dimension. One disadvantage of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ is that $\mathbf{i}$ can be confused with $i := \sqrt{-1}$.

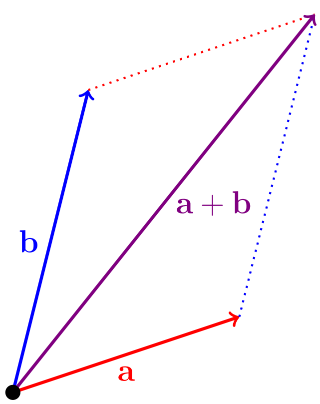
If P is a point in space, then the position vector $\mathbf{P}$ is the vector pointing from $(0,0,0)$ to P. (Alternative notations: OP or $\overrightarrow{OP}$.) Sometimes we'll think of the point $(2,3,5)$ and its position vector $\begin{pmatrix}2\\3\\5\end{pmatrix}$ as being the same.

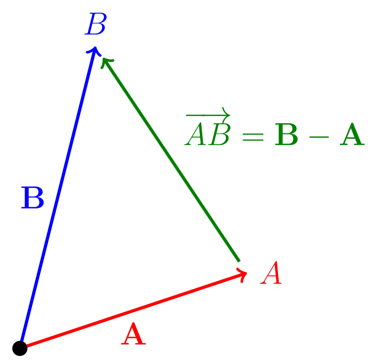
The length of $\mathbf{v} = \begin{pmatrix}a\\b\\c\end{pmatrix}$ is $|\mathbf{v}| := \sqrt{a^2+b^2+c^2}$ (also called magnitude). This formula can be explained by using the Pythagorean theorem twice.

A unit vector is a vector of length 1.

1.2. **Vector operations.** Addition: $\begin{pmatrix}3\\1\end{pmatrix} + \begin{pmatrix}1\\4\end{pmatrix} = \begin{pmatrix}4\\5\end{pmatrix}$. Subtraction: $\begin{pmatrix}1\\4\end{pmatrix} - \begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}-2\\3\end{pmatrix}$.

Geometrically: parallelogram law for $+$, triangle law for $-$:





Important: If A and B are two points, and $\mathbf{A}$ and $\mathbf{B}$ are their position vectors, then the vector from A to B is $\overrightarrow{AB} = \mathbf{B} - \mathbf{A}$, because this is what you have to add to $\mathbf{A}$ to get to $\mathbf{B}$.

Scalar multiplication: $\underbrace{-10}_{\text{scalar}} \underbrace{\begin{pmatrix}3\\1\end{pmatrix}}_{\text{vector}} = \underbrace{\begin{pmatrix}-30\\-10\end{pmatrix}}_{\text{vector}}$. Scalar means number (use this word to emphasize that it is not a vector). Scalar multiplication is a scalar times a vector, and the result is a vector. Geometrically: $c\mathbf{v}$ has the same (or opposite) direction as $\mathbf{v}$, but possibly a different length. Two vectors $\mathbf{v}$ and $\mathbf{w}$ are parallel if one of them is a scalar multiple of the other.

Problem 1.1. Fill in the blanks:

$$\begin{pmatrix}3\\4\end{pmatrix} = \underbrace{\boxed{}}_{\text{positive scalar}} \underbrace{\boxed{}}_{\text{unit vector}}.$$

Solution: The positive scalar must be the length, 5. So the answer is

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \underset{\text{positive scalar}}{5} \, \underset{\text{unit vector}}{\begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}}.$$

In general, for any nonzero $\mathbf{v}$:

$$\mathbf{v} = \underset{\text{length}}{|\mathbf{v}|} \quad \underset{\text{unit vector in the direction of } \mathbf{v}}{\frac{\mathbf{v}}{|\mathbf{v}|}}.$$

Question 1.2. Does the zero vector $\mathbf{0} := \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ have a direction? It is best to say that it has *every* direction, and hence to say that it is parallel to every other vector, and perpendicular to every other vector.

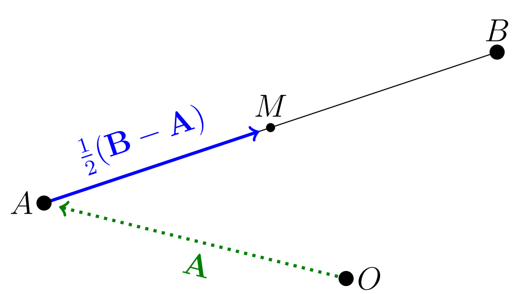
Another example, involving addition and scalar multiplication:

$$\underset{\text{linear combination of } \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3}{2\,\mathbf{e}_1 + 3\,\mathbf{e}_2 + 5\,\mathbf{e}_3} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}.$$

In general, a linear combination of vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ is any vector obtained by multiplying the vectors by (possibly different) scalars and adding the results, i.e., an expression of the form $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$ for some scalars $c_1, \dots, c_n$.

1.3. **Solving geometry problems with vectors.**

Question 1.3. Suppose that M is the midpoint of segment AB. In terms of the position vectors $\mathbf{A}$ and $\mathbf{B}$, what is the position vector $\mathbf{M}$?



To get to M from the origin, first go to A and then go halfway from A to B. The position vector $\mathbf{A}$ gets you to A. The vector from A to B is $\mathbf{B} - \mathbf{A}$, so the vector from A to M is $\frac{1}{2}(\mathbf{B} - \mathbf{A})$. Thus

$$\mathbf{M} = \mathbf{A} + \frac{1}{2}(\mathbf{B} - \mathbf{A}) = \frac{\mathbf{A} + \mathbf{B}}{2}.$$

Question 1.4. Explain why the midpoints of the sides of a space quadrilateral form a parallelogram.

How should one approach a problem like this?

1. Give variable names to the objects given in the problem.
   Let A, B, C, D be the vertices of the quadrilateral in order. Let A' be the midpoint of AB, let B' be the midpoint of BC, let C' be the midpoint of CD, and let D' be the midpoint of DA. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{A}', \mathbf{B}', \mathbf{C}', \mathbf{D}'$ be the corresponding position vectors.
2. Write down known equations relating the variables.
   We know that
   $$\mathbf{A}' = \frac{\mathbf{A}+\mathbf{B}}{2}, \quad \mathbf{B}' = \frac{\mathbf{B}+\mathbf{C}}{2}, \quad \mathbf{C}' = \frac{\mathbf{C}+\mathbf{D}}{2}, \quad \mathbf{D}' = \frac{\mathbf{D}+\mathbf{A}}{2}.$$
3. See if these equations imply the desired conclusion.
   We compute
   $$\mathbf{B}' - \mathbf{A}' = \frac{\mathbf{C}-\mathbf{A}}{2}$$
   $$\mathbf{C}' - \mathbf{D}' = \frac{\mathbf{C}-\mathbf{A}}{2}.$$
   Thus segments $B'A'$ and $C'D'$ are parallel and have the same length. This means that $A'B'C'D'$ is a parallelogram.

1.4. **Some advice for success in 18.02.**

- Read the Information pages on Canvas.
- Reading assignments are posted on Canvas. Do the reading *before* lecture!
- Come to office hours! (Office hours generally consist of a small group of students discussing additional examples not covered in lecture or recitation, asking questions, getting started on difficult homework problems, etc. The recitation leaders for this class are some of the best math postdocs and grad students worldwide, and office hours are your best chance to learn from them.)
- Homework:
  - It's long and has difficult problems, so start now!
  - The problems indicate the date after which you should have the knowledge to do them.
  - Work together in groups! It's OK if other people tell you how to solve a problem, but don't be looking at their solution as you write your own.
  - Do what it takes (come to office hours, discuss problems with others) so that when you submit an assignment you are pretty sure that it is complete and correct.

1.5. **Dot product.** Do we multiply vectors coordinate-wise? No! Why not? This does not give a notion with useful geometric meaning. Instead:

Dot product (also called scalar product or inner product):

$$\underbrace{\begin{pmatrix}2\\3\\5\end{pmatrix}}_{\text{vector}} \cdot \underbrace{\begin{pmatrix}7\\8\\9\end{pmatrix}}_{\text{vector}} = (2)(7)+(3)(8)+(5)(9)$$

$$= 14+24+45$$

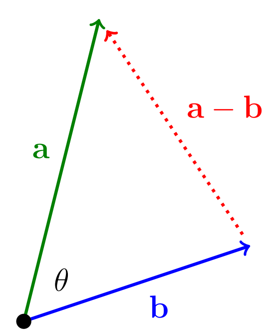
$$= \underbrace{83}_{\text{scalar}}.$$

Important special case: $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$.

Theorem 1.5 (Geometric interpretation of the dot product)**.** *If θ is the angle between nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, then*

$$\boxed{\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta.}$$

(If $\mathbf{a}$ or $\mathbf{b}$ is $\mathbf{0}$, then θ can be taken to be any real number, and the formula still holds, with both sides being 0*.)*



Why?

Proof (=explanation). Let $a = |\mathbf{a}|$, $b = |\mathbf{b}|$, $c = |\mathbf{a}-\mathbf{b}|$. Then

$$\begin{aligned} c^2 &= (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) \\ &= \mathbf{a}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{b} - \mathbf{b}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b} \quad \text{(since $\cdot$ satisfies the distributive law)} \\ &= a^2 + b^2 - 2\,\mathbf{a}\cdot\mathbf{b}. \end{aligned}$$

On the other hand, the law of cosines says

$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$

Comparing shows that $\mathbf{a}\cdot\mathbf{b} = ab\cos\theta$. □

Corollary 1.6. *Vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular $\iff$ $\mathbf{a}\cdot\mathbf{b} = 0$.*

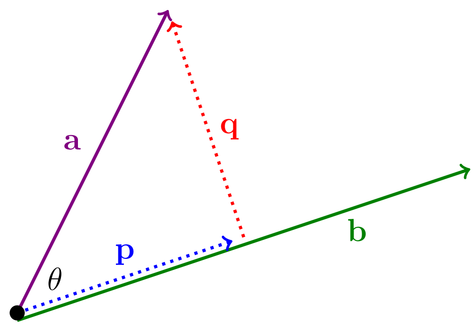
1.6. **Scalar component of a vector in the direction of another vector.** Suppose that $\mathbf{a}$ is a vector, and $\mathbf{b}$ is a nonzero vector. The question we want to answer is

"How much of $\mathbf{a}$ is in the direction of $\mathbf{b}$?"

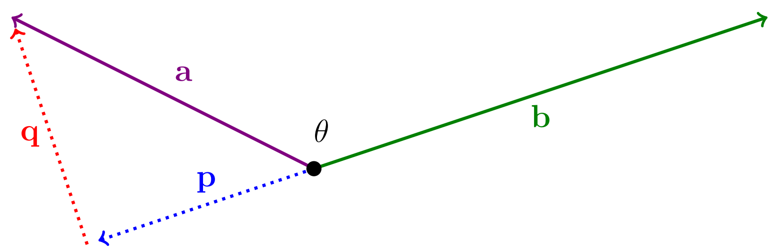
Start by dropping a perpendicular from $\mathbf{a}$ to the line spanned by $\mathbf{b}$. This gives a decomposition

$$\mathbf{a} = \underbrace{\mathbf{p}}_{\text{parallel to } b} + \underbrace{\mathbf{q}}_{\text{perpendicular to } b}.$$

Then $\mathbf{p}$ is in the *same* direction as in $\mathbf{b}$



or $\mathbf{p}$ is in the opposite direction



or $\mathbf{p} = \mathbf{0}$.

The vector $\mathbf{p}$ is called the projection of $\mathbf{a}$ onto the line spanned by $\mathbf{b}$.

The vector $\mathbf{p}$ is also called the vector component of $\mathbf{a}$ in the direction of $\mathbf{b}$.

To get the scalar component, take its length with an appropriate sign:

Definition 1.7. The scalar component of $\mathbf{a}$ in the direction of $\mathbf{b}$ is

$$\operatorname{comp}_{\mathbf{b}} \mathbf{a} := \pm|\mathbf{p}|,$$

where the sign is $+$ or $-$ according to whether $\mathbf{p}$ is in the same or opposite direction as $\mathbf{b}$. (If $\mathbf{p} = \mathbf{0}$, either sign gives 0.)

Although $\mathbf{p}$ is a vector, $\operatorname{comp}_{\mathbf{b}} \mathbf{a}$ is a scalar, as the name suggests!

Another geometric interpretation: Set up a new coordinate system in which the x-axis is in the direction of $\mathbf{b}$; then $\operatorname{comp}_{\mathbf{b}} \mathbf{a}$ is the new x-coordinate of $\mathbf{a}$.

Computing the scalar component: The diagrams show that

$$\operatorname{comp}_{\mathbf{b}} \mathbf{a} = |\mathbf{a}| \cos\theta. \tag{1}$$

But the easiest way to compute $\operatorname{comp}_{\mathbf{b}} \mathbf{a}$ is usually to dot $\mathbf{a}$ with the direction unit vector $\dfrac{\mathbf{b}}{|\mathbf{b}|}$:

$$\boxed{\operatorname{comp}_{\mathbf{b}} \mathbf{a} = \mathbf{a} \cdot \frac{\mathbf{b}}{|\mathbf{b}|} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|}.}$$

(This is equivalent to formula (1), since $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\,\cos\theta$.)

Example 1.8. $\operatorname{comp}_{\mathbf{e}_2} \begin{pmatrix} -2 \\ -3 \\ -5 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ -5 \end{pmatrix} \cdot \mathbf{e}_2 = -3.$

(In this example, the $\mathbf{b}$ was already a unit vector, so there was no need to compute $\dfrac{\mathbf{b}}{|\mathbf{b}|}$.)

2. Matrices

2.1. Matrix notation.

Definition 2.1. An $m \times n$ matrix is a rectangular array of numbers with m rows and n columns.

Example:

$$A := \begin{pmatrix} 3 & 5 & \pi \\ 4 & 0 & 0 \\ 6 & 7 & 9 \\ 1 & -2 & 3 \end{pmatrix}$$

is a 4×3 matrix. (The dimensions are always given in the order "height $\times$ width".)

The notation a_{ij} means the number in the ith row and jth column. In the example above, $a_{32} = 7$. (One could write $a_{3,2}$ but people often omit the comma.)

Two matrices A and B are equal if A has the same dimensions as B and $a_{ij} = b_{ij}$ for all i and j.

An $m \times 1$ matrix is the same as a (column) vector in $\mathbb{R}^m$. Example: $\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} \in \mathbb{R}^3$.

2.2. Matrix times a vector.

Example 2.2. The product

$$\underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \end{pmatrix}}_{\text{matrix}} \underbrace{\begin{pmatrix} 100 \\ 1 \\ 10 \end{pmatrix}}_{\text{vector}}$$

is defined by taking the dot product of each matrix row with the vector:

$$\langle 1,2,3\rangle \cdot \begin{pmatrix} 100 \\ 1 \\ 10 \end{pmatrix} = 132$$

$$\langle 2,3,5\rangle \cdot \begin{pmatrix} 100 \\ 1 \\ 10 \end{pmatrix} = 253,$$

so the final result is

$$\underset{\text{matrix}}{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \end{pmatrix}} \underset{\text{vector}}{\begin{pmatrix} 100 \\ 1 \\ 10 \end{pmatrix}} = \underset{\text{vector}}{\begin{pmatrix} 132 \\ 253 \end{pmatrix}}.$$

In general, a matrix-vector product $A\mathbf{x}$ is defined when the dot products are defined, which is when the width of A equals the height of $\mathbf{x}$.

Example 2.3.

$$\begin{pmatrix} 6 & 7 & 8 \\ 2 & 3 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}.$$

In general:

$$\boxed{A\mathbf{e}_1 = (\text{1st column of } A)}, \boxed{A\mathbf{e}_2 = (\text{2nd column of } A)}, \text{ and so on.}$$

Example 2.4. $\begin{pmatrix} 6 & 7 & 8 \\ 2 & 3 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6x+7y+8z \\ 2x+3y+5z \end{pmatrix} = x\begin{pmatrix} 6 \\ 2 \end{pmatrix} + y\begin{pmatrix} 7 \\ 3 \end{pmatrix} + z\begin{pmatrix} 8 \\ 5 \end{pmatrix}.$

In general: $\boxed{A\mathbf{x} = (\text{some linear combination of the columns of } A)}$.

2.3. **Matrix operations.**

Addition, subtraction, and scalar multiplication are defined entrywise, as for vectors. (For addition and subtraction, the two matrices have to have the same dimensions.)

Transpose:

$$\begin{pmatrix} 2 & 3 & 5 \\ 7 & 11 & 13 \end{pmatrix}^T = \begin{pmatrix} 2 & 7 \\ 3 & 11 \\ 5 & 13 \end{pmatrix}.$$

Each row of the original matrix corresponds to a column of the transpose matrix. The ij-entry of A equals the ji-entry of A^T.

Multiplication: If A is $m \times n$ and B is $n \times p$, then AB is the $m \times p$ matrix whose ij-entry is the dot product

$$(i\text{th row of } A) \cdot (j\text{th column of } B).$$

(If the second dimension of A does not match the first dimension of B, then the dot products are not defined, so AB is not defined.)

Example 2.5. Multiplying a 2×3 matrix by a 3×3 matrix gives a 2×3 matrix:

$$\begin{pmatrix} 2 & 3 & 7 \\ 0 & 5 & 1 \end{pmatrix} \begin{pmatrix} 10 & 0 & 9 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 23 & 13 & 18 \\ 5 & 11 & 0 \end{pmatrix}.$$

For example, the $2,1$ entry is computed as follows:

$$\underbrace{\langle 0, 5, 1 \rangle}_{\text{2nd row}} \cdot \underbrace{\begin{pmatrix} 10 \\ 1 \\ 0 \end{pmatrix}}_{\text{1st column}} = 0(10) + 5(1) + 1(0) = \underbrace{5}_{\text{2,1 entry}}.$$

Warning 2.6. Even when AB and BA both make sense, they might be unequal. (In other words, matrix multiplication is not commutative.)

Friday, September 10

2.4. **Determinants.** To each *square* matrix A is associated a number called its determinant.

$$\begin{aligned}
\det \begin{pmatrix} a \end{pmatrix} &:= a \\
&= \pm \text{ length of segment in the real line } \mathbb{R} \text{ determined by } a \\
\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} &:= ad - bc \\
&= \pm \text{ area of parallelogram in } \mathbb{R}^2 \text{ formed by } \langle a, b \rangle \text{ and } \langle c, d \rangle \\
&\qquad \text{(the sign is + if } \langle c, d \rangle \text{ is counterclockwise from } \langle a, b \rangle\text{)} \\
\det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} &:= a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - c_1 b_2 a_3 - c_2 b_3 a_1 - c_3 b_1 a_2 \\
&= \pm \text{ volume of parallelepiped in } \mathbb{R}^3 \text{ formed by the rows } \mathbf{a}, \mathbf{b}, \mathbf{c} \\
&\qquad \text{(the sign is + if } \mathbf{a}, \mathbf{b}, \mathbf{c} \text{ agree with the right hand rule)}
\end{aligned}$$

(In the formula for the 3×3 determinant, each $+$ term is the product along a "southeast" diagonal with wraparound and each $-$ term is the product along a "northeast" diagonal with wraparound.)

The geometric interpretations are true also for the segment/parallelogram/parallelepiped formed by the *columns* instead of the rows.

Alternative notation for determinant: $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$. (This is a scalar, not a matrix!)

Laplace expansion (along the first row) for a 3×3 determinant:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = +a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

The general rule leading to the formula above is this:

(1) Move your finger along the entries in a row.

(2) At each position, compute the minor, defined as the smaller determinant obtained by crossing out the row and the column through your finger; then multiply the minor by the number you are pointing at, and adjust the sign according to the checkerboard pattern

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

(the pattern always starts with $+$ in the upper left corner).

(3) Add up the results.

There is a similar expansion for a determinant of any size, computed along any row or column.

Properties of determinants:

D-1: Interchanging two rows changes the sign of $\det A$.

D-2: If one of the rows is all 0, then $\det A = 0$.

D-3: Multiplying an entire row by a scalar c multiples $\det A$ by c.

D-4: Adding a multiple of a row to another row does not change $\det A$.

These properties can all be interpreted geometrically. The same properties hold for *column* operations.

Example 2.7. Let

$$A := \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}.$$

If we add 10 times the first row to the second row (and leave the first row unchanged), we get a new matrix

$$B := \begin{pmatrix} 2 & 3 \\ 25 & 37 \end{pmatrix}.$$

Property D-4 says that $\det B = \det A$.

Question 2.8. Suppose that A is a 3×3 matrix such that $\det A = 5$. Doubling every entry of A gives a matrix $2A$. What is $\det(2A)$?

Solution. Each time we multiply a row by 2, the determinant gets multiplied by 2. We need to do this three times to double the whole matrix A, so the determinant gets multiplied by $2 \cdot 2 \cdot 2 = 8$. Thus $\det(2A) = 8\det(A) = 40$. $\square$

2.5. Cross product.

2.5.1. *Definition.*

Definition 2.9 (Cross product, also called vector product). The cross product of vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$ is another vector in $\mathbb{R}^3$:

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &:= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\
&:= + \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{e}_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{e}_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{e}_3 \\
&= \left\langle a_2b_3 - a_3b_2, \quad a_3b_1 - a_1b_3, \quad a_1b_2 - a_2b_1 \right\rangle.
\end{aligned}$$

The cross product is defined for vectors in $\mathbb{R}^3$ only!

2.5.2. *Geometric interpretation of the cross product.*

If $\mathbf{a}$ and $\mathbf{b}$ are nonzero vectors forming an angle θ, then $\mathbf{a} \times \mathbf{b}$ is the vector determined by the following three conditions:

- It is perpendicular to $\mathbf{a}$ and $\mathbf{b}$.
- Its length is $|\mathbf{a}||\mathbf{b}|\sin\theta$ (which equals the area of the parallelogram formed by $\mathbf{a}$ and $\mathbf{b}$).
- Its direction is given by the right hand rule: if you point the fingers of your right hand in the direction of $\mathbf{a}$ so that bending your fingers makes them point in the direction of $\mathbf{b}$, then your thumb shows the direction of $\mathbf{a} \times \mathbf{b}$.

(We won't explain in 18.02 *why* the determinant and cross product have the geometric interpretations claimed.)

2.5.3. *Properties of cross products.*

For vectors $\mathbf{a}$ and $\mathbf{b}$,

$$\mathbf{a} \text{ and } \mathbf{b} \text{ are parallel} \iff \mathbf{a} \times \mathbf{b} = \mathbf{0} \quad (\text{these happen when } \theta \text{ is } 0 \text{ or } \pi)$$

$$\mathbf{a} \times \mathbf{a} = \mathbf{0}$$

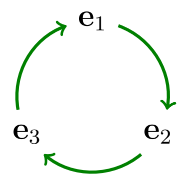
$$\mathbf{b} \times \mathbf{a} = -\,\mathbf{a} \times \mathbf{b}.$$

2.5.4. *Cross products involving* $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$.

Problem 2.10. What is $\mathbf{e}_1 \times \mathbf{e}_3$?

Solution. It is a vector perpendicular to $\mathbf{e}_1$ and $\mathbf{e}_3$, and its length is $(1)(1)\sin\frac{\pi}{2} = 1$, so it must be $\mathbf{e}_2$ or $-\mathbf{e}_2$. The right hand rule tells you which: $\mathbf{e}_1 \times \mathbf{e}_3 = -\mathbf{e}_2$. $\square$

If you like, you can use the following mnemonic: Start by drawing



The cross product of any one of these with the next one is the third vector in the sequence: for example, $\mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2$. But if the two input vectors are in the order opposite to the order specified by the arrows, then insert a negative sign: $\mathbf{e}_1 \times \mathbf{e}_3 = -\mathbf{e}_2$. (And if the two input vectors are the same, the cross product is $\mathbf{0}$.)

3. Matrices as linear transformations

3.1. Functions with vector input and/or vector output.

Scalar input, scalar output: Alternative notation for the function $f(x) := \sin x$:

$$\begin{aligned} f\colon \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto \sin x. \end{aligned}$$

The first line specifies that

- the set of allowable inputs is $\mathbb{R}$ (the domain), and
- every output is an element of $\mathbb{R}$ (the codomain).

The second line shows a typical input x in the domain, and shows which element of the codomain it is mapped to.

(Note: The codomain specifies only the *type* of output; every output is an element of the codomain, but there might be other elements of the codomain that are not outputs. In contrast, the range of f (also called the image of f) is the set of all actual outputs of f, which in this example is the interval $[-1, 1]$. In general, the range is a subset of the codomain.)

Scalar input, vector output (vector-valued function):

$$\begin{aligned} \mathbf{r}\colon \mathbb{R} &\longrightarrow \mathbb{R}^2 \\ t &\longmapsto \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}. \end{aligned}$$

(We use a bold letter $\mathbf{r}$ instead of r since its values are vectors.) For example, $\mathbf{r}(\pi/3) = \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix}$.

Vector input, scalar output (multivariable function):

$$f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}$$
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto x^2 + y^2 - z^2.$$

It could also be written $f(x, y, z) := x^2 + y^2 - z^2$. For example, $f(1, 2, 3) = 1 + 4 - 9 = -4$.

Vector input, vector output: Can we have functions with vector input *and* vector output? Yes! See Example 3.1 below.

3.2. **Going from a matrix to a linear transformation.**

Warm-up: Given a number, say 3, we get a function

$$f\colon \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto 3x$$

that multiplies each input by 3. What is the higher-dimensional analogue?

Example 3.1. Given the 2×3 matrix $\begin{pmatrix} 6 & 7 & 8 \\ 2 & 3 & 5 \end{pmatrix}$, we get a function

$$\mathbf{f}\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$
$$\mathbf{x} \longmapsto \begin{pmatrix} 6 & 7 & 8 \\ 2 & 3 & 5 \end{pmatrix} \mathbf{x}$$

with vector input and vector output! Explicitly,

$$\mathbf{f} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6x + 7y + 8z \\ 2x + 3y + 5z \end{pmatrix}.$$

Each coordinate of the output is a linear combination of the input variables.

What does this function do to $\mathbf{e}_2$? Answer: $\mathbf{f}(\mathbf{e}_2) = \mathbf{f} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$.

In general, an $m \times n$ matrix A gives rise to a function

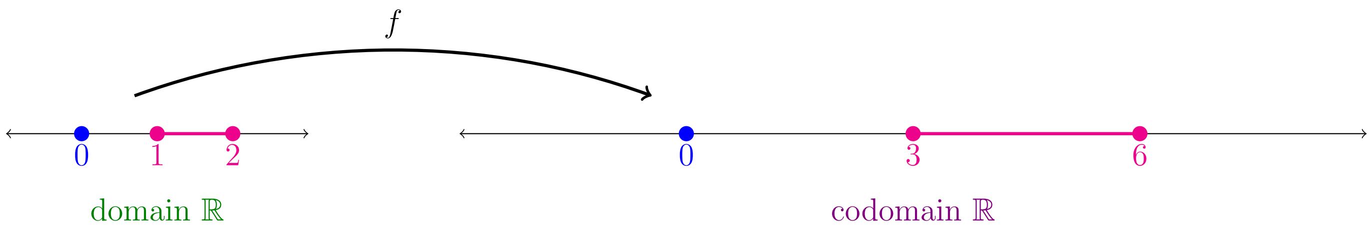
$$\mathbf{f}\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$$
$$\mathbf{x} \longmapsto A\mathbf{x}$$

(note that m and n get reversed). Functions arising in this way are called linear transformations. Sometimes we use the matrix A itself instead of $\mathbf{f}$ to denote the linear transformation.

3.3. **Depicting a linear transformation with a domain-codomain diagram.**

3.3.1. *A single-variable function.* To visualize a function like $f(x) := 3x$, we would usually draw its graph in $\mathbb{R}^2$, the line $y = 3x$.

But there is another way: Draw the domain and the codomain (two copies of the real line), and show what certain features in the domain get transformed to:



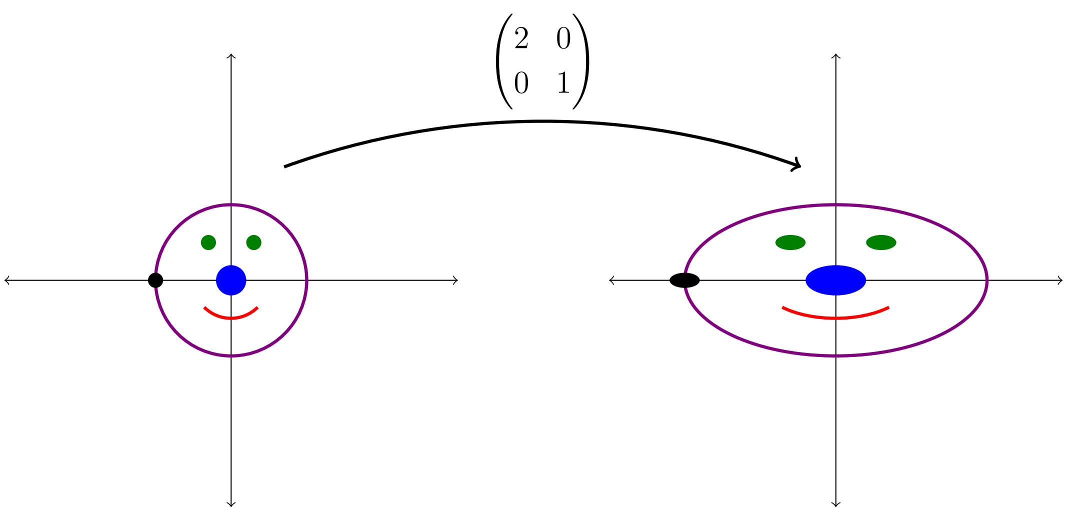
For example, f maps the point 2 to the point 6. The diagram shows how f expands everything by a factor of 3.

3.3.2. *Higher-dimensional analogue.* Consider the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and the associated linear transformation

$$\begin{aligned} \mathbf{f} \colon \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ \begin{pmatrix} x \\ y \end{pmatrix} &\longmapsto \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ y \end{pmatrix}. \end{aligned}$$

Drawing a graph of $\mathbf{f}$ would require 4 dimensions (2 for the input, 2 fo the output), so instead let's draw a domain-codomain diagram. How does $\mathbf{f}$ transform the van Gogh unit smiley?

Tuesday, September 14



For example, the ear at $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ is mapped to $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$. The linear transformation $\mathbf{f}$ stretches in the horizontal direction only.

3.4. **Going from a linear transformation to a matrix.** Given a linear transformation $\mathbf{f}$, how do we reconstruct the matrix A?

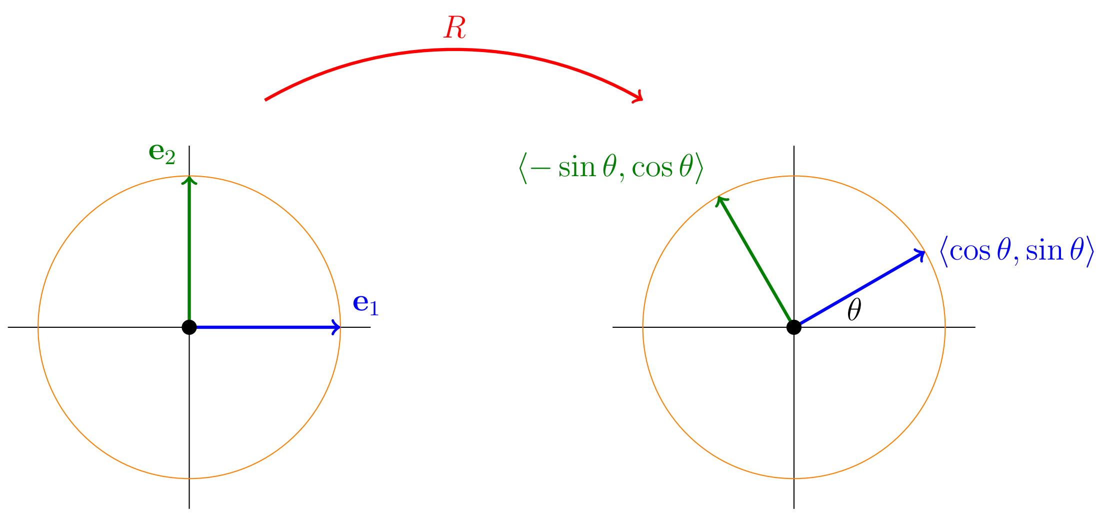
Answer 1: If you know a *formula* for $\mathbf{f}$, just read off the entries of the matrix. For example, if $\mathbf{f}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6x + 7y & \\ 2x & + 5z \end{pmatrix}$, then $A = \begin{pmatrix} 6 & 7 & 0 \\ 2 & 0 & 5 \end{pmatrix}$.

Answer 2: If you know only a *geometric description* of the linear transformation $\mathbf{f}$, then get A as the matrix whose columns are $\mathbf{f}(\mathbf{e}_1)$, $\mathbf{f}(\mathbf{e}_2)$, etc.

Here is an example illustrating Answer 2:

Question 3.2. Given θ, there is a 2×2 matrix R that rotates each vector in $\mathbb{R}^2$ counterclockwise by the angle θ. What is it?

Solution.



Thus

$$(\text{first column of } R) = R\mathbf{e}_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$(\text{second column of } R) = R\mathbf{e}_2 = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix},$$

so

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \qquad \square$$

3.5. **Area scaling factor.**

Example 3.3. What does the dilation given by $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ do to area?
(Hint: Consider what it does to a 1×1 square.)

Answer: It multiplies area by 9. Notice that $\det \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = 9$ too.

In general, for any 2×2 matrix A, the associated linear transformation has

$$\boxed{\text{area scaling factor} = |\det A|.}$$

The area scaling factor is always nonnegative, while $\det A$ could be negative, so it is necessary to take the absolute value.

Why is the formula correct? If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$\text{the square formed by } \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

is transformed into

$$\text{the parallelogram formed by } \begin{pmatrix} a \\ c \end{pmatrix} \text{ and } \begin{pmatrix} b \\ d \end{pmatrix};$$

the square has area 1 and the parallelogram has area $|\det A|$.

3.6. **Composition.** Matrix multiplication corresponds to composition of the linear transformations. More explicitly, if

- $\mathbf{f} \colon \mathbb{R}^n \to \mathbb{R}^m$ corresponds to a matrix A, and
- $\mathbf{g} \colon \mathbb{R}^p \to \mathbb{R}^n$ corresponds to a matrix B,

then the composition

$$\begin{tikzcd} \mathbb{R}^p \arrow[r, "\mathbf{g}"] \arrow[rr, bend right, "\mathbf{f} \circ \mathbf{g}"'] & \mathbb{R}^n \arrow[r, "\mathbf{f}"] & \mathbb{R}^m \end{tikzcd}$$

$$\mathbf{x} \longmapsto \mathbf{f}(\mathbf{g}(\mathbf{x}))$$

corresponds to the matrix product AB. In other words, if you multiply $\mathbf{x}$ by B (apply the inner function $\mathbf{g}$ first!) and then multiply the result by A, you get the same output vector as if you multiplied $\mathbf{x}$ by AB:

$$A(B\mathbf{x}) = (AB)\mathbf{x}.$$

Matrix multiplication is defined as it is to make this true.

Example 3.4. Let α and β be any two angles. Let R_α be the 2×2 rotation matrix for α, and so on. Then

$$R_\alpha \, R_\beta = R_{\alpha+\beta}$$

because rotating by β and then rotating by α has the same effect as rotating by $\alpha + \beta$. (In this example the order did not matter, but in other examples it does.) What famous formulas do you get if you compare entries in this identity?

3.7. **Identity matrix.** The identity transformation

$$\mathbf{f} \colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1x + 0y + 0z \\ 0x + 1y + 0z \\ 0x + 0y + 1z \end{pmatrix}.$$

is associated to the 3×3 identity matrix

$$I := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(You can guess what the 4×4 identity matrix looks like. The positions of the 1s form what is called the diagonal of the matrix.)

The matrix I acts like the number 1:

$$IA = A \qquad \text{and} \qquad AI = A$$

whenever the multiplication makes sense.

3.8. **Inverse matrices.**

3.8.1. *Motivation: solving linear systems.* To solve $3x = 5$, multiply both sides by 3^{-1}.

Similarly, one way to solve

$$2x_1 + 3x_2 = 4$$
$$4x_1 + 5x_2 = 6,$$

is to rewrite as

$$\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix},$$

which has the shape $A\mathbf{x} = \mathbf{b}$, and left multiply both sides by A^{-1} to get $\mathbf{x} = A^{-1}\mathbf{b}$.

3.8.2. *Definition.*

Definition 3.5. The inverse of an $n \times n$ matrix A is another $n \times n$ matrix, A^{-1}, such that

$$AA^{-1} = I \qquad \text{and} \qquad A^{-1}A = I.$$

It exists if and only if $\det A \neq 0$; in that case, A is called invertible, or nonsingular.

If A corresponds to the linear transformation $\mathbf{f}$, then A^{-1} corresponds to the inverse function $\mathbf{f}^{-1}$ (if it exists). For example, the inverse of the rotation matrix R_θ is $R_{-\theta}$.

3.8.3. *Formula for 2×2 matrices.*

Theorem 3.6. *Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.*

- *If* $\det A \neq 0$, *then* $\boxed{A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$.
- *If* $\det A = 0$, *then A^{-1} does not exist.*

For square matrices larger than 2×2, there is still a formula, but it is much more complicated (see M.2), and there are better algorithms for computing A^{-1}, and anyway one would typically use a computer for this.

3.9. **Equations of planes.**

Question 3.7. What does the set of vectors perpendicular to $\langle 1,2,3\rangle$ look like?

Solution. It's a plane through the origin. Its equation is

$$\langle 1,2,3\rangle \cdot \langle x,y,z\rangle = 0,$$

which is

$$x + 2y + 3z = 0. \qquad \square$$

The vector $\mathbf{n} := \langle 1,2,3\rangle$ is called a normal vector to the plane. (*Normal* is another word for perpendicular.)

Question 3.8. What is the plane with normal vector $\langle 1,2,3\rangle$ passing through $(4,5,6)$?

Solution. A point (x,y,z) lies on this plane if the vector from $(4,5,6)$ to (x,y,z) (not the position vector of (x,y,z)!) is perpendicular to $\langle 1,2,3\rangle$, so its equation is

$$\langle 1,2,3\rangle \cdot (\langle x,y,z\rangle - \langle 4,5,6\rangle) = 0,$$

which is

$$(x-4) + 2(y-5) + 3(z-6) = 0. \qquad \square$$

Question 3.9. (Followup question) What is the distance from $(2,3,5)$ to that plane?

Solution. If we choose any point on the plane, say $(4,5,6)$, and form the vector

$$\mathbf{v} := \langle 2,3,5\rangle - \langle 4,5,6\rangle = \langle -2,-2,-1\rangle$$

between the two points, then the desired distance is *not* the length of $\mathbf{v}$, because the straight line path from $(2,3,5)$ to $(4,5,6)$ is not the shortest path from $(2,3,5)$ to the plane. Instead we want "the amount of $\mathbf{v}$ in the direction parallel to the normal vector $\mathbf{n} := \langle 1,2,3\rangle$", taking the absolute value if necessary. Thus the desired distance is the *absolute value* of the scalar component of $\mathbf{v}$ in the direction of $\mathbf{n}$. That scalar component is

$$\begin{aligned}\operatorname{comp}_{\mathbf{n}} \mathbf{v} &= \mathbf{v} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \\ &= \frac{\langle -2,-2,-1\rangle \cdot \langle 1,2,3\rangle}{\sqrt{1^2+2^2+3^2}} \\ &= \frac{-2-4-3}{\sqrt{14}} \\ &= -\frac{9}{\sqrt{14}},\end{aligned}$$

so the distance is $9/\sqrt{14}$. ☐

Question 3.10. Are the vector $\langle -5,1,1\rangle$ and the plane $x+2y+3z=6$

- parallel,
- perpendicular,
- both,
- or neither?

Hint: $\langle -5,1,1\rangle \cdot \langle 1,2,3\rangle = 0$.

Solution. The vector $\langle -5,1,1\rangle$ is perpendicular not to the plane, but to a *normal vector* of the plane. So $\langle -5,1,1\rangle$ is parallel to the plane. The vectors perpendicular to the plane are the scalar multiples of the normal vector, so $\langle -5,1,1\rangle$ is not like this. So the answer is that the vector and plane are parallel. ☐

Thursday, September 16

4. Linear algebra

4.1. **Linear combinations.** Suppose that you have a list of vectors. To get a linear combination of them,

1. multiply each vector by a scalar and
2. add the results.

vectors	typical linear combination
$\mathbf{v}$	$-3\mathbf{v}$
$\mathbf{v}, \mathbf{w}$	$2\mathbf{v} + (-5)\mathbf{w}$
$\mathbf{v}, \mathbf{w}, \mathbf{x}$	$4\mathbf{v} + 0\mathbf{w} + 7\mathbf{x}$

4.2. **Span.**

Definition 4.1. Given one vector $\mathbf{v}$, the span of $\mathbf{v}$ is *the set of all linear combinations of* $\mathbf{v}$, which is the set of all scalar multiples of $\mathbf{v}$:

$$\mathrm{Span}(\mathbf{v}) := \{\text{all vectors } c\mathbf{v}, \text{ where } c \text{ ranges over all real numbers}\}.$$

Example 4.2. Suppose that $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. What are some scalar multiples of $\mathbf{v}$? Well, there's

$$2\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix},$$
$$3\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix},$$
$$-\frac{1}{2}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1/2 \end{pmatrix},$$

and so on. Then $\mathrm{Span}(\mathbf{v})$ is an infinite set whose elements are *all* of these vectors.

Geometric interpretation: Think of each of these vectors as a point in the plane:

$$(4,2), \quad (6,3), \quad (-1,-1/2), \quad \text{and so on;}$$

then $\mathrm{Span}(\mathbf{v})$ is the set of all these points. So $\mathrm{Span}(\mathbf{v})$ is a line through the origin, specifically, the line $y = \frac{1}{2}x$.

Definition 4.3. Given *two* vectors, $\mathbf{v}$ and $\mathbf{w}$, the span of $\mathbf{v}$ and $\mathbf{w}$ is the set of all linear combinations of $\mathbf{v}$ and $\mathbf{w}$:

$$\mathrm{Span}(\mathbf{v},\mathbf{w}) := \{\text{all vectors } c_1\mathbf{v} + c_2\mathbf{w}, \text{ where } c_1 \text{ and } c_2 \text{ range over all pairs of real numbers}\}.$$

Example 4.4. Consider $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. What are some linear combinations of $\mathbf{e}_1$ and $\mathbf{e}_2$? Well, there's

$$3\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix},$$
$$5\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (-7)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ -7 \\ 0 \end{pmatrix},$$

and so on. Then $\mathrm{Span}(\mathbf{e}_1,\mathbf{e}_2)$ is the set of all such vectors: this is the set of all the vectors in $\mathbb{R}^3$ whose 3rd coordinate is 0.

Geometric interpretation: $\mathrm{Span}(\mathbf{e}_1,\mathbf{e}_2)$ is the xy-plane. It is defined by the equation $z = 0$.

Question 4.5. Is the span of two vectors in $\mathbb{R}^3$ always a plane?

Answer: Usually it is, but not always. For example, the span of $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\begin{pmatrix}2\\4\\6\end{pmatrix}$ is a line because the linear combinations are all scalar multiples of the first vector. An even more degenerate case: if both vectors are $\mathbf{0}$, then their span is the set $\{\mathbf{0}\}$ containing only the one vector $\mathbf{0}$.

Problem 4.6. Let $\mathbf{a} = \begin{pmatrix}2\\1\\0\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}0\\1\\3\end{pmatrix}$. What is the equation of the plane $\operatorname{Span}(\mathbf{a}, \mathbf{b})$?

Solution. To give the equation of a plane through the origin, all we need is a normal vector. It needs to be perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. One possible normal vector is

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix}3\\-6\\2\end{pmatrix}.$$

The equation of the plane is then $\begin{pmatrix}3\\-6\\2\end{pmatrix} \cdot \mathbf{x} = 0$, which is

$$3x - 6y + 2z = 0.$$

(To check the answer, verify that $\mathbf{a}$ and $\mathbf{b}$ satisfy the equation.) $\square$

One can also take the span of *more than two* vectors: again it is defined to be the set of all linear combinations that one can form with the given vectors.

Example 4.7. $\operatorname{Span}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is the whole space $\mathbb{R}^3$.

4.3. Linearly dependent vectors.

Definition 4.8. Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent if one of them is a linear combination of the others. Equivalent definition: Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent if there exist scalars $c_1, \ldots, c_n$ *not all zero* such that $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$.

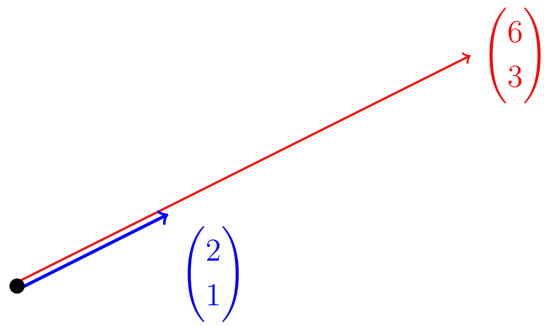
Example 4.9. The vectors $\begin{pmatrix}2\\1\end{pmatrix}$ and $\begin{pmatrix}6\\3\end{pmatrix}$ are linearly dependent because the second is a scalar multiple of the first:

$$\begin{pmatrix}6\\3\end{pmatrix} = 3\begin{pmatrix}2\\1\end{pmatrix}.$$

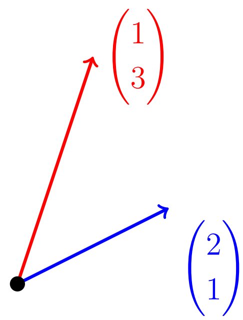
According to the equivalent definition, they are linearly dependent because

$$3\begin{pmatrix}2\\1\end{pmatrix} + (-1)\begin{pmatrix}2\\1\end{pmatrix} = \mathbf{0}.$$

The span of $\begin{pmatrix}2\\1\end{pmatrix}$ and $\begin{pmatrix}6\\3\end{pmatrix}$ is only a line.



Example 4.10. The vectors $\begin{pmatrix}2\\1\end{pmatrix}$ and $\begin{pmatrix}1\\3\end{pmatrix}$ are linearly *independent*. Their span is the whole plane $\mathbb{R}^2$.



In general,

$$\dim \operatorname{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n) \text{ is } \begin{cases} n, & \text{if } \mathbf{v}_1, \ldots, \mathbf{v}_n \text{ are linearly } \textit{independent}, \\ \text{less than } n, & \text{if } \mathbf{v}_1, \ldots, \mathbf{v}_n \text{ are linearly dependent.} \end{cases}$$

(We haven't defined dimension mathematically, but you probably have an intuitive sense of what it is.)

Example 4.11. The vectors $\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\3\\5\end{pmatrix}, \begin{pmatrix}4\\5\\7\end{pmatrix}$, are linearly dependent since the 3rd vector is a linear combination of the first two:

$$\begin{pmatrix}4\\5\\7\end{pmatrix} = 2\begin{pmatrix}1\\1\\1\end{pmatrix} + 1\begin{pmatrix}2\\3\\5\end{pmatrix}.$$

The span of the three vectors is the same as the span of the first two, which is a plane, only 2-dimensional.

Example 4.12. The vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are linearly *independent* and their span is the whole space $\mathbb{R}^3$, which is 3-dimensional.

4.4. Basis.

Definition 4.13. A list of vectors is a basis of $\mathbb{R}^2$ if

1. the span of the vectors is the whole plane $\mathbb{R}^2$, and
2. they are linearly independent.

Example 4.14. $\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ is a basis of $\mathbb{R}^2$.

Every basis of $\mathbb{R}^2$ has exactly 2 vectors, as we'll explain later.

Given a basis $\mathbf{b}_1, \mathbf{b}_2$, we can form the 2×2 basechange matrix $B := \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{pmatrix}$ whose columns are $\mathbf{b}_1$ and $\mathbf{b}_2$. In the example,

$$B = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}.$$

4.5. Coordinates with respect to a basis.

A basis $\mathbf{b}_1, \mathbf{b}_2$ sets up a new coordinate system on $\mathbb{R}^2$ (this is why bases are important). Specifically, given any vector $\mathbf{v}$ in $\mathbb{R}^2$, the coordinates of $\mathbf{v}$ with respect to the basis $\mathbf{b}_1, \mathbf{b}_2$ are the numbers c_1, c_2 such that

$$c_1\mathbf{b}_1 + c_2\mathbf{b}_2 = \mathbf{v}.$$

Problem 4.15. Find the coordinates of $\begin{pmatrix} 7 \\ 6 \end{pmatrix}$ with respect to the basis $\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

Solution. We need to find numbers c_1, c_2 such that

$$c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 6 \end{pmatrix}.$$

Solving the resulting system

$$2c_1 + c_2 = 7$$
$$c_1 + 3c_2 = 6$$

yields $c_1 = 3$ and $c_2 = 1$. □

The two conditions in the definition of basis are what guarantee that the coordinates exist and are unique:

- First, since $\operatorname{Span}(\mathbf{b}_1, \mathbf{b}_2) = \mathbb{R}^2$, every vector $\mathbf{v}$ in $\mathbb{R}^2$ is a linear combination $c_1\mathbf{b}_1 + c_2\mathbf{b}_2$.

- If some $\mathbf{v}$ could be so expressed in two different ways, say,

  $$10\mathbf{b}_1 + 40\mathbf{b}_2 = \mathbf{v} = 2\mathbf{b}_1 + 3\mathbf{b}_2$$

  then subtracting would give

  $$8\mathbf{b}_1 + 37\mathbf{b}_2 = 0,$$

  which is impossible since $\mathbf{b}_1$ and $\mathbf{b}_2$ are *linearly independent*.

4.6. **Changing coordinates.** The plane $\mathbb{R}^2$ has its original coordinate system given by $\mathbf{e}_1, \mathbf{e}_2$ and the new coordinate system given by $\mathbf{b}_1, \mathbf{b}_2$. Any given point can have coordinates (x_1, x_2) in the original system and different coordinates (c_1, c_2) in the new system:

$$x_1\mathbf{e}_1 + x_2\mathbf{e}_2 = c_1\mathbf{b}_1 + c_2\mathbf{b}_2.$$

Let's rewrite the equation in terms of $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $B := \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$:

$$\mathbf{x} = B\mathbf{c}.$$

This shows how, given $\mathbf{c}$, to get $\mathbf{x}$. Because it is also possible to convert the other way, B is automatically invertible, and

$$\mathbf{c} = B^{-1}\mathbf{x}.$$

Friday, September 17

4.7. **Basis of $\mathbb{R}^n$.** All the statements about $\mathbb{R}^2$ have analogues for $\mathbb{R}^n$.

For a list of vectors in $\mathbb{R}^n$, the small-text label on each sloped equality in

$$\begin{tikzcd}
& \text{dimension of the span} \arrow[dl, equal, "\text{linearly independent}"'] \arrow[dr, equal, "\text{span is } \mathbb{R}^n"] & \\
\text{number of vectors} \arrow[rr, equal] & & n
\end{tikzcd}$$

is equivalent to the equality holding. If any two of the three equalities hold, then so does the third.

Conclusion: To check that a list of vectors is a basis of $\mathbb{R}^n$, it is enough to check *any two* of the following three conditions:

- the span of the vectors is the whole space $\mathbb{R}^n$,
- the vectors are linearly independent,
- the number of vectors is n;

then the other one will hold too. (The definition of basis is that the first two hold.)

Another way to test whether vectors $\mathbf{b}_1, \ldots, \mathbf{b}_n \in \mathbb{R}^n$ form a basis:

1. Form the matrix B whose columns are $\mathbf{b}_1, \ldots, \mathbf{b}_n$.
2. Calculate $\det B$.
3. Then

$$\mathbf{b}_1, \ldots, \mathbf{b}_n \text{ is a basis of } \mathbb{R}^n \iff \det B \neq 0$$

4.8. **Square systems of linear equations.** A square system of linear equations is one with the same number of equations as variables.

4.8.1. *Homogeneous systems.* The system

$$\begin{aligned} x + 2y + 3z &= 0 \\ 8x + 4z &= 0 \\ 7x + 6y + 5z &= 0 \end{aligned}$$

is called homogeneous[3] since all the right sides are 0. In matrix form, it is

$$A\mathbf{x} = \mathbf{0}$$

for some square matrix A.

4.8.2. *Inhomogeneous systems.* The system

$$\begin{aligned} x + 2y + 3z &= 0 \\ 8x + 4z &= 37 \\ 7x + 6y + 5z &= 23 \end{aligned}$$

is inhomogeneous since some of the right sides are nonzero. It has the form

$$A\mathbf{x} = \mathbf{b}$$

for some square matrix A and column vector $\mathbf{b}$.

4.8.3. *Behavior of solution set.* What are the solutions to a square system? The behavior depends on whether the system is homogeneous, and on whether $\det A$ is nonzero:

	if $\det A \neq 0$	if $\det A = 0$
homogeneous system $A\mathbf{x} = \mathbf{0}$	only $\mathbf{x} = \mathbf{0}$	infinitely many
inhomogeneous system $A\mathbf{x} = \mathbf{b}$	only $\mathbf{x} = A^{-1}\mathbf{b}$	infinitely many or none

In the $\det A \neq 0$ column, why is there only one solution? Answer: If $\det A \neq 0$, then A^{-1} exists, so the system can be solved for $\mathbf{x}$ by multiplying by A^{-1}.

[3] Both of the letters e in homogeneous are pronounced ee, and the stress is on the third syllable!

Example 4.16. The homogeneous system

$$\begin{aligned} x + 2y &= 0 \\ 2x + 4y &= 0 \end{aligned}$$

has infinitely many solutions (they form a line).

Example 4.17. The inhomogeneous system

$$\begin{aligned} x + 2y &= 3 \\ 2x + 4y &= 6 \end{aligned}$$

has infinitely many solutions (again a line).

Example 4.18. The inhomogeneous system

$$\begin{aligned} x + 2y &= 3 \\ 2x + 4y &= 7 \end{aligned}$$

has no solutions.

4.8.4. *Geometric interpretation in the* 3×3 *case.* For any 3×3 matrix A, the solution set to the homogeneous system $A\mathbf{x} = \mathbf{0}$ is the intersection of 3 planes passing through $\mathbf{0}$ (assuming that no row of A is all 0).

- If $\det A \neq 0$, the intersection contains only $\mathbf{0}$ (this is what usually happens).
- If $\det A = 0$, the intersection is either a line or a plane (through $\mathbf{0}$).

For a 3×3 inhomogeneous system $A\mathbf{x} = \mathbf{b}$, the solution set is again the intersection of 3 planes, but they may be shifted away from $\mathbf{0}$.

5. Parametric lines and curves

5.1. **Lines.** Two ways to describe lines in $\mathbb{R}^3$:

- intersection of two planes
- parametric equation (to be introduced now)

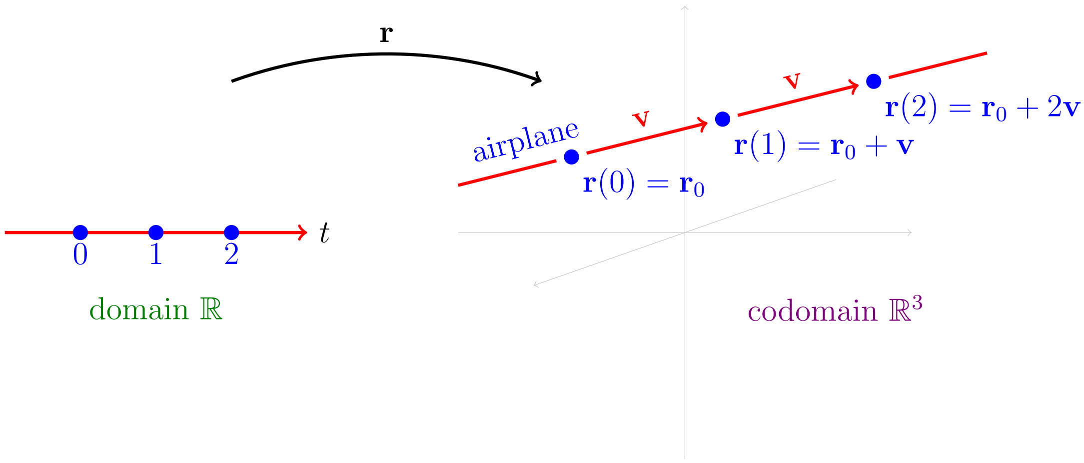
Think of the trajectory of an airplane moving at constant velocity. Let $\mathbf{r}_0$ be the position vector of the airplane at time $t = 0$. Let $\mathbf{v}$ be the velocity.

Where is the airplane at time $t = 1$? Answer: $\mathbf{r}_0 + \mathbf{v}$.

At time $t = 2$? Answer: $\mathbf{r}_0 + 2\mathbf{v}$.

In general, at time t the airplane is at

$$\mathbf{r}(t) := \mathbf{r}_0 + t\mathbf{v}.$$



This is a function with scalar input, vector output:

$$\begin{aligned} \mathbf{r} \colon \mathbb{R} &\longrightarrow \mathbb{R}^3 \\ t &\longmapsto \mathbf{r}_0 + t\mathbf{v}. \end{aligned}$$

Each input real number t gives one output point $\mathbf{r}(t)$ on the line, and as t varies, these points trace out the whole line. (The codomain is $\mathbb{R}^3$, but the range is a *line* in $\mathbb{R}^3$.) The input variable t is called the parameter.

Problem 5.1. Parametrize the line L through $(1, 2, 3)$ and $(4, 1, 3)$.

Solution. Use initial position vector $\mathbf{r}_0 := \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and velocity vector $\mathbf{v} := \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$ so that at time $t = 1$ the point reaches $(4, 1, 3)$. So L is given by

$$\mathbf{r} := \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}.$$

Another way to express the answer is to give the parametric equations of L in terms of coordinate functions:

$$x = 1 + 3t, \qquad y = 2 - t, \qquad z = 3.$$ □

Question 5.2. The lines

$$x = 1 + 3t, \qquad y = 2 - t, \qquad z = 3$$

and

$$x = 2t, \qquad y = -1 + t, \qquad z = 1 + t$$

(1) are the same,
(2) are parallel,
(3) intersect in one point, or
(4) are skew (i.e., do not intersect, but are not parallel either)?

Answer. The velocity vectors $\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ are not scalar multiples of each other, so the lines are not the same, and are not parallel. They intersect if the system

$$\begin{aligned} 1 + 3t &= 2s \\ 2 - t &= -1 + s \\ 3 &= 1 + s. \end{aligned}$$

is solvable. Why did we use a different variable on the right side? Imagine airplanes moving along the lines. If we used the same t on both sides, a solution would be a time when both airplanes are at the same place. If we use t on the left and s on the right, a solution would mean that airplane #1 at some time is at the same point as airplane #2 at a possibly different time, meaning that their *paths* still cross, and this is what we're trying to test! OK, let's now solve the system. The last equation implies $s = 2$. The first equation then implies $t = 1$. These values make all three equations true. So the lines intersect, at the point $\mathbf{r}_1(1) = \mathbf{r}_2(2) = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$. □

5.2. **Parametric equations of curves.**

Question 5.3. As t ranges through all real numbers,

$$x = 2\cos t, \qquad y = \sin t$$

describes

1) A circle
2) An ellipse
3) A line of slope 1/2
4) A point.

Answer. Without the 2, it would be the unit circle. The 2 stretches it in the x-direction, to make an ellipse. It is given by the implicit equation

$$x^2 + 4y^2 = 4.$$

To find the implicit equation, one must eliminate t from the parametric equations; how to do this depends on the shape of the parametric equations, and may require some guesswork. In this problem, we know that $\sin^2 t + \cos^2 t = 1$, and this can be rewritten as $y^2 + (x/2)^2 = 1$ in which t does not appear, which is equivalent to $x^2 + 4y^2 = 4$. □

The parametric equations and the implicit equation are completely different ways of describing the same curve: the parametric equations tell you the curve one point at a time, one point for each value of t; the implicit equation gives you a test for when a given point in $\mathbb{R}^2$ lies on the curve.

Slope of tangent line to this ellipse at a given time? Use the chain rule

$$\underbrace{\frac{dy}{dt}}_{\text{known}} = \underbrace{\frac{dy}{dx}}_{???}\ \underbrace{\frac{dx}{dt}}_{\text{known}}$$

to get

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t}{-2\sin t} = -\frac{1}{2}\cot t.$$

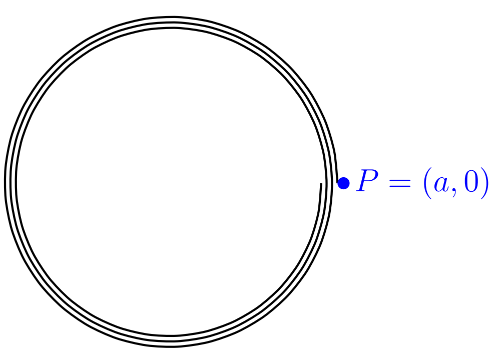
The following question was not discussed in lecture.

What is $\dfrac{d^2y}{dx^2}$? It's

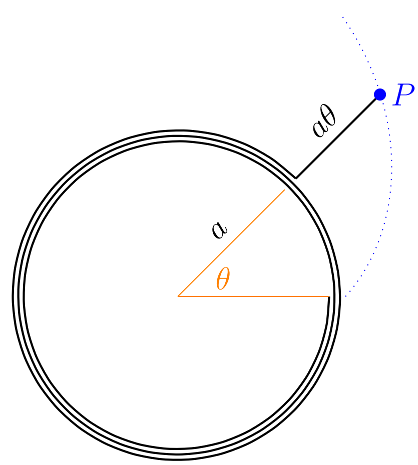
$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{dy'/dt}{dx/dt} = \frac{\frac{1}{2}\csc^2 t}{-2\sin t} = -\frac{1}{4\sin^3 t}.$$

5.3. **Parametric equations of curves: an example.** This example will be done in recitation on Monday, September 20.

Problem 5.4 (Supplementary notes, Exercise 1I-4)**.** A roll of plastic tape of outer radius a is held in a fixed position while the tape is being unwound counterclockwise. The end P of the unwound tape is always held so that the unwound portion is perpendicular to the roll. Taking the center of the roll to be the origin O, and the end P to be initially at $(a, 0)$, write parametric equations for the motion of P.



Initial position



After unwinding θ radians

Solution.

Parameter: The radian measure θ of the amount of tape unwound so far.
The length of tape unwound so far is $a\theta$.
The position of the point where the unwound tape meets the roll: $(a\cos\theta, a\sin\theta)$.
Final answer:

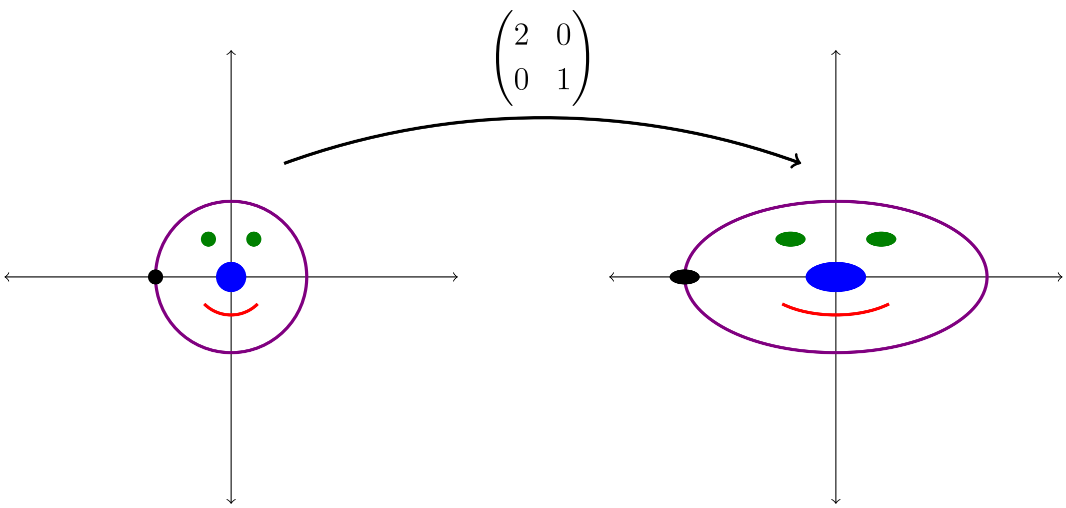
$$\begin{aligned}\mathbf{r} &= \langle a\cos\theta, a\sin\theta\rangle + a\theta\langle\cos\theta, \sin\theta\rangle \\ &= \langle a(1+\theta)\cos\theta,\ a(1+\theta)\sin\theta\rangle.\end{aligned}$$

□

Tuesday, September 21

6. Eigenvalues and eigenvectors

6.1. **Introduction.** Recall that the matrix $\begin{pmatrix}2&0\\0&1\end{pmatrix}$ stretches in the horizontal direction by a factor of 2.



In particular,

$$\begin{pmatrix}2&0\\0&1\end{pmatrix}\underset{\text{eigenvector}}{\mathbf{e}_1} = \underset{\text{eigenvalue}}{2}\,\mathbf{e}_1.$$

The stretching factor (here 2) is called an eigenvalue, and the vector in the stretching direction (here $\mathbf{e}_1$) is called an eigenvector.

Definition 6.1. Suppose that A is a square matrix.

- An eigenvalue of A is a scalar λ such that $A\mathbf{v} = \lambda\mathbf{v}$ for some nonzero vector $\mathbf{v}$.
  (The scalar λ may be 0.)
- An eigenvector of A associated to a given λ is a vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$.
  (Warning: Some authors require an eigenvector to be not the zero vector.)

Try the "Matrix Vector" mathlet

`http://mathlets.org/mathlets/matrix-vector/`

Problem 6.2. Let $A = \begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix}$ and let $\mathbf{v} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$. Is $\mathbf{v}$ an eigenvector of A?

Solution. The calculation

$$A\mathbf{v} = \begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} = 2\mathbf{v},$$

shows that $\mathbf{v}$ is an eigenvector, and that the associated eigenvalue is 2. □

Given A, how can we find the eigenvalues and eigenvectors? For this we need some more concepts of linear algebra.

6.2. **Trace.**

Definition 6.3. The trace of an $n \times n$ matrix A is the sum of the entries along the diagonal:

$$\operatorname{tr} A := a_{11} + a_{22} + \cdots + a_{nn}$$

Example 6.4. If $A = \begin{pmatrix} 4 & 6 & 9 \\ 1 & 7 & 8 \\ 2 & 3 & 5 \end{pmatrix}$, then $\operatorname{tr} A = 4 + 7 + 5 = 16$.

6.3. **Characteristic polynomial.** Use λ to denote a scalar-valued variable.

Definition 6.5. The characteristic polynomial of an $n \times n$ matrix A is $\det(\lambda I - A)$.

The reason for this definition will be clear in the next section when we show how to compute eigenvalues.

Remark 6.6. We often calculate the characteristic polynomial using $\det(A - \lambda I)$ instead. This turns out to be the same as $\det(\lambda I - A)$, except negated when n is odd. (The reason is that changing the signs of all n rows of the matrix $A - \lambda I$ flips the sign of the determinant n times.) Usually we care only about the *roots* of the polynomial, so negating the whole polynomial doesn't make a difference. In any case, $\det(A - \lambda I) = \det(\lambda I - A)$ for 2×2 matrices (since 2 is even).

Problem 6.7. What is the characteristic polynomial of $A := \begin{pmatrix} 7 & 2 \\ 3 & 5 \end{pmatrix}$?

Solution. We have

$$A - \lambda I = \begin{pmatrix} 7 & 2 \\ 3 & 5 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 7-\lambda & 2 \\ 3 & 5-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = (7-\lambda)(5-\lambda) - 2(3) = \boxed{\lambda^2 - 12\lambda + 29}.$$ □

Here is a shortcut for 2×2 matrices:

Theorem 6.8. *If A is a 2×2 matrix, then the characteristic polynomial of A is*

$$\lambda^2 - (\operatorname{tr} A)\lambda + (\det A).$$

Proof. Write $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then the characteristic polynomial is

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \\ &= (a-\lambda)(d-\lambda) - bc \\ &= \lambda^2 - (a+d)\lambda + (ad-bc) \\ &= \lambda^2 - (\operatorname{tr} A)\lambda + (\det A). \end{aligned}$$ □

We can solve Problem 6.7 again, using this shortcut: the matrix $A := \begin{pmatrix} 7 & 2 \\ 3 & 5 \end{pmatrix}$ has $\operatorname{tr} A = 12$ and $\det A = 29$, so the characteristic polynomial of A is $\lambda^2 - 12\lambda + 29$.

For an $n \times n$ matrix, the characteristic polynomial is a degree n polynomial in the variable λ and its leading coefficient is 1, so the polynomial looks like $\lambda^n + \dots$.

Remark 6.9. Suppose that $n > 2$. Then, for an $n \times n$ matrix A, the characteristic polynomial has the form

$$\lambda^n - (\operatorname{tr} A)\lambda^{n-1} + \cdots \pm \det A$$

where the $\pm$ is $+$ if n is even, and $-$ if n is odd. So knowing $\operatorname{tr} A$ and $\det A$ determines some of the coefficients of the characteristic polynomial, but not all of them.

6.4. **Computing all the eigenvalues.**

Warm-up problem: Given a square matrix A, how can we test if 5 is an eigenvalue?

Solution. The following are equivalent:

- 5 is an eigenvalue.
- There exists a nonzero solution to

$$\begin{aligned} A\mathbf{v} &= 5\mathbf{v} \\ 5\mathbf{v} - A\mathbf{v} &= \mathbf{0} \\ 5I\mathbf{v} - A\mathbf{v} &= \mathbf{0} \\ (5I - A)\mathbf{v} &= \mathbf{0}. \end{aligned}$$

- $\det(5I - A) = 0$.
- Evaluating the characteristic polynomial $\det(\lambda I - A)$ at 5 gives 0.

- 5 is a root of the characteristic polynomial. $\square$

The same test works for any number in place of 5. (Now that we know how this works, we never again have to go through the argument above.) Conclusion:

$$\boxed{\text{eigenvalues} = \text{roots of the characteristic polynomial}}.$$

Steps to find all the eigenvalues of a square matrix A:

1. Calculate the characteristic polynomial $\det(\lambda I - A)$ or $\det(A - \lambda I)$.
2. The roots of this polynomial are all the eigenvalues of A.

Problem 6.10. Find all the eigenvalues of $A := \begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix}$.

Solution. We have $\operatorname{tr} A = 1 + 0 = 1$ and $\det A = 0 - 2 = -2$, so the characteristic polynomial is

$$p(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + (\det A) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1).$$

Its roots are 2 and -1; these are the eigenvalues. $\square$

6.5. **Computing eigenvectors.**

Problem 6.11. Find all the eigenvectors of $A := \begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix}$ associated with the eigenvalue 2.

Solution. By definition, an eigenvector associated to the eigenvalue 2 is a vector $\mathbf{v} = \begin{pmatrix} v \\ w \end{pmatrix}$ satisfying

$$\begin{aligned} A\mathbf{v} &= 2\mathbf{v} \\ A\mathbf{v} - 2\mathbf{v} &= \mathbf{0} \\ (A - 2I)\mathbf{v} &= \mathbf{0} \\ \left(\begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right) \mathbf{v} &= \mathbf{0} \\ \begin{pmatrix} -1 & -2 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \end{aligned}$$

which is equivalent to

$$-v - 2w = 0.$$

One solution is $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$; the other solutions are the scalar multiples of this one. Conclusion: The eigenvectors with eigenvalue 2 are all the scalar multiples of $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$. □

Remark 6.12. In this example, the matrix equation became two copies of the same equation $-v - 2w = 0$. More generally, for any 2×2 matrix A and eigenvalue λ, one of the two equations will be a scalar multiple of the other, so again we need to consider only one of them. In particular, the system of two equations will always have a nonzero solution (as there must be, by definition of eigenvalue).

A similar calculation shows that the eigenvectors of A associated with the eigenvalue -1 are the scalar multiples of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

To summarize:

Steps to find all the eigenvectors associated to a given eigenvalue λ of a 2×2 matrix A:

1. Calculate $A - \lambda I$.
2. Expand $(A - \lambda I)\mathbf{v} = \mathbf{0}$ using $\mathbf{v} = \begin{pmatrix} v \\ w \end{pmatrix}$; this gives a system of two equations in x and y.
3. Solve the system; one of the equations will be redundant, so nonzero solutions will exist.
4. The solution vectors $\begin{pmatrix} v \\ w \end{pmatrix}$ are the eigenvectors associated to λ.

Remark 6.13. Let A be a 2×2 matrix.

- If A is the matrix $3I = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$, then the only eigenvalue is 3, and *every* vector is an eigenvector with eigenvalue 3. If $A = aI$ for some other scalar a, a similar thing happens.
- Otherwise, for each eigenvalue λ, the system $(A - \lambda I)\mathbf{v} = 0$ amounts to one nontrivial equation (the other is redundant), so the eigenvectors associated to λ will be the scalar multiples of a single nonzero vector. In this case, if λ is real, then the set of all real eigenvectors forms a line through the origin, called the eigenline of λ.

Thursday, September 23

7. Derivatives and integrals of functions with scalar input, vector output

7.1. **Derivative of a vector-valued function.**

7.1.1. *Definition and physical interpretation.*

Definition 7.1. The derivative of a vector-valued function $\mathbf{r}(t)$ is the vector-valued function

$$\mathbf{r}'(t) := \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}.$$

(Alternative notation: $\dfrac{d\mathbf{r}}{dt}$.) It measures the rate of change of $\mathbf{r}$ at each time t. For example, if $\mathbf{r}(t)$ is the position of a moving particle at time t, then $\mathbf{r}'(t)$ is its velocity at time t.

7.1.2. *Calculating the derivative.* Derivatives can be calculated coordinate-wise:

$$\text{For example, if } \mathbf{r}(t) = \begin{pmatrix} t^3 \\ \cos t \end{pmatrix}, \text{ then } \mathbf{r}'(t) = \begin{pmatrix} 3t^2 \\ -\sin t \end{pmatrix}.$$

This holds because all the ingredients used in the definition of derivative (vector subtraction, scalar multiplication by $1/h$, and limits) can be calculated coordinate-wise. It's the same in 3D.

Advice: Use the definition to understand the derivative physically, but work coordinate-wise to calculate it.

7.2. **Integration of vector-valued functions.**

Similarly, the definite integral

$$\int_a^b \mathbf{f}(t)\, dt := \lim_{\Delta t \to 0} \sum_{i=1}^{n} \mathbf{f}(t_i^*)\, \Delta t$$

(where $[a, b]$ is subdivided into n subintervals of width Δt, and $t_1^*, \ldots, t_n^*$ are sample points, one in each subinterval) can be computed coordinate-wise: for example,

$$\text{For example, if } \mathbf{f}(t) = \begin{pmatrix} t^3 \\ \cos t \end{pmatrix}, \text{ then } \int_0^1 \mathbf{f}(t)\, dt = \begin{pmatrix} t^4/4\big|_0^1 \\ \sin t\big|_0^1 \end{pmatrix} = \begin{pmatrix} 1/4 \\ \sin 1 \end{pmatrix}.$$

Also, there is a fundamental theorem of calculus for vector-valued functions: if $\mathbf{f}(t) = \mathbf{F}'(t)$, then

$$\underbrace{\int_a^b \mathbf{f}(t)\, dt}_{\text{integral of a rate of change}} = \underbrace{\mathbf{F}(b) - \mathbf{F}(a)}_{\text{total change}}.$$

It follows from the scalar function version.

7.3. **Acceleration.** Recall: If $\mathbf{r}(t)$ is the position of an object at time t, then its velocity at time t is $\mathbf{v}(t) := \mathbf{r}'(t)$. Next, its acceleration at time t is $\mathbf{a}(t) := \mathbf{v}'(t) = \mathbf{r}''(t)$. All of these are vector-valued functions.

On the other hand, its speed is $|\mathbf{v}(t)|$; this is a scalar function whose values are nonnegative.

Question 7.2. What does it mean if a particle's position vector $\mathbf{r}(t)$ satisfies $\frac{d|\mathbf{r}|}{dt} = 0$ for all t?

Answer: The *length* of the position vector is constant, so the particle is staying on a sphere centered at the origin.

7.4. **Differentials.** In 18.01, for a function $u(x)$, one defines $du := u'(x)\,dx$ Here du and dx are differentials (not numbers, not vectors, not matrices, but a new kind of object — it does act like a tiny number, though). Similarly, for a vector-valued function $\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$, one defines the "vector-valued differential"

$$d\mathbf{r} = \mathbf{r}'(t)\,dt,$$

which is the same as

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} x'(t)\,dt \\ y'(t)\,dt \\ z'(t)\,dt \end{pmatrix}.$$

Informally, one imagines dt as being a very tiny number, and $d\mathbf{r}$ is a tiny vector measuring the change in position during the short time interval from time t to time $t + dt$.

7.5. **Arc length.** Next define

$$ds = |d\mathbf{r}| := |\mathbf{r}'(t)|\,dt = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt.$$

Informally, ds measures the *distance travelled* during the same time interval $[t, t + dt]$. To get the *total* distance $s(t)$ travelled along the trajectory from a starting point $\mathbf{r}(0)$ to a variable end point $\mathbf{r}(t)$, one "adds up" the distance travelled over all the tiny time subintervals, by integrating:

$$s(t) = \int_0^t ds = \int_0^t |\mathbf{r}'(u)|\,du.$$

(The u in the integral can be thought of as a variable number increasing from 0 up to t; we couldn't just use t, because the name t is already being used, as the upper limit of the integration.) This scalar-valued function $s(t)$ is called the arc length function.

Question 7.3. What are the physical meanings of $\displaystyle\int_0^t |d\mathbf{r}|$ and $\displaystyle\left|\int_0^t d\mathbf{r}\right|$?

Solution. The first one is the definition of arc length $s(t)$, which measures distance *along the trajectory* from the starting point $\mathbf{r}(0)$ to the end point $\mathbf{r}(t)$. On the other hand, "adding up" the infinitesimal changes in position vector gives the total change in position vector, $\int_0^t d\mathbf{r} = \mathbf{r}(t) - \mathbf{r}(0)$, by the fundamental theorem of calculus, and the length of this vector is the distance *as the crow flies.* □

7.6. **Unit tangent vector.** Suppose that $\mathbf{v}(t) \neq \mathbf{0}$. Then $\mathbf{v}(t)$ is a tangent vector to the curve.

Definition 7.4. The unit tangent vector at $\mathbf{r}(t)$ is the unit vector in the direction of $\mathbf{v}(t)$:

$$\mathbf{T} := \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}.$$

Informally, $\mathbf{T}$ is also the unit vector in the direction of $d\mathbf{r} = \mathbf{v}(t)\,dt$; then the identity

$$d\mathbf{r} = \mathbf{T}\,ds$$

is like the expression of a nonzero vector as a unit vector multiplied by a length.

7.7. **Foci of an ellipse.** (This section is just to help make sense of Kepler's first law, and to help with one of the homework problems.)

One way to write down an ellipse is to write an equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Here a is half the width, and b is half the height.

But if you were to ask an ancient Greek what an ellipse is, the answer would be:

> Fix two points P and Q and fix a number ℓ greater than PQ. Then the locus (possible positions) of a point R such that $PR + RQ = \ell$ is an ellipse.

The points P and Q are the foci (plural of focus) of the ellipse.

Another property of the foci: Inside an elliptical room with mirror walls, if you place a light at one focus, the rays will reflect and meet again at the other focus. (Same for sound.)

7.8. **Kepler's second law.** In the early 1600s, Johannes Kepler noticed that Tycho Brahe's data on planetary motion was consistent with three laws:

(1) The orbit of a planet is an ellipse with the sun at one focus.
(2) A planet moves in a plane containing the sun, and the line segment connecting the sun to the planet sweeps out area at a constant rate.
(3) The square of the period of revolution of a planet about the sun is proportional to the cube of the major semiaxis of its elliptical orbit.

Another law: "Gravitational force $\mathbf{F}$ is central", which in mathematical terms says

(2′) Acceleration is central, that is, the planet's acceleration vector $\mathbf{a}$ is always parallel to the vector from the sun to the planet.

Theorem 7.5. *Kepler's second law* (2) *is equivalent to the "acceleration is central" law* (2′)*.*

Proof. Let the origin be where the sun is. Let $\mathbf{r} = \mathbf{r}(t)$ be the position vector of the planet. Let $A = A(t)$ be the area swept out from time 0 to time t.

Between time t and $t + dt$,

$$dA \approx \text{Area(triangle)} = \frac{1}{2}\,|\mathbf{r} \times d\mathbf{r}|.$$

Divide by dt to get a rate:

$$\frac{dA}{dt} = \frac{1}{2}\left|\mathbf{r} \times \frac{d\mathbf{r}}{dt}\right| = \frac{1}{2}|\mathbf{r} \times \mathbf{v}|.$$

So far, this was just setting things up. Now, to prove equivalence, we prove that each law implies the other.

$\Longrightarrow$: Suppose that Kepler's second law holds: $\dfrac{dA}{dt}$ is constant. Then $|\mathbf{r} \times \mathbf{v}|$ is constant. But the direction of $\mathbf{r} \times \mathbf{v}$ is also constant since it is perpendicular to the plane of motion (and by continuity cannot suddenly switch to the opposite direction). Thus $\mathbf{r} \times \mathbf{v}$ is constant. So

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{v}) = \mathbf{0}.$$

On the other hand, by a rule for differentiating a cross product (from the textbook reading for today),

$$\begin{aligned} \frac{d}{dt}(\mathbf{r} \times \mathbf{v}) &= \frac{d\mathbf{r}}{dt} \times \mathbf{v} + \mathbf{r} \times \frac{d\mathbf{v}}{dt} \qquad \text{(important: keep the } \mathbf{r} \text{ and } \mathbf{v} \text{ in order)} \\ &= \mathbf{v} \times \mathbf{v} + \mathbf{r} \times \mathbf{a} \\ &= \mathbf{r} \times \mathbf{a}. \end{aligned}$$

Combining the previous two equations gives

$$\mathbf{r} \times \mathbf{a} = \mathbf{0}.$$

This means that $\mathbf{a}$ is parallel to $\mathbf{r}$.

$\Longleftarrow$: The converse, that the acceleration being central implies Kepler's second law, can be proved by reversing the steps of the previous paragraph. □

Friday, September 24

8. Derivatives of multivariable functions

8.1. **Graphs and level curves of two-variable functions.** Three ways to depict a 2-variable function $f(x, y)$:

(1) Map of its values: At many points (x, y) in the plane, write the value $f(x, y)$. For example, if $f(x, y) := y(y + 1)/2 - x + 10$, then the values at integer points are

30	29	28	27	26	25	24	23	22	21	20
25	24	23	22	21	20	19	18	17	16	15
21	20	19	18	17	16	15	14	13	12	11
18	17	16	15	14	13	12	11	10	9	8
16	15	14	13	12	11	10	9	8	7	6
15	14	13	12	11	10	9	8	7	6	5
15	14	13	12	11	10	9	8	7	6	5
16	15	14	13	12	11	10	9	8	7	6
18	17	16	15	14	13	12	11	10	9	8
21	20	19	18	17	16	15	14	13	12	11
25	24	23	22	21	20	19	18	17	16	15

(the colored entry is $f(0, 0)$).

(2) Graph: Above each point (x, y) in the plane, plot a point in $\mathbb{R}^3$ whose z-coordinate is the value $f(x, y)$. Taken together, these points form a surface in $\mathbb{R}^3$ called the graph of f. It is the set of points in space satisfying the equation $z = f(x, y)$.

(3) Level curves: For each number h, the level curve at height h is the set of points (x, y) in the xy-plane such that $f(x, y) = h$. In the example above, each level curve is a parabola.

Question 8.1. Consider $f(x, y) = x^2 + y^2$. Its graph is a paraboloid. Draw level curves for equally spaced values of h; these are circles in $\mathbb{R}^2$ centered at $(0, 0)$, namely $x^2 + y^2 = h$. Then as one goes out, are the circles

(1) getting closer together
(2) occurring at equally spaced radii
(3) getting farther apart?

Answer: Getting closer together. The farther out you are, the steeper the paraboloid is, so the shorter you have to go horizontally to get a given increase in height.

8.2. Partial derivatives.

8.2.1. *Introduction via an example.*

Here is a map showing values of a function $f(x, y)$ at integer points, with the colored value at $(0, 0)$:

20	19	18	17	16	15
16	16	16	16	16	16
12	13	14	15	16	17
8	10	12	14	16	18
4	7	10	13	16	19

If one starts at $(0, 0)$ and moves to the right, the value increases by 2 for each increase of x by 1; thus the rate of change, denoted $\frac{\partial f}{\partial x}(0, 0)$, equals 2. (Note: $\frac{\partial f}{\partial x}$ is often pronounced "partial f partial x".) Similarly, $\frac{\partial f}{\partial y}(0, 0) = 3$, and $\frac{\partial f}{\partial x}(2, -1) = 3$. Since $\frac{\partial f}{\partial x}$ has potentially different values at different points, it is itself a function defined on the plane.

8.2.2. *Definition.*

Definition 8.2. The partial derivative of $f(x, y)$ with respect to x is a function $\frac{\partial f}{\partial x}$ whose value at (x_0, y_0) is

- the rate of change of $f(x, y)$ when x is varying near $x = x_0$ and y is held constant at the value y_0, or,
- more precisely,

$$\frac{\partial f}{\partial x}(x_0, y_0) := \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} \qquad \text{(if the limit exists).}$$

Other notations:

$$f_x(x_0, y_0), \qquad \left(\frac{\partial f}{\partial x}\right)_0 .$$

The 0 subscript on the derivative means "evaluate me at (x_0, y_0)".

Question 8.3. What is the difference between $\frac{df}{dx}$ and $\frac{\partial f}{\partial x}$?

- $\frac{df}{dx}$ is used if f is a function of x alone.
- $\frac{\partial f}{\partial x}$ is used if f is a function of several variables but we are measuring the rate of change of f arising from a change in only the variable x.

The definition of $\frac{\partial f}{\partial y}$ is similar to the definition of $\frac{\partial f}{\partial x}$.

8.2.3. *How to compute* $\frac{\partial f}{\partial x}$.

View y as a constant and differentiate with respect to x. (And then evaluate at (x_0, y_0) if desired.) Example: If $f(x, y) = x^3 y^5$, then $f_x = 3x^2 y^5$ and $f_x(2, 1) = 12$.

Question 8.4. Let $f(x,y) = x^y$ for (x,y) in the half-plane $x > 0$. What is $\dfrac{\partial f}{\partial y}$?

Possible answers:

(1) $x^y \ln x$
(2) yx^{y-1}
(3) 0
(4) None of the above

Answer: (1). Computing $\dfrac{\partial f}{\partial y}$ is like computing

$$\frac{d}{dy} 2^y = \frac{d}{dy} e^{y \ln 2} = e^{y \ln 2} \ln 2 = 2^y \ln 2$$

except with a "constant" x in place of the 2. So the answer is $x^y \ln x$.

8.3. **Second partial derivatives.** To calculate the second partial derivative

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y} f_x = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f\right) = \frac{\partial^2 f}{\partial y\, \partial x},$$

first take the x-derivative of f, and then take the y-derivative of the result. The other second partial derivatives are f_{xx}, f_{yx}, and f_{yy}. For example,

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}.$$

For most functions arising in practice, including any function for which all the second partial derivatives are continuous, $f_{xy} = f_{yx}$. So usually you don't have to worry about the *order* in which you take derivatives.

8.4. **Total derivative.** If $f \colon \mathbb{R}^2 \to \mathbb{R}$ is a function of x and y, then the total derivative is the 1×2 matrix of functions

$$f'(x,y) := \begin{pmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \end{pmatrix},$$

whose value at each input point (x_0, y_0) is the 1×2 matrix of partial derivatives at that point. (Alternative notation: $(Df)(\mathbf{x})$.) The dimensions of the matrix are the same as they would be for a linear transformation $\mathbb{R}^2 \to \mathbb{R}$ (but f here doesn't have to be a linear transformation)

Example 8.5. Let $f(x,y) := x^3 y^5$. This is a function $f \colon \mathbb{R}^2 \to \mathbb{R}$, but it is not a linear transformation since $x^3 y^5$ is not just a linear combination of the input variables x and y. The total derivative of f is

$$f'(x,y) = \begin{pmatrix} 3x^2 y^5 & 5x^3 y^4 \end{pmatrix}$$

and its value at $(2,1)$ is the 1×2 matrix of numbers

$$f'(2,1) = \begin{pmatrix} 12 & 40 \end{pmatrix}.$$

In general, for a function

$$\mathbf{f}\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$$
$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longmapsto \begin{pmatrix} f_1(x_1,\dots,x_n) \\ \vdots \\ f_m(x_1,\dots,x_n) \end{pmatrix},$$

the total derivative of $\mathbf{f}$ is the $m \times n$ matrix of functions

$$\mathbf{f}'(\mathbf{x}) := \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix},$$

whose value at any input point in $\mathbb{R}^n$ is an $m \times n$ matrix of numbers. The m rows correspond to the coordinate functions $f_1, \dots, f_m$ of $\mathbf{f}$, and the n columns correspond to partial derivatives with respect to different input variables $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$.

Example 8.6. For the function

$$\mathbf{f}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x^2 y \\ x \sin y \end{pmatrix}$$

the total derivative is

$$\mathbf{f}'(x,y) = \begin{pmatrix} 2xy & x^2 \\ \sin y & x \cos y \end{pmatrix},$$

and its value at any input point (x_0, y_0) like $(2,3)$ is a 2×2 matrix of numbers.

8.5. **Linear approximation.**

8.5.1. *Warm-up: single-variable function.*

Question 8.7. How do you estimate $f(x)$ for x *near* 3?

Another way to state the question:

Question 8.8. How does $f(3 + \Delta x)$ compare to $f(3)$, when Δx is a small number?

(Below we'll use green for constants that do not depend on Δx.)

Answer (assuming that f is differentiable). The change Δx in the input gets approximately magnified by the derivative $f'(3)$:

$$\begin{aligned} \Delta f &\approx f'(3)\,\Delta x \\ f(3+\Delta x) - f(3) &\approx f'(3)\,\Delta x \\ f(3+\Delta x) &\approx f(3) + f'(3)\,\Delta x. \end{aligned}$$

□

More generally, for any initial input number x_0, there is the relative change formula

$$\boxed{\Delta f \approx f'(x_0)\,\Delta x.}$$

and the linear approximation formula

$$\boxed{f(x_0+\Delta x) \approx \underbrace{f(x_0)}_{\text{starting value}} + \underbrace{f'(x_0)\,\Delta x}_{\text{adjustment}}}$$

that gives the best linear approximation to $f(x)$ for $x := x_0 + \Delta x$ near x_0.

Another way to write the linear approximation formula (substitute $x = x_0 + \Delta x$, so $\Delta x = x - x_0$): When x is close to the number x_0,

$$f(x) \approx f(x_0) + f'(x_0)\,(x - x_0).$$

The following problem was not discussed in lecture.

Problem 8.9. Let $f(x) := \sqrt{x+1}$. Estimate $f(3.1)$.

Solution. We have $f(3) = 2$, $f'(x) = \frac{1}{2}(x+1)^{-1/2}$, $f'(3) = 1/4$, so $f(3.1) \approx f(3) + f'(3)(0.1) = 2 + (1/4)(0.1) = 2.025$. □

8.5.2. *Two-variable function.* For $f(x,y)$, the boxed formulas are the same except that $\mathbf{x}_0$ and $\Delta\mathbf{x}$ are vectors, and $f'(\mathbf{x}_0)$ is now the *total derivative*. For example, the relative change formula for $f(x,y)$ says

$$\begin{aligned} \Delta f &\approx f'(\mathbf{x}_0)\,\Delta\mathbf{x} \\ &= \begin{pmatrix} \left(\dfrac{\partial f}{\partial x}\right)_0 & \left(\dfrac{\partial f}{\partial y}\right)_0 \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \\ &= \left(\dfrac{\partial f}{\partial x}\right)_0 \Delta x + \left(\dfrac{\partial f}{\partial y}\right)_0 \Delta y, \end{aligned}$$

so the relative change formula expands to become

$$\boxed{\Delta f \approx \left(\dfrac{\partial f}{\partial x}\right)_0 \Delta x + \left(\dfrac{\partial f}{\partial y}\right)_0 \Delta y}$$

(it tells you approximately *how much* f changes in response to changes in x and y).
The new value $f(x_0+\Delta x, y_0+\Delta y)$ is the old value $f(x_0,y_0)$ plus the change Δf; this gives the linear approximation formula

$$\boxed{f(x_0+\Delta x, y_0+\Delta y) \approx \underbrace{f(x_0,y_0)}_{\text{starting value}} + \underbrace{\left(\frac{\partial f}{\partial x}\right)_0 \Delta x}_{\text{adjustment from } \Delta x} + \underbrace{\left(\frac{\partial f}{\partial y}\right)_0 \Delta y}_{\text{adjustment from } \Delta y}.}$$

Another way to write the linear approximation formula:

$$f(x,y) \approx f(x_0,y_0) + \left(\frac{\partial f}{\partial x}\right)_0 (x-x_0) + \left(\frac{\partial f}{\partial y}\right)_0 (y-y_0).$$

The approximation can be expected to be reasonably good if (x,y) is close to (x_0,y_0) and the partial derivatives are continuous in a neighborhood of (x_0,y_0), so that they don't change too suddenly.

Tuesday, September 28

Problem 8.10. A point P in $\mathbb{R}^2$ is near $(-12,5)$, but its coordinates could be off by as much as 0.1 each. How far is P from the origin? Estimate the maximum error.

Solution. Let $f(x,y)=\sqrt{x^2+y^2}$. Then $f_x = \frac{x}{\sqrt{x^2+y^2}}$ and $f_y = \frac{y}{\sqrt{x^2+y^2}}$, so

$$\begin{aligned} f(x,y) &\approx f(-12,5) + f_x(-12,5)\,\Delta x + f_y(-12,5)\,\Delta y \\ &\approx 13 - \frac{12}{13}\,\Delta x + \frac{5}{13}\,\Delta y, \end{aligned}$$

which is 13 with maximum error of absolute value

$$\approx \frac{12}{13}(0.1) + \frac{5}{13}(0.1) \approx 0.13.$$

(The worst error occurs when $(\Delta x,\Delta y)=(-0.1,0.1)$ or $(\Delta x,\Delta y)=(0.1,-0.1)$.) □

8.5.3. *Three-variable function and beyond.* This section was not explicitly covered in lecture.

Approximation can be done for functions of more than two variables too.

Question 8.11. What is the best linear polynomial approximation to a 3-variable function $f(x,y,z)$ for inputs near a starting point (x_0,y_0,z_0)?

As one moves from (x_0,y_0,z_0) to (x,y,z), the relative change formula says that Δf is caused by Δx, Δy, Δz, each magnified by the corresponding partial derivative:

$$\underbrace{\Delta f}_{\text{change in } f} \approx (f_x)_0\,\Delta x + (f_y)_0\,\Delta y + (f_z)_0\,\Delta z.$$

This gives the linear approximation formula

$$\begin{aligned} f(x,y,z) &= \underbrace{f(x_0,y_0,z_0)}_{\text{starting value}} + \underbrace{\Delta f}_{\text{change in } f} \\ f(x,y,z) &\approx f(x_0,y_0,z_0) + (f_x)_0\,(x-x_0) + (f_y)_0\,(y-y_0) + (f_z)_0\,(z-z_0). \end{aligned}$$

9. Maximum and minimum

9.1. **Review of 1-variable max/min.**

1. Identify the function $f(x)$ to be maximized.
2. Identify the domain I on which f is to be maximized. This is the set of inputs of f that are under consideration. It is not necessarily the whole real line $\mathbb{R}$. For a 1-variable function, usually the domain I is an interval in the real line, such as $(-\infty, 3]$ or $(-4, -3)$ or ....
3. Check all of the following to find *potential* maxima:
   A. critical points in I (points where $f'(x) = 0$)
   B. points in I where $f'(x)$ is undefined
   C. behavior of $f(x)$ as $x \to \infty$ or $x \to -\infty$ (if applicable)
   D. boundary behavior (value/limit at the endpoints of I).

   The global max is to be found among these points, so evaluate f at these points to find the maximum value. (*Warning*: If the values of f become larger and larger as x approaches $\pm\infty$ or an endpoint of I, then the global max *does not exist*.)

Here are some examples illustrating A–D above:

- The global max of $f(x) := -x^2$ on $\mathbb{R}$ occurs at $x = 0$ where the derivative $f'(x) = -2x$ becomes zero.
- The function $f(x) := 5 - |x|$ on $\mathbb{R}$ has a global max at $x = 0$, where $f'(x)$ is undefined.
- Consider the function $f(x) := x^3 - 3x$ on $\mathbb{R}$. Its derivative $f'(x) = 3x^2 - 3$ becomes 0 at $x = \pm 1$. But neither $x = -1$ nor $x = 1$ is a global max; in fact, $f(x) \to +\infty$ as $x \to +\infty$, so a global max does not exist.
- Consider $f(x) := x^3 - 3x$ again, but this time *restricted to the interval* $[-10, 10]$. As before, $f'(x) = 0$ if and only if $x = \pm 1$. We have $f(-1) = 2$ and $f(1) = -2$, but the global max on $[-10, 10]$ actually occurs on the boundary of the interval, where $f(10) = 970$.

9.2. **Constraint inequalities and constraint equations.** Not covered at this point in lecture, though it should have been.

For a function in more than one variable, the domain is typically more complicated than an interval.

Example 9.1. The set of points (x, y) in $\mathbb{R}^2$ satisfying the constraint inequalities $x \geq 0$, $y \geq 0$, and $2x + 3y < 6$ is a solid right triangle T (with the hypotenuse missing). A real-valued function f could have domain T, in which case one would write $f\colon T \to \mathbb{R}$.

Example 9.2. The set of points (x, y, z) in $\mathbb{R}^3$ satisfying the constraint equation $x^2+y^2+z^2 = 4$ is a sphere of radius 2. In contrast, $x^2 + y^2 + z^2 \leq 4$ defines a solid *ball* in $\mathbb{R}^3$. Similarly, $x^2 + y^2 = 9$ defines a circle in $\mathbb{R}^2$, while $x^2 + y^2 \leq 9$ defines a disk.

9.3. **Dimension.** Not covered at this point in lecture, though it should have been.

The dimension of a set is how many parameters are needed to specify a point in the set. (To make this definition precise would require more care; we won't worry about this in 18.02.)

Examples:

- Any given circle is 1-dimensional (need only one parameter θ to distinguish different points on it).
- A disk is 2-dimensional (need r and θ).
- The plane $\mathbb{R}^2$ itself is 2-dimensional (need x and y).
- A point is 0-dimensional.

Usually, each constraint equation reduces the dimension by 1. This leads to the following:

> Rule of thumb: Usually, if a set defined by e constraint equations in n variables, it will be $(n-e)$-dimensional.

Example 9.3. The constraint equation

$$x + 2y + 3z = 5$$

defines a 2-dimensional set (a plane).

Example 9.4. The constraint equations

$$x^2 + y^2 + z^2 = 100$$
$$x + 2y + 3z = 5$$

define a 1-dimensional set (a circle, arising as the intersection of a sphere and a plane).

Warning 9.5. It is not *always* true that each constraint equation reduces the dimension by 1.

Constraint inequalities usually do not affect the dimension.

9.4. **Boundary.** Not covered at this point in lecture, though it should have been.

Rather than define the boundary of a set precisely, let's give some examples.

Question 9.6. What are the boundaries of the following sets?

- The disk $x^2 + y^2 \leq 1$.
  Answer: The circle $x^2 + y^2 = 1$.
- The first quadrant, where $x \geq 0$ and $y \geq 0$.
  Answer: The nonnegative parts of the x- and y-axes.
- The interval consisting of numbers x such that $3 \leq x \leq 5$.
  Answer: The two points $x = 3$ and $x = 5$.
- The circle $x^2 + y^2 = 1$.
  Answer: No constraint inequalities, so no boundary!

- The arc of the circle $x^2 + y^2 = 1$ given by $0 \leq \theta \leq \pi/4$.
  Answer: The points $(1,0)$ (where $\theta = 0$) and $(\sqrt{2}/2, \sqrt{2}/2)$ (where $\theta = \pi/4$).

General rule of thumb: To find the boundary, take one of the constraint inequalities and change it to $=$; this gives one piece of the boundary. Then do this for each constraint inequality to get all the pieces of the boundary.

In particular, *if there are no constraint inequalities, there is no boundary.*

Another rule of thumb: For a set R,

$$\dim(\text{boundary of } R) = \dim R - 1$$

(if there is a boundary at all).

Example 9.7. A disk in $\mathbb{R}^2$ is 2-dimensional. Its boundary is a circle, which is 1-dimensional.

9.5. **Critical points.**

Definition 9.8. A critical point for $f(x,y)$ is a point (a,b) such that $f_x(a,b) = 0$ and $f_y(a,b) = 0$.

Question 9.9. What are the critical points of $f(x,y) := x^4 + y^3 - 3y$?

Solution: Compute the partial derivatives, and set them equal to 0:

$$4x^3 = 0$$
$$3y^2 - 3 = 0.$$

Solving this system gives $x = 0$ and $y = \pm 1$. So the critical points are $(0,1)$ and $(0,-1)$.

9.6. **Global and local extrema.**

Definition 9.10. The function $f(x,y)$ has a global maximum at the point (a,b) if $f(x,y) \leq f(a,b)$ for all points (x,y) in the domain of f (that is, for all points (x,y) where $f(x,y)$ is defined).

Global minimum is similar.

Example 9.11. If $f(x,y) = (x-y)^2 + 5$, then every point along the line $x = y$ is a global minimum.

Definition 9.12. The function $f(x,y)$ has a local maximum at the point (a,b) if there exists some number $\epsilon > 0$ such that at every point (x,y) in the domain whose distance to (a,b) is less than ϵ, the inequality $f(x,y) \leq f(a,b)$ holds. In other words, (a,b) might not be a global maximum, but it becomes a maximum if one restricts the domain to a small neighborhood of (a,b).

Local minimum is similar.

Every global max is automatically a local max.

Drew a level curve diagram, waved the chalk over it, and asked the students to yell whenever it crossed a local min or local max.

Theorem 9.13. *Every local max (or local min) is a critical point, assuming that the partial derivatives exist at the point being tested.*

Proof. The point has to be a local max for the partial functions obtained by plugging a number into one of the variables, so by one-variable calculus, the derivatives of the partial functions are 0 there if they exist. □

Question 9.14. True or false: Every critical point of a function $f(x, y)$ is either a local min or a local max. (Hint: What happens for functions of 1 variable?)

Answer: False. Here are two counterexamples.

(1) $f(x, y) := x^3$ has a critical point at $(0, 0)$, but it is not a local max (because there are nearby points to the right where $f(x, y) > 0$), and not a local min (because there are nearby points to the left where $f(x, y) < 0$).

$$\begin{array}{ccccccc} -27 & -8 & -1 & 0 & 1 & 8 & 27 \\ -27 & -8 & -1 & 0 & 1 & 8 & 27 \\ -27 & -8 & -1 & 0 & 1 & 8 & 27 \\ -27 & -8 & -1 & 0 & 1 & 8 & 27 \\ -27 & -8 & -1 & 0 & 1 & 8 & 27 \\ -27 & -8 & -1 & 0 & 1 & 8 & 27 \\ -27 & -8 & -1 & 0 & 1 & 8 & 27 \end{array}$$

(2) $f(x, y) := x^2 - y^2$ has a critical point at $(0, 0)$, but it is neither a local max nor a local min, because there are nearby points (on one axis or the other) where the value is larger or smaller than 0.

$$\begin{array}{ccccccc} 0 & -5 & -8 & -9 & -8 & -5 & 0 \\ 5 & 0 & -3 & -4 & -3 & 0 & 5 \\ 8 & 3 & 0 & -1 & 0 & 3 & 8 \\ 9 & 4 & 1 & 0 & 1 & 4 & 9 \\ 8 & 3 & 0 & -1 & 0 & 3 & 8 \\ 5 & 0 & -3 & -4 & -3 & 0 & 5 \\ 0 & -5 & -8 & -9 & -8 & -5 & 0 \end{array}$$

9.7. Solving unconstrained max/min problems.

General method for solving a max/min problem (without constraint equations):

1. Identify the function $f(x, y)$ to be maximized.

2. Identify the domain $\mathcal{R}$ on which f is to be maximized.
   - Easy case: only constraint inequalities or no constraints, so that the dimension of $\mathcal{R}$ equals the number of variables, which for this function is 2. (Example: constraint inequalities $x, y > 0$ defining the first quadrant.)
   - Hard case: constraint equation(s), so that the dimension of $\mathcal{R}$ is less than the number of variables. (Example: constraint equation $x^2 + y^2 = 1$ defining the unit circle, which is of dimension only 1.)

   Assume that we are in the easy case. (In the hard case, a more sophisticated method is needed: *Lagrange multipliers.*)
3. Check all of the following to find potential maxima:
   A. critical points in $\mathcal{R}$ (points where $f_x = 0$ and $f_y = 0$ simultaneously)
   B. points in $\mathcal{R}$ where f_x or f_y is undefined
   C. behavior at ∞ (what happens to f as $(x, y) \to \infty$ in $\mathcal{R}$?)
   D. boundary behavior (if there are constraint inequalities) — this may lead to another, lower-dimensional max/min problem (and, even in the "easy" case, this might require Lagrange multipliers if you can't parametrize the boundary).

   The global max (a, b), if it exists, is to be found among these points, so evaluate f at these points to find the maximum value. (*Warning*: If the values of f become larger and larger as (x, y) approaches the boundary or ∞, then the global max *does not exist.*)

The method also works for finding max/min of functions in 3 variables on a 3-dimensional region, and so on.

Terminology: the global maximum is the location (a, b), but the maximum value is $f(a, b)$. In any max/min problem, you will need to determine which is being asked for.

Midterm 1 on Thursday, September 30.

Friday, October 1

Example 9.15. Find the point on the surface $xyz^2 = 2$ closest to the origin.

Solution: There is a constraint equation (3 variables, but only 2-dimensional domain), but fortunately we can eliminate z by solving for z.

Exploit symmetry: The surface lies in the regions where $xy > 0$, which means that $x, y > 0$ or $x, y < 0$. Because of the symmetries

$$(x, y, z) \mapsto (-x, -y, z)$$
$$(x, y, z) \mapsto (x, y, -z)$$

that preserve the equation of the surface, it's enough to consider the parts with $x, y > 0$ and $z > 0$: this is the set of points of the form

$$\left(x, y, \sqrt{\frac{2}{xy}}\right)$$

for $x, y > 0$.

1. What function do we want to minimize? Shortcut: The point where distance is minimized is the same as the point where distance^2 is minimized; let's use distance^2 since it has a simpler formula, namely

$$f(x, y) := x^2 + y^2 + \frac{2}{xy}.$$

2. What is the region $\mathcal{R}$? Since we are considering $x, y > 0$ only, $\mathcal{R}$ is the first quadrant. (It's 2-dimensional, so we don't need Lagrange multipliers.)
3. A. Critical points: solve

$$\begin{aligned} f_x &= 2x - \frac{2}{x^2 y} = 0 \\ f_y &= 2y - \frac{2}{xy^2} = 0. \end{aligned}$$

   These lead to

$$\begin{aligned} x^3 y &= 1 \\ xy^3 &= 1, \end{aligned}$$

   which imply $x^3 y = xy^3$, and we may divide by xy to get $x^2 = y^2$, so $x = y$ or $x = -y$. Since $x, y > 0$, we must have $x = y$. Then $x^4 = 1$, and $x > 0$ so $x = 1$. Thus the only critical point is $(x, y) = (1, 1)$.

   B. Points in $\mathcal{R}$ where f_x or f_y is undefined: none. (The points where $x = 0$ or $y = 0$ are not part of $\mathcal{R}$.)

   C. Behavior as $(x, y) \to \infty$: as x or y grows, the function $f(x, y)$ tends to $+\infty$ (because of the x^2 and y^2 terms in $f(x, y)$), so it's not approaching a minimum out there.

   D. Boundary behavior: As $x \to 0$ (from the right, while y is bounded), the function $f(x, y)$ tends to $+\infty$ (because of the $\frac{2}{xy}$ term in $f(x, y)$), so it's not approaching a minimum there. Same if $y \to 0$.

Conclusion: $f(x, y)$ is minimized at $(x, y) = (1, 1)$, and $z = \sqrt{2}$ there, so the point is $(1, 1, \sqrt{2})$. The other symmetric points are $(1, 1, -\sqrt{2})$ and $(-1, -1, \sqrt{2})$ and $(-1, -1, -\sqrt{2})$. (And the minimum *value* of the distance is $\sqrt{D(1, 1)} = \sqrt{1 + 1 + 2} = 2$.)

The following good question came from a student:

Question 9.16. The point $(1, 1, \sqrt{2})$ we computed seems to be the point with minimum height, closest to the xy-plane (among those on the piece of the surface above $\mathcal{R}$), but how do we know that there isn't another point on the surface that is even closer to the origin?

Answer: The function we minimized was not the z-coordinate of a point on the surface, but the squared distance to the origin, so the point we computed is truly the one closest to the origin.

Another way of saying this: we found the lowest point on the graph $z = f(x, y)$ above $\mathcal{R}$, but this graph is different from the piece of the original surface, which is given by $z = \sqrt{\dfrac{2}{xy}}$.

If we had wanted the point closest to the xy-plane, we would have instead minimized the z-coordinate, which is given by the function $\sqrt{\dfrac{2}{xy}}$; in that case, the minimum does not exist since $\sqrt{\dfrac{2}{xy}}$ can be made arbitrarily close to 0 by taking x and y to be large positive numbers.

9.8. **Least squares interpolation.** Problem: Given data points (x_1, y_1), …, (x_n, y_n) that approximately lie on an unknown line $y = ax + b$, find the line.

What are the unknowns here? a and b!

Given a candidate line $y = ax + b$, how do we measure how good of an approximation it is? For each input x_i, the line predicts an output of $ax_i + b$, but the actual output was y_i, so the error in the prediction is $|y_i - (ax_i + b)|$. Then the total error from all the data points would be

$$\sum_{i=1}^{n} |y_i - (ax_i + b)|,$$

and we want to find a, b that make this small. Because of the absolute values, the partial derivatives of this function do not exist everywhere, which complicates the minimization problem, so instead we try to minimize the sum of the *squares* of the errors.

Definition 9.17. The least squares line (the "best" line) is the one for which

$$D := \sum_{i=1}^{n} (y_i - (ax_i + b))^2$$

is minimum.

This D is a function $D(a, b)$.

The minimum occurs where $\frac{\partial D}{\partial a}$ and $\frac{\partial D}{\partial b}$ are both 0. Instead of expanding out D, use the chain rule:

$$\frac{\partial D}{\partial a} = \sum_{i=1}^{n} 2(y_i - (ax_i + b))(-x_i) = 0$$
$$\frac{\partial D}{\partial b} = \sum_{i=1}^{n} 2(y_i - (ax_i + b))(-1) = 0.$$

This is a system of two linear equations in a and b (remember: the x_i and y_i are given numbers). Solving for a and b gives the best line.

The same method can be used to approximate data points by the graph of an unknown quadratic function $y = ax^2 + bx + c$.

How do you know what kind of function to use? Maybe the source of the data suggests that a particular shape of function is the right answer. Or maybe the data themselves look as if they can be fitted with a parabola, or ....

If you suspect that x and y are related by a power law, a function $y = cx^d$, then plot $\ln x$ versus $\ln y$ so that the relationship is linear again:

$$\ln y = C + d \ln x,$$

where $C := \ln c$ and d are constants to be solved for. In other words, find the line that best approximates the points $(\ln x_i, \ln y_i)$; this gives C and d.

9.9. **Second derivative test for two-variable functions.**

9.9.1. *Introduction: estimating a two-variable polynomial near* $(0,0)$. The degree of a monomial is the sum of the exponents; for example, the degree of x^2y^3 is 5.

The monomials in a two-variable polynomial $f(x,y)$ can be grouped by degree, and in estimating $f(x,y)$ near $(0,0)$, it is the *lowest* degree terms that matter most (x is much larger than x^2, for instance).

Example 9.18. Suppose that

$$\begin{aligned} f(x,y) = \quad & 5 && \text{(the constant term is } f(0,0)) \\ & + 2x + 7y && \text{(these coefficients are } f_x(0,0),\ f_y(0,0)) \\ & + x^2 + 6xy + 11y^2 \\ & + \cdots . \end{aligned}$$

After the 5, the terms that matter most for (x,y) near $(0,0)$ are the $2x$ and $7y$. The best linear approximation is $5 + 2x + 7y$. As one moves from $(0,0)$ to the right, $f(x,y)$ increases; as one moves to the left $f(x,y)$ decreases; and so on — the monomials of degree ≥ 2 are too small to interfere with this when (x,y) is close to $(0,0)$.

A linear combination of the variables, like $2x + 7y$, is called a linear form. A linear combination of the degree 2 monomials, like $x^2 + 6xy + 11y^2$, is called a quadratic form.

Example 9.19. Suppose that

$$\begin{aligned} f(x,y) = \quad & 5 && \text{(the constant term is } f(0,0)) \\ & + 0x + 0y && (f_x(0,0) \text{ and } f_y(0,0) \text{ are } 0) \\ & + x^2 + 6xy + 11y^2 \\ & + \cdots . \end{aligned}$$

This time $f(x,y)$ has a critical point at $(0,0)$. To determine whether nearby values are greater or less than the value $f(0,0) = 5$, what matters most is whether the nearby values of the quadratic form $x^2 + 6xy + 11y^2$ are positive or negative.

To understand functions like this, we'll develop a general method for testing whether a quadratic form is positive/negative near $(0,0)$.

9.9.2. *Behavior of quadratic forms.* Examples of critical point behavior at $(0,0)$:

- $x^2 + y^2$ has a local min,
- $-x^2 - y^2$ has a local max,
- $x^2 - y^2$ has a saddle point.

(A saddle point is a critical point that is neither a local min nor a local max.[4])

Example 9.20. $f(x,y) = x^2 + 6xy + 11y^2$. Complete the square: it's $(x+3y)^2 + 2y^2$, so it has a local min at $(0,0)$.

A general quadratic form can be written in the form

$$f(x,y) := \frac{1}{2}Ax^2 + Bxy + \frac{1}{2}Cy^2.$$

(The reason for putting $\frac{1}{2}$ in the formula is so that

$$f_{xx} = A, \qquad f_{xy} = f_{yx} = B, \qquad f_{yy} = C$$

at $(0,0)$.) Let's suppose that $A \neq 0$. Completing the square rewrites $f(x,y)$ as

$$\frac{A}{2}\left(x + \frac{B}{A}y\right)^2 + \left(\frac{AC - B^2}{2A}\right)y^2,$$

and its behavior depends on the signs of the coefficients in front of the squares.

[4]Sometimes people require more before they call a critical point a "saddle point": they may require the graph to curve upward along one line through the point, and downwards along a line in a different direction, like a saddle on a horse or like a Pringles potato chip.

Case	Conclusion
$AC - B^2 > 0$ and $A > 0$	local min
$AC - B^2 > 0$ and $A < 0$	local max
$AC - B^2 < 0$	saddle point
$AC - B^2 = 0$	inconclusive

The formula $AC - B^2$ can be remembered as a determinant:

$$AC - B^2 = \begin{vmatrix} A & B \\ B & C \end{vmatrix} = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}.$$

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9.9.3. *Second derivative test.* The same table lets one analyze functions that are more complicated than quadratic forms:

Second derivative test: Suppose that $f(x,y)$ has a critical point at (a,b) (that is, $f_x(a,b) = 0$ and $f_y(a,b) = 0$), and that all second derivatives exist and are continuous in a neighborhood of (a,b). Define

$$A := f_{xx}(a,b), \qquad B := f_{xy}(a,b), \qquad C := f_{yy}(a,b).$$

Then the type of the critical point is given by the table above. This works even when $A = 0$. "Inconclusive" means that it could be anything: a local min, a local max, or a saddle point (possibly a weird one).

Warning 9.21*.* It's not OK to use the second derivative test when there is a constraint equation!

9.9.4. *An example.*

Question 9.22. What kind of critical point does

$$f(x,y) := xy(x-y) = x^2y - xy^2$$

have at $(0,0)$?

Calculate all the partial derivatives up to second order:

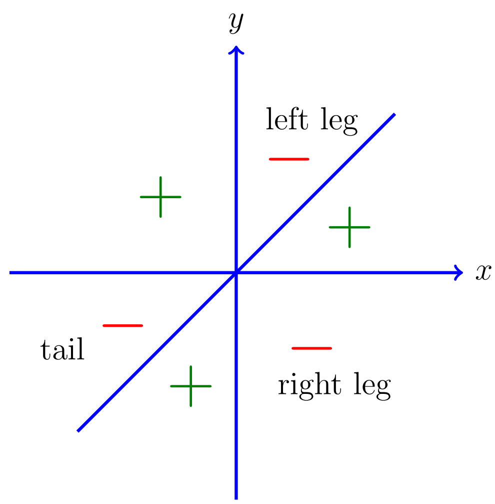
$$\begin{aligned} f_x &= 2xy - y^2 \\ f_y &= x^2 - 2xy \\ f_{xx} &= 2y \\ f_{xy} = f_{yx} &= 2x - 2y \\ f_{yy} &= -2x. \end{aligned}$$

Since f_x and f_y are 0 at $(0,0)$, there is a critical point there. Next,

$$\begin{aligned} A &:= f_{xx}(0,0) = 0 \\ B &:= f_{xy}(0,0) = 0 \\ C &:= f_{yy}(0,0) = 0. \end{aligned}$$

Since $AC - B^2 = 0$, the second derivative test is inconclusive.

So, what next? The lines $x = 0$, $y = 0$ and $x - y = 0$ divide the plane into six regions on which $f(x, y)$ is alternately positive or negative (to see which are which, evaluate f at a point in each region). So $f(x, y)$ has neither a local min nor a local max at $(0,0)$. In fact, it has what is called a *monkey saddle*: there are three negative regions, two for the legs and one for the tail.



Challenge: Find a surface that has an octopus saddle!

10. More on derivatives of a multivariable function

10.1. Differentials. The total differential of $f(x, y)$ is

$$\begin{aligned} df &:= \underbrace{f'(\mathbf{x})}_{\text{total derivative}} \, d\mathbf{x} \\ &= \begin{pmatrix} f_x & f_y \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}. \end{aligned}$$

In other words,

$$\boxed{df := f_x \, dx + f_y \, dy.}$$

This should remind you of the relative change formula

$$\Delta f \approx f_x \, \Delta x + f_y \, \Delta y.$$

10.2. **Chain rule.** In 18.01, if f is a function of x, and x is a function of t, then the chain rule says that

$$\frac{d}{dt} f(x(t)) = f'(x(t))\, x'(t).$$

If instead $f = f(x, y)$, where $x = x(t)$ and $y = y(t)$, then in terms of the vector quantity $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ one has $f = f(\mathbf{x})$ and $\mathbf{x} = \mathbf{x}(t)$, and the analogous formula is

$$\begin{aligned} \frac{d}{dt} f(\mathbf{x}(t)) &= \underset{\text{total derivative}}{f'(\mathbf{x}(t))} \quad \mathbf{x}'(t) \\ &= \begin{pmatrix} f_x & f_y \end{pmatrix} \begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix}. \end{aligned}$$

Thus the (multivariable) chain rule says

$$\boxed{\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}.}$$

One way to remember this: "divide $df := f_x\, dx + f_y\, dy$ by dt".

The chain rule can also be written as

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

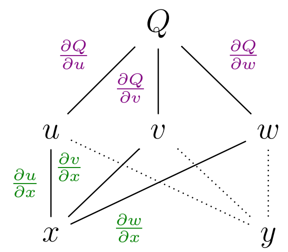
One can also use a dependency diagram to remember the chain rule:

$$\begin{tikzcd} & f \arrow[dl, dash, "\frac{\partial f}{\partial x}"'] \arrow[dr, dash, "\frac{\partial f}{\partial y}"] & \\ x \arrow[dr, dash, "\frac{dx}{dt}"'] & & y \arrow[dl, dash, "\frac{dy}{dt}"] \\ & t & \end{tikzcd}$$

This shows that f is a function of x, y, that x is a function of t, and that y is a function of t. A change in t is magnified by the factor dx/dt to produce a change in x, which is magnified by $\partial f/\partial x$ to produce a change in f; simultaneously, the change in t is magnified by the factor dy/dt to produce a change in y, which is magnified by $\partial f/\partial y$ to produce a change in f. Thus the total magnification factor by which changes in t produce changes in f is

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Example 10.1 (more complicated version)**.** Suppose that $Q = Q(u, v, w)$ where $u = u(x, y)$, $v = v(x, y)$, and $w = w(x, y)$. The dependency diagram



shows that

$$\frac{\partial Q}{\partial x} = \frac{\partial Q}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial Q}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial Q}{\partial w}\frac{\partial w}{\partial x}$$

and $\dfrac{\partial Q}{\partial y}$ is similar.

Most general chain rule: If $\mathbf{f}$ and $\mathbf{g}$ are differentiable functions with vector input, vector output, such that $\mathbf{f}(\mathbf{g}(\mathbf{x}))$ is defined, then the total derivative of $\mathbf{f}(\mathbf{g}(\mathbf{x}))$ is the matrix product $\mathbf{f}'(\mathbf{g}(\mathbf{x}))\,\mathbf{g}'(\mathbf{x})$.

10.3. **Using the chain rule to derive the product and quotient rules.**

Example 10.2 (Product rule)**.** If $f = uv$ where $u = u(x)$ and $v = v(x)$, then

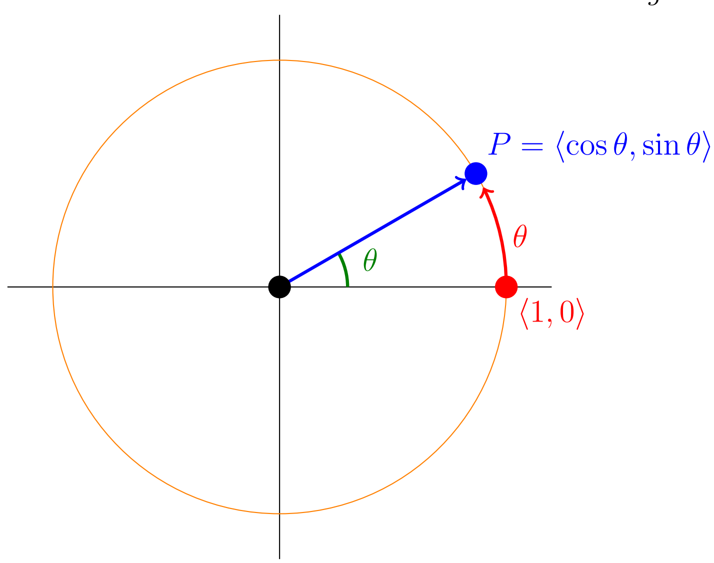
$$\frac{df}{dx} = f_u\frac{du}{dx} + f_v\frac{dv}{dx}.$$

In other words,

$$\frac{d}{dx}(uv) = vu' + uv'.$$

The quotient rule can also be obtained this way.

10.4. **Review: definitions of cos and sin.** What is the *meaning* of $\cos\theta$ and $\sin\theta$?



Definition 10.3. Draw a unit circle centered at the origin. Let P be the point reached by going θ units counterclockwise from $(1, 0)$. (If θ is negative, this means going clockwise.) Then

$$\cos\theta := x\text{-coordinate of } P$$
$$\sin\theta := y\text{-coordinate of } P.$$

Example 10.4. $\cos\pi = (x\text{-coordinate of } (-1, 0)) = -1$.

10.5. **Review: polar coordinates.** A point P in the plane can be specified in rectangular coordinates x, y or in polar coordinates r, θ. Here r means OP, the distance from P to the origin, and θ means the angle measured counterclockwise from the positive x-axis to the ray $\overrightarrow{OP}$. The value of θ is not completely determined by P, since one can add any integer times 2π.

Then

$$\mathbf{P} = \begin{pmatrix} x \\ y \end{pmatrix} = r \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix},$$

and we get the conversion formulas

$$x = r\cos\theta$$
$$y = r\sin\theta.$$

for going from r, θ to x, y.

Problem 10.5. In reverse, given x, y, how can we find r, θ?

Solution. Finding r is easy: $r = \sqrt{x^2 + y^2}$.

Finding θ is trickier. If $x = 0$ or $y = 0$, then (x, y) is on one of the axes and θ will be an appropriate integer multiple of $\pi/2$. So assume that x and y are nonzero. The correct θ satisfies $\tan\theta = y/x$, but there are also other angles that satisfy this equation, namely $\theta + k\pi$ for any integer k. Some of these other angles point in the opposite direction. In particular, $\tan^{-1}(y/x)$ might be in the opposite direction. By definition, the angle $\tan^{-1}(y/x)$ always lies in $(-\pi/2, \pi/2)$, pointing into the right half plane, so it will be wrong when (x, y) lies in the left half plane; in that case, adjust $\tan^{-1}(y/x)$ by adding or subtracting π to get a possible θ. Finally, if desired, add an integer multiple of 2π to get the principal value, which is the θ in the interval $(-\pi, \pi]$. The "2-variable arctangent function" in Mathematica and MATLAB looks not only at y/x, but also at the point (x, y), to calculate a correct θ:

$$\theta = \underbrace{\texttt{ArcTan[x,y]}}_{\text{Mathematica}} = \underbrace{\texttt{atan2(y,x)}}_{\text{MATLAB}}.$$

Warning: Some people require $\theta \in [0, 2\pi)$ instead.

Warning: In MATLAB, be careful to use (y, x) and not (x, y).

An alternative approach to finding θ is to use $\cos\theta = x/r$ or $\sin\theta = y/r$, but again one may need to adjust to get the quadrant right. $\square$

Problem 10.6. Convert $(x, y) = (-1, -1)$ to polar coordinates.

Solution. First, $r = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$. Evaulating $\tan^{-1}(y/x)$ at $(-1, -1)$ gives $\tan^{-1}(1) = \pi/4$, pointing in the direction *opposite* to $(-1, -1)$. Subtracting π gives $-3\pi/4$ as a possible θ. (The other possible angles θ are the numbers $-3\pi/4 + 2\pi k$, where k can be any integer.) $\square$

Question 10.7. At a time when a particle is at $(4, 3)$ and has velocity vector $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ (in rectangular coordinates), what is $\frac{dr}{dt}$?

Solution 1. View $r = r(x, y)$ and $x = x(t)$, $y = y(t)$. Namely, start with $r = \sqrt{x^2 + y^2}$, and apply the chain rule

$$\frac{dr}{dt} = \frac{\partial r}{\partial x}\frac{dx}{dt} + \frac{\partial r}{\partial y}\frac{dy}{dt}.$$

Here

$$\begin{aligned}
\frac{\partial r}{\partial x} &= \frac{1}{2}(x^2 + y^2)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 + y^2}} \\
\frac{\partial r}{\partial y} &= \frac{1}{2}(x^2 + y^2)^{-1/2}(2y) = \frac{y}{\sqrt{x^2 + y^2}},
\end{aligned}$$

which at $(4, 3)$ are $4/5$ and $3/5$. Also, at the given time $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, so

$$\begin{aligned}
\frac{dr}{dt} &= \frac{\partial r}{\partial x}\frac{dx}{dt} + \frac{\partial r}{\partial y}\frac{dy}{dt} \\
&= \frac{4}{5}(3) + \frac{3}{5}(-1) \\
&= \frac{9}{5}.
\end{aligned}$$ $\square$

Solution 2 (mostly avoiding square roots). Start with

$$r^2 = x^2 + y^2.$$

Apply $\frac{d}{dt}$:

$$2r\frac{dr}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}.$$

At the given time, $r = 5$, so this becomes

$$2(5)\frac{dr}{dt} = 2(4)(3) + 2(3)(-1),$$

which again leads to

$$\frac{dr}{dt} = \frac{9}{5}. \qquad \square$$

11. Complex numbers

Complex numbers are expressions of the form $a + bi$, where a and b are real numbers, and i is a new symbol. Multiplication of complex numbers will eventually be defined so that $i^2 = -1$. (Electrical engineers sometimes write j instead of i, because they want to reserve i for current, but everybody else thinks that's weird.)

Just as the set of all real numbers is denoted $\mathbb{R}$, the set of all complex numbers is denoted $\mathbb{C}$. The notation "$\alpha \in \mathbb{C}$" means literally that α is an element of the set of complex numbers, so it is a short way of saying "α is a complex number".

Question 11.1. Is 9 a real number or a complex number?

Possible answers:

1. real number
2. complex number
3. both
4. neither

Answer: Both, because 9 can be identified with $9 + 0i$.

11.1. Operations on complex numbers.

real part $\quad \operatorname{Re}(a + bi) := a$

imaginary part $\quad \operatorname{Im}(a + bi) := b \qquad$ (Note: It is b, not bi, so $\operatorname{Im}(a + bi)$ is real!)

complex conjugate $\quad \overline{a + bi} := a - bi \qquad$ (negate the imaginary component)

Question 11.2. What is $\operatorname{Im}(17 - 83i)$?

Possible answers:

- 17
- $17i$
- 83
- -83
- $83i$
- $-83i$

Answer: The imaginary part is -83, without the i.

(In lecture there was a joke about the Greek letter Ξ; you had to be there.)

One can add, subtract, multiply, and divide complex numbers (except for division by 0). Addition, subtraction, and multiplication are defined as for polynomials, except that after multiplication one simplifies by using $i^2 = -1$; for example,

$$\begin{aligned}(2+3i)(1-5i) &= 2 - 7i - 15i^2 \\ &= 17 - 7i.\end{aligned}$$

To divide z by w, multiply z/w by $\overline{w}/\overline{w}$ so that the denominator becomes real; for example,

$$\frac{2+3i}{1-5i} = \frac{2+3i}{1-5i} \cdot \frac{1+5i}{1+5i} = \frac{2+13i+15i^2}{1-25i^2} = \frac{-13+13i}{26} = -\frac{1}{2} + \frac{1}{2}i.$$

The arithmetic operations on complex numbers satisfy the same properties as for real numbers ($zw = wz$ and so on). The mathematical jargon for this is that $\mathbb{C}$, like $\mathbb{R}$, is a field. In particular, for any complex number z and integer n, the nth power z^n can be defined in the usual way (need $z \neq 0$ if $n < 0$); e.g., $z^3 := zzz$, $z^0 := 1$, $z^{-3} := 1/z^3$. (Warning: Although there is a way to define z^n also for a *complex* number n, when $z \neq 0$, it turns out that z^n has more than one possible value for non-integral n, so it is ambiguous notation.)

If you change every i in the universe to $-i$ (that is, take the complex conjugate everywhere), then all true statements remain true. For example, $i^2 = -1$ becomes $(-i)^2 = -1$. Another example: If $z = v + w$, then $\overline{z} = \overline{v} + \overline{w}$; in other words,

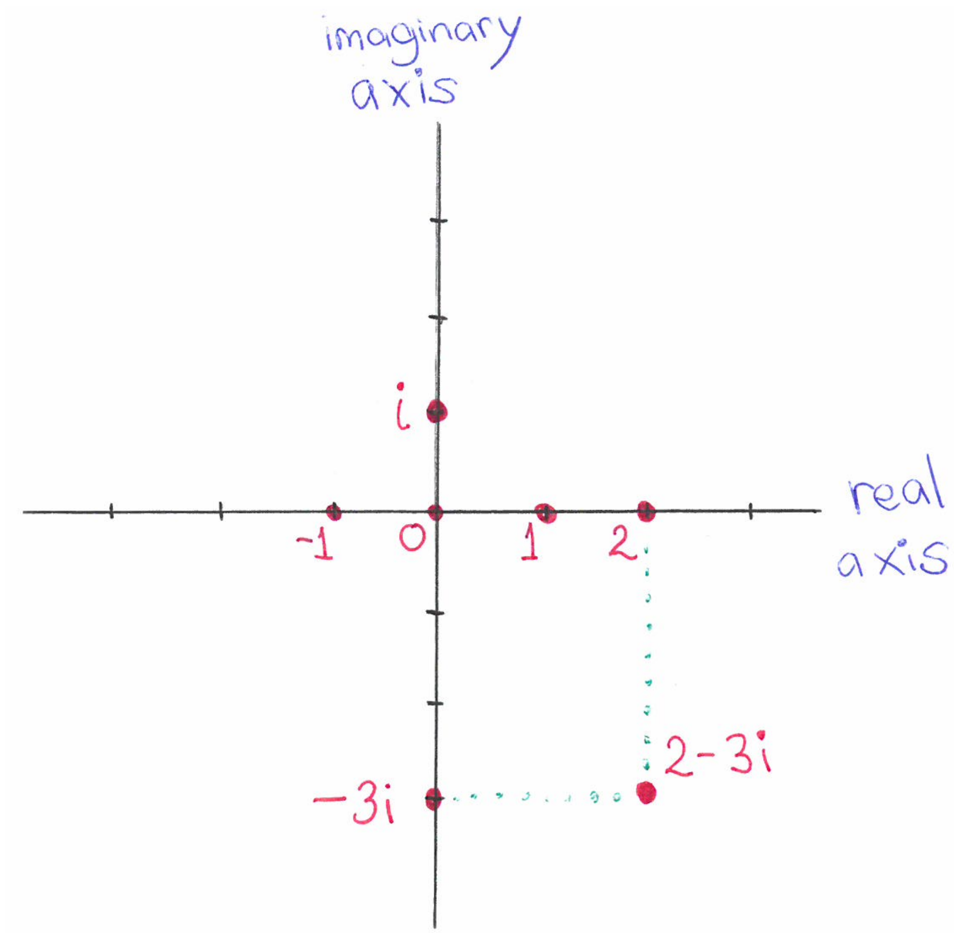
$$\overline{v+w} = \overline{v} + \overline{w}$$

for any complex numbers v and w. Similarly,

$$\overline{v\,w} = \overline{v}\,\overline{w}.$$

These two identities say that complex conjugation respects addition and multiplication.

11.2. **The complex plane.** Just as real numbers can be plotted on a line, complex numbers can be plotted on a plane: plot $a + bi$ at the point (a, b).

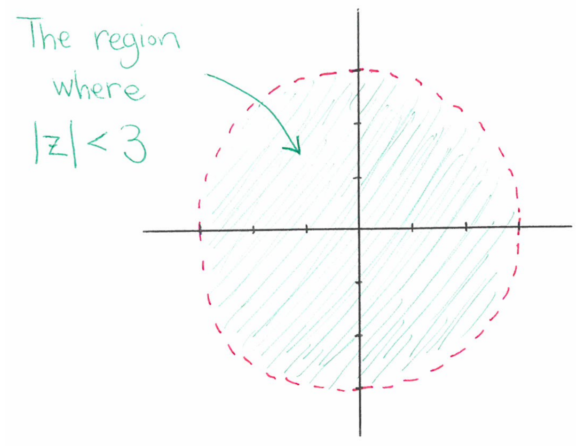


Addition and subtraction of complex numbers have the same geometric interpretation as for vectors. The same holds for scalar multiplication by a *real* number. (The geometric interpretation of multiplication by a *complex* number is different; we'll explain it soon.) Complex conjugation reflects a complex number in the real axis.

The absolute value (also called magnitude or modulus) $|z|$ of a complex number $z = a + bi$ is its distance to the origin:

$$|a + bi| := \sqrt{a^2 + b^2} \qquad \text{(this is a real number)}.$$

For a complex number z, inequalities like $z < 3$ do not make sense, but inequalities like $|z| < 3$ do, because $|z|$ is a real number. The complex numbers satisfying $|z| < 3$ are those in the open disk of radius 3 centered at 0 in the complex plane. (Open disk means the disk without its boundary.)



11.3. **Some useful identities.** The following are true for all complex numbers z:

$$\operatorname{Re} z = \frac{z + \overline{z}}{2}, \qquad \operatorname{Im} z = \frac{z - \overline{z}}{2i}, \qquad \overline{\overline{z}} = z, \qquad z\overline{z} = |z|^2.$$

Also, for any *real* number c and complex number z,

$$\operatorname{Re}(cz) = c \operatorname{Re} z, \qquad \operatorname{Im}(cz) = c \operatorname{Im} z.$$

(These can fail if c is not real.)

Proof of the first identity. Write z as $a + bi$. Then

$$\operatorname{Re} z = a,$$
$$\frac{z + \overline{z}}{2} = \frac{(a + bi) + (a - bi)}{2} = a,$$

so $\operatorname{Re} z = \dfrac{z + \overline{z}}{2}$. □

The proofs of the others are similar.

Many identities have a geometric interpretation too. For example, $\operatorname{Re} z = \dfrac{z + \overline{z}}{2}$ says that $\operatorname{Re} z$ is the midpoint between z and its reflection $\overline{z}$.

11.4. **Complex roots of polynomials.**

real polynomial: polynomial with real coefficients

complex polynomial: polynomial with complex coefficients

Example 11.3. How many roots does the polynomial $z^3 - 3z^2 + 4$ have? It factors as $(z - 2)(z - 2)(z + 1)$, so it has only two distinct roots (2 and -1). But if we count 2 twice, then the number of roots *counted with multiplicity* is 3, equal to the degree of the polynomial.

Some real polynomials, like $z^2 + 9$, cannot be factored completely into degree 1 real polynomials, but do factor into degree 1 complex polynomials: $(z + 3i)(z - 3i)$. In fact, *every* complex polynomial factors completely into degree 1 complex polynomials — this is proved in advanced courses in complex analysis. This implies the following:

Fundamental theorem of algebra. *Every degree n complex polynomial $f(z)$ has exactly n complex roots, if counted with multiplicity.*

Since real polynomials are special cases of complex polynomials, the fundamental theorem of algebra applies to them too. For real polynomials, the non-real roots can be paired off with their complex conjugates.

Example 11.4. The degree 3 polynomial z^3+z^2-z+15 factors as $(z+3)(z-1-2i)(z-1+2i)$, so it has three distinct roots: -3, $1+2i$, and $1-2i$. Of these roots, -3 is real, and $1+2i$ and $1-2i$ form a complex conjugate pair.

Example 11.5. Want a fourth root of i? The fundamental theorem of algebra guarantees that $z^4 - i = 0$ has a complex solution (in fact, four of them).

The fundamental theorem of algebra is useful for understanding *all* the eigenvalues, and this is used in 18.03 for constructing solutions to linear ordinary differential equations (ODEs) with constant coefficients.

11.5. **Real and imaginary parts of complex-valued functions.** Suppose that $y(t)$ is a complex-valued function of a real variable t. It can be expressed as

$$\begin{aligned} y\colon \mathbb{R} &\longrightarrow \mathbb{C} \\ t &\longmapsto y(t) = f(t) + i\,g(t) \end{aligned}$$

for some *real-valued* functions $f(t)$ and $g(t)$, namely $f(t) := \operatorname{Re} y(t)$ and $g(t) := \operatorname{Im} y(t)$. Differentiation and integration can be done component-wise, as for vector-valued functions of a single real variable:

$$\begin{aligned} y'(t) &= f'(t) + i\,g'(t) \\ \int y(t)\,dt &= \int f(t)\,dt + i \int g(t)\,dt. \end{aligned}$$

Example 11.6. Suppose that $y(t) = \dfrac{2+3i}{1+it}$. Then

$$y(t) = \frac{2+3i}{1+it} = \frac{2+3i}{1+it} \cdot \frac{1-it}{1-it} = \frac{(2+3t)+i(3-2t)}{1+t^2} = \underbrace{\left(\frac{2+3t}{1+t^2}\right)}_{f(t)} + i\underbrace{\left(\frac{3-2t}{1+t^2}\right)}_{g(t)}.$$

The functions in parentheses labelled $f(t)$ and $g(t)$ are real-valued, so these are the real and imaginary parts of the function $y(t)$. $\square$

Friday, October 8

11.6. **The complex exponential function.** Raising e to a complex number has no *a priori* meaning; it needs to be defined. People long ago tried to define it so that the key properties of the function e^t for real numbers t would be true for complex numbers too. They succeeded, and we will too!

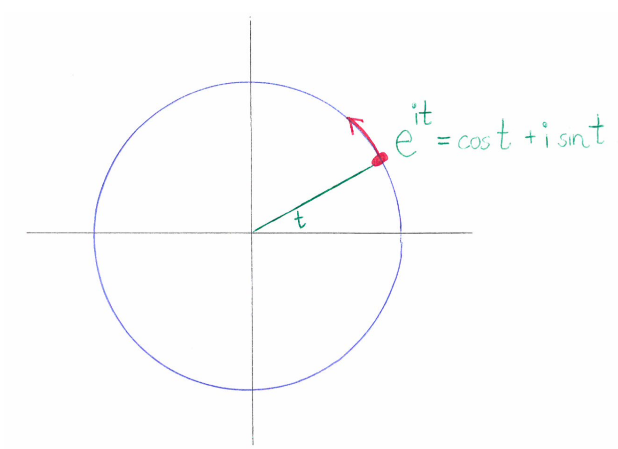
Definition 11.7 (Euler's formula)**.** For each real number t,

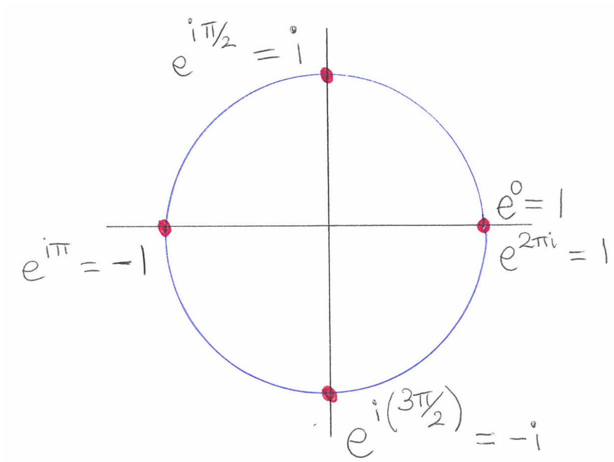
$$\boxed{e^{it} := \cos t + i \sin t.}$$

(The reason for this definition will be explained in 18.03. Briefly, it is that a function $y(t) = e^{it}$ should have the properties $y'(t) = i\,y(t)$ and $y(0) = 1$, and the function $\cos t + i \sin t$ is the unique function with these properties.)

Remark 11.8. Some older books use the awful abbreviation $\operatorname{cis} t := \cos t + i \sin t$, but this belongs in a cispool [*sic*], since e^{it} is a more useful expression for the same thing.

As t increases, the complex number $e^{it} = \cos t + i \sin t$ travels counterclockwise around the unit circle.





Definition 11.9. For any complex number $a + bi$, where a and b are real numbers,

$$e^{a+bi} := e^a e^{bi} = e^a(\cos b + i \sin b).$$

Properties:

- $e^0 = 1$.
- $e^z e^w = e^{z+w}$ for all complex numbers z and w.
- $\dfrac{1}{e^z} = e^{-z}$ for every complex number z.
- $(e^z)^n = e^{nz}$ for every complex number z and integer n.
- $e^{-it} = \cos t - i \sin t = \overline{e^{it}}$ for every real number t.
- $|e^{it}| = 1$ for every real number t.

11.7. **Polar forms of a complex number.** Given a nonzero complex number $z = x + yi$, we can express the point (x, y) in polar coordinates r and θ:

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

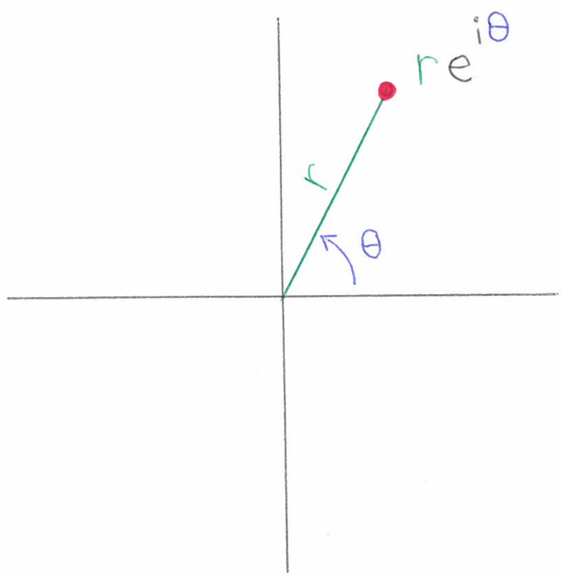
Then

$$x + yi = (r\cos\theta) + (r\sin\theta)i = r(\cos\theta + i\sin\theta).$$

In other words,

$$\boxed{z = re^{i\theta}}.$$

The expression $re^{i\theta}$ is called a polar form of the complex number z. Here r is required to be a *positive* real number (assuming $z \neq 0$), so $r = |z|$.

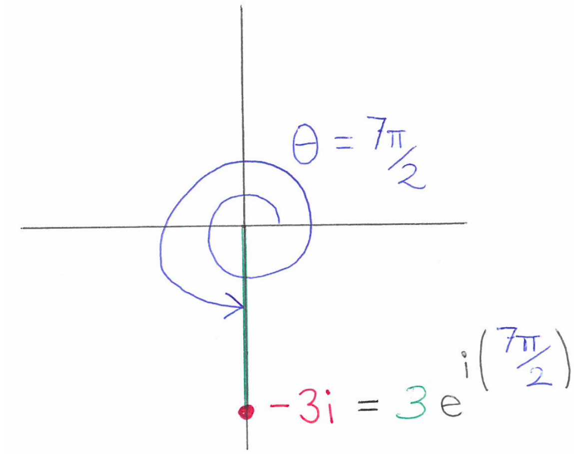


Any possible θ for z (a possible value for the angle or argument of z) may be called $\arg z$, but this is dangerously ambiguous notation since there are many values of θ for the same z: this means that $\arg z$ is not a function. The θ in $(-\pi, \pi]$ is called the principal value of the argument and is denoted in various ways:

$$\theta = \operatorname{Arg} z = \underbrace{\texttt{Arg[z]}}_{\text{Mathematica}} = \underbrace{\texttt{ArcTan[x,y]}}_{\text{Mathematica}} = \underbrace{\texttt{atan2(y,x)}}_{\text{MATLAB}}.$$

Example 11.10. Suppose that $z = -3i$. So z corresponds to the point $(0, -3)$. Then $r = |z| = 3$, but there are infinitely many possibilities for the angle θ. One possibility is $-\pi/2$; all the others are obtained by adding integer multiples of 2π:

$$\arg z \quad = \quad \ldots,\ -5\pi/2,\ -\pi/2,\ 3\pi/2,\ 7\pi/2,\ \ldots.$$



So z has many polar forms:

$$\cdots = 3e^{i(-5\pi/2)} = 3e^{i(-\pi/2)} = 3e^{i(3\pi/2)} = 3e^{i(7\pi/2)} = \cdots . \quad \square$$

Test for equality of two nonzero complex numbers in polar form:

$$r_1 e^{i\theta_1} = r_2 e^{i\theta_2} \quad \iff \quad r_1 = r_2 \text{ and } \theta_1 = \theta_2 + 2\pi k \text{ for some integer } k.$$

This assumes that r_1 and r_2 are positive real numbers, and that θ_1 and θ_2 are real numbers, as you would expect for polar coordinates.

Question 11.11. How do you convert a nonzero complex number $z = x + yi$ to polar form?

Solution. Convert (x, y) to polar coordinates (r, θ). Then $z = re^{i\theta}$. $\square$

11.8. **Operations in polar form.** Some arithmetic operations on complex numbers are easy in polar form:

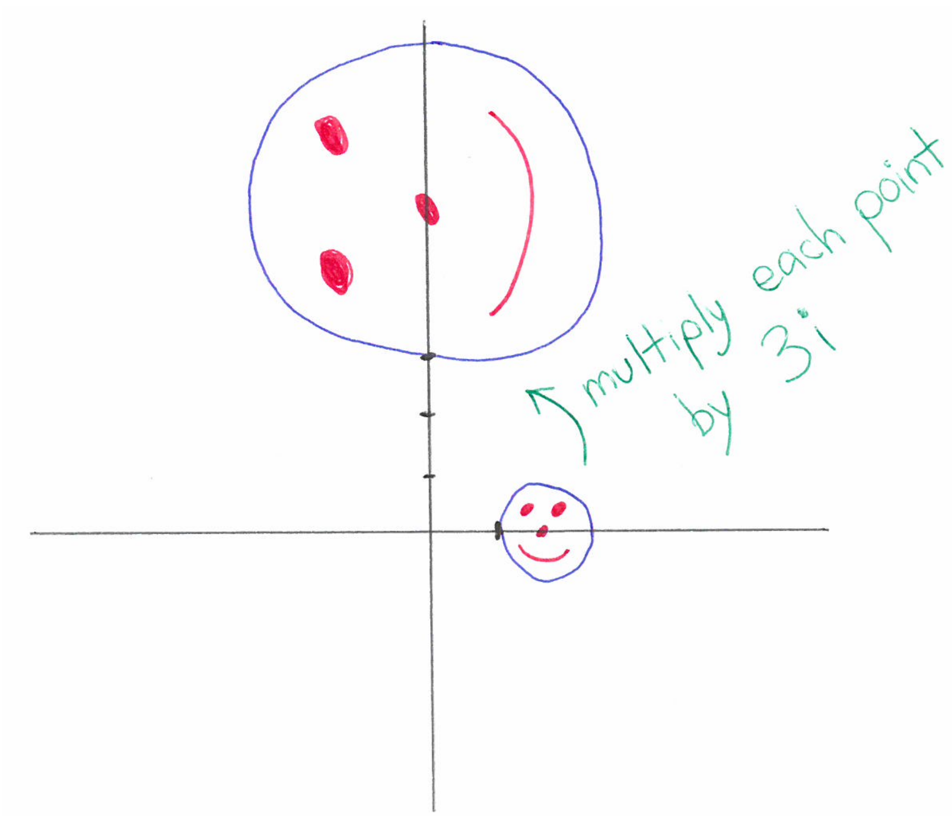
$$\begin{aligned}
&\text{multiplication: } (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1+\theta_2)} \quad \text{(multiply absolute values, add angles)}\\
&\text{reciprocal: } \frac{1}{re^{i\theta}} = \frac{1}{r} e^{-i\theta}\\
&\text{division: } \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)} \quad \text{(divide absolute values, subtract angles)}\\
&n^{\text{th}} \text{ power: } (re^{i\theta})^n = r^n e^{in\theta} \quad \text{for any integer } n\\
&\text{complex conjugation: } \overline{re^{i\theta}} = re^{-i\theta}.
\end{aligned}$$

Taking absolute values gives identities:

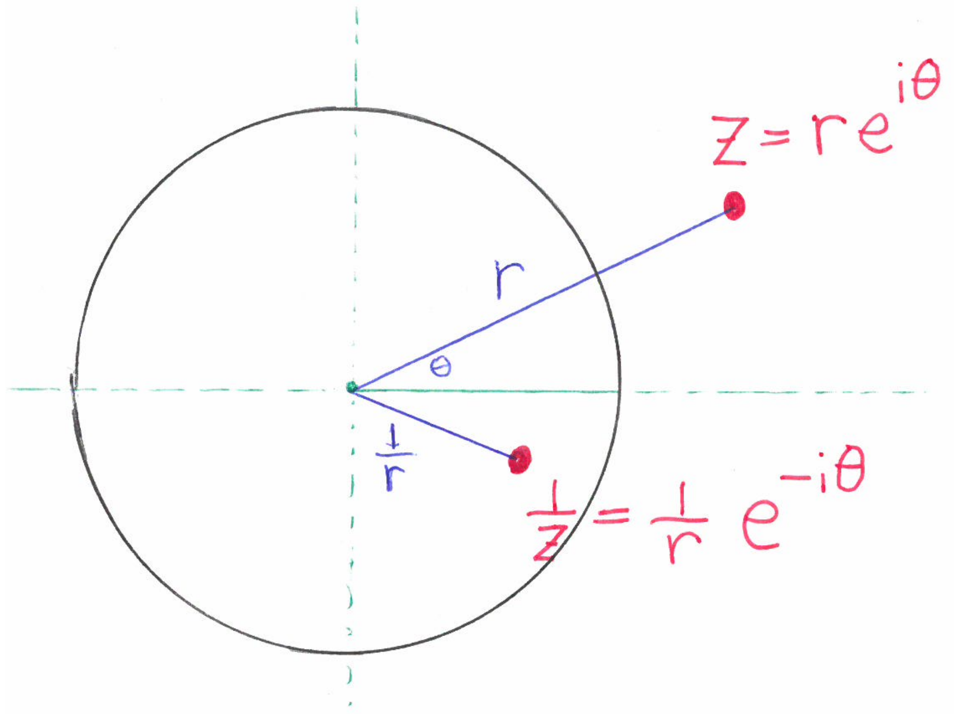
$$|z_1 z_2| = |z_1|\,|z_2|, \quad \left|\frac{1}{z}\right| = \frac{1}{|z|}, \quad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \quad |z^n| = |z|^n, \quad |\overline{z}| = |z|.$$

Question 11.12. What happens if you take a smiley in the complex plane and multiply each of its points by $3i$?

Solution: Since $i = e^{i\pi/2}$, multiplying by i adds $\pi/2$ to the angle of each point; that is, it rotates counterclockwise by $90°$ (around the origin). Next, multiplying by 3 does what you would expect: dilate by a factor of 3. Doing both leads to...



For example, the nose was originally on the real line, a little less than 2, so multiplying it by $3i$ produces a big nose close to $(3i)2 = 6i$. □



Question 11.13. How do you trap a lion?

Answer: Build a cage in the shape of the unit circle $|z| = 1$. Get inside the cage. Make sure that the lion is outside the cage. Apply the function $1/z$ to the whole plane. Voilà! The lion is now inside the cage, and you are outside it. (Only problem: There's a lot of other stuff inside the cage too. Also, don't stand too close to $z = 0$ when you apply $1/z$.)

Question 11.14. Why not always write complex numbers in polar form?

Answer: Because addition and subtraction are difficult in polar form!

12. Gradient and its applications

12.1. **Gradient.**

Definition 12.1. The gradient of a scalar-valued function $f(x, y, z)$ is the vector-valued function

$$\nabla f := \begin{pmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \\ \dfrac{\partial f}{\partial z} \end{pmatrix}.$$

It contains the same information as the total derivative

$$f'(\mathbf{x}) := \begin{pmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{pmatrix};$$

in fact, ∇f is just the *transpose* of the total derivative $f'(\mathbf{x})$.

There is a similar definition for a function in any number of variables.

Problem 12.2. Let $f(x, y, z) = x^2y + 7z$, and let $P = (2, 3, 5)$. What is $\nabla f(P)$?

Solution. $\nabla f = \begin{pmatrix} 2xy \\ x^2 \\ 7 \end{pmatrix}$, so its value at $(2, 3, 5)$ is $\begin{pmatrix} 12 \\ 4 \\ 7 \end{pmatrix}$. □

We can restate certain theorems and definitions in terms of ∇:

- Chain rule: If $f = f(x, y, z)$ and $\mathbf{r}(t) = (x(t), y(t), z(t))$, then

$$\begin{aligned} \frac{d}{dt} f(\mathbf{r}(t)) &= f'(\mathbf{r}(t))\, \mathbf{r}'(t) \\ &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \\ &= \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t). \end{aligned}$$

- Critical points of $f(x, y)$: These are points P where $\nabla f(P) = \mathbf{0}$.

A geometric property of ∇:

Theorem 12.3.

(a) *In 2D, $\nabla f(P)$ is perpendicular to the level curve $f(x, y) = c$ through P.*

(b) *In 3D, $\nabla f(P)$ is perpendicular to the* level surface *$f(x, y, z) = c$ through P.*

Proof. Both statements follow from the following claim:

> If $\mathbf{r}(t)$ is any parametric curve on which f is constant, then at each time t, the vector $\nabla f(\mathbf{r}(t))$ is perpendicular to the tangent vector (velocity vector) $\mathbf{r}'(t)$.

Let's prove this claim. To say that f is constant on the curve means that there is a constant c such that

$$f(\mathbf{r}(t)) = c$$

for all t. Take $\dfrac{d}{dt}$ and use the chain rule:

$$\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 0.$$

This says that $\nabla f(\mathbf{r}(t))$ is perpendicular to $\mathbf{r}'(t)$. □

Tuesday, October 12

Examples 12.4.

- If $f(x,y,z) = 2x + 3y + 5z$, then at every point P in $\mathbb{R}^3$, the vector $\nabla f = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$ is perpendicular to the level surface $2x + 3y + 5z = c$ through P (which is a plane).
- If $f(x,y) = x^2 + y^2$, then $\nabla f = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$, which points out from the origin, perpendicular to the level curves (which are circles).

12.2. **Tangent plane.**

Definition 12.5. The tangent plane to the surface $f(x,y,z) = c$ at $P = (x_0, y_0, z_0)$ is the plane with normal vector $\nabla f(P)$ through P:

$$\frac{\partial f}{\partial x}(P)\ (x - x_0) + \frac{\partial f}{\partial y}(P)\ (y - y_0) + \frac{\partial f}{\partial z}(P)\ (z - z_0) = 0.$$

(This makes sense only if $\nabla f(P) \neq \mathbf{0}$.)

Example 12.6. If $f(x,y,z) = x^2 + y^2 - z^2$ then $\nabla f = \begin{pmatrix} 2x \\ 2y \\ -2z \end{pmatrix}$, so the cone $x^2 + y^2 - z^2 = 0$ has a tangent plane at each point except $(0,0,0)$. At $P = (3,4,5)$, the gradient vector $\nabla f(P)$ is $\begin{pmatrix} 6 \\ 8 \\ -10 \end{pmatrix}$, so the tangent plane is

$$6(x-3) + 8(y-4) - 10(z-5) = 0.$$

(Check the answer: $(3,4,5)$ is on this plane, and the normal vector is correct.)

12.3. **Directional derivatives.** Recall: Given $f(x,y)$, the partial derivatives f_x and f_y measure the rate of change of f as one moves in the direction of $\mathbf{e}_1$ or $\mathbf{e}_2$.

What about other directions?

Definition 12.7. Let $\mathbf{u}$ be any unit vector. The directional derivative of $f(\mathbf{x})$ in the direction $\mathbf{u}$ is

$$\begin{aligned} D_{\mathbf{u}} f(\mathbf{x}) &:= \left. \frac{d}{ds} f(\mathbf{x} + s\mathbf{u}) \right|_{s=0} \\ &= \lim_{s \to 0} \frac{f(\mathbf{x} + s\mathbf{u}) - f(\mathbf{x})}{s}. \end{aligned}$$

This is because $\mathbf{r}(s) := \mathbf{x} + s\mathbf{u}$ describes the position of a point starting at $\mathbf{x}$ and moving at speed 1 in the direction $\mathbf{u}$.

Why s, and not t? Answer: speed is 1, so distance traveled s equals time t.

Alternative notation: $\left.\frac{df}{ds}\right|_{\mathbf{u}}(P)$. Sometimes also $\left.\frac{df}{ds}\right|_{P}$ or just $\frac{df}{ds}$ (this presumes that P and $\mathbf{u}$ are understood).

Question 12.8. If $f(x, y) := x^2 + y^2$ and $\mathbf{x} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$, in which directions $\mathbf{u}$ is the directional derivative maximum?

Possible answers:

(1) $\mathbf{e}_1$
(2) $\mathbf{e}_2$
(3) $\begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}$
(4) $\begin{pmatrix} -3/5 \\ -4/5 \end{pmatrix}$
(5) $\begin{pmatrix} 4/5 \\ -3/5 \end{pmatrix}$ or $\begin{pmatrix} -4/5 \\ 3/5 \end{pmatrix}$.
(6) None of these.

To figure out the answer, we'll use the following:

Theorem 12.9 (Formula for calculating directional derivatives)**.**

$$D_{\mathbf{u}} f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{u}.$$

Proof.

$$\begin{aligned} D_{\mathbf{u}} f(\mathbf{x}) &= \left.\frac{d}{ds} f(\mathbf{x} + s\mathbf{u})\right|_{s=0} \\ &= \left.\nabla f(\mathbf{x} + s\mathbf{u}) \cdot \frac{d}{ds}(\mathbf{x} + s\mathbf{u})\right|_{s=0} \\ &= \nabla f(\mathbf{x}) \cdot \mathbf{u}. \end{aligned}$$ □

Given f and $\mathbf{x}$, we maximize $D_{\mathbf{u}} f(\mathbf{x})$ by choosing $\mathbf{u}$ to point in the direction of $\nabla f(\mathbf{x})$, so that the $\cos\theta$ factor in the geometric formula for a dot product is $\cos 0 = 1$. And if $\mathbf{u}$ is so chosen, then $D_{\mathbf{u}} f(\mathbf{x}) = |\nabla f(\mathbf{x})|$, by that geometric formula. Hence:

Geometric interpretation of ∇f:

- direction of ∇f = direction in which f is increasing the fastest (perpendicular to level curve/surface)
- length of ∇f = directional derivative of f in that direction

Example 12.10. Back to $f(x,y) := x^2 + y^2$ and $P = (6,8)$. We have $\nabla f(x,y) = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$, which at P is $\begin{pmatrix} 12 \\ 16 \end{pmatrix}$, which is 20 times the unit vector $\mathbf{u} := \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}$. So the function is increasing the fastest in the direction of $\boxed{\begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}}$ (and its rate of increase in that direction is 20). So the answer to Question 12.8 is (3).

Question 12.11. In Question 12.8, in which directions $\mathbf{v}$ is the directional derivative 0?

Solution. These are the directions $\mathbf{v}$ such that

$$\nabla f(P) \cdot \mathbf{v} = 0.$$

We already calculated that $\nabla f(P) = \begin{pmatrix} 12 \\ 16 \end{pmatrix}$, so we want the unit vectors $\mathbf{v}$ satisfying

$$\begin{pmatrix} 12 \\ 16 \end{pmatrix} \cdot \mathbf{v} = 0.$$

These are the unit vectors $\mathbf{v}$ perpendicular to the vector $\mathbf{u} := \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}$ in the previous question, so $\boxed{\mathbf{v} = \begin{pmatrix} 4/5 \\ -3/5 \end{pmatrix} \text{ or } \mathbf{v} = \begin{pmatrix} -4/5 \\ 3/5 \end{pmatrix}}$. (Whereas $\mathbf{u}$ is perpendicular to the level curve, these vectors $\mathbf{v}$ are tangent to the level curve. This makes sense, since the value of f is constant along the level curve, so the directional derivative of f should be 0 in either direction along the level curve.) □

12.4. **Constraint equations vs. constraint inequalities.**

(1) Find the minimum value of $f(x,y) := 2x^2 - 7xy$ subject to the constraint inequalities $x \geq 1$, $y \geq 1$, and $x + y \leq 4$.

This is a standard max/min problem with boundaries to check.

(2) Find the minimum value of $f(x,y,z) := x^2 + y^2 + z^2$ subject to the constraint equation $3x + 5y + z = 9$.

Eliminate z to get an equivalent standard max/min problem: Find the minimum value of $F(x,y) := x^2 + y^2 + (9 - 3x - 5y)^2$ with no constraint equation. (For other such problems, however, you might need Lagrange multipliers to study the *boundary*.)

(3) Find the minimum value of $f(x,y,z) := x^2 + y^2 + z^2$ subject to the constraint equation $e^{x+y+2z} = x^9yz + 2$.

When there is a constraint equation, and you can't or don't want to eliminate it, you need Lagrange multipliers!

12.5. **Constraint equations and dimension.** (This was discussed in lecture here, but I have moved it earlier in these notes, to a more logical place.)

12.6. **Lagrange multipliers.**

Lagrange multipliers — a method for finding max/min of $f(x, y)$ when x and y are required to satisfy a constraint $g(x, y) = c$ (the "hard" case):

1. Identify the function $f(x, y)$ to be maximized (or minimized).
2. Identify the constraint equation $g(x, y) = c$ and constraint inequalities that define the domain $\mathcal{R}$ of f. Usually the presence of the constraint equation means that the dimension of $\mathcal{R}$ is one less than the number of variables (but constraint inequalities do not reduce the dimension). The constraint inequalities will be useful later, to determine what boundaries need to be checked.
3. Compute ∇f and ∇g.
4. Solve the system

$$g = c$$
$$\nabla f = \lambda \, \nabla g$$

   in (x, y, λ) to find the possible pairs (x, y) (we don't care about the value of λ, so a good strategy is to eliminate it).
5. Check also
   A. points on $g = c$ where $\nabla g = 0$
   B. points on $g = c$ where f or g is not differentiable
   C. behavior at ∞ (what happens to f as (x, y) approaches ∞ *along the constraint curve* $g = c$?)
   D. boundary points (if there are constraint inequalities)

   The global max (a, b) is to be found among these points from steps 4 and 5, so evaluate f at these points to find the maximum value.
   (*Warning*: If the values of f become larger and larger as (x, y) approaches the boundary or ∞ along $g(x, y) = c$, then the global max *does not exist.*)

Thursday, October 14

We drew a level diagram for a function $f(x, y)$ representing height of a point on a mountain, and a level curve $g(x, y) = c$ representing a smooth train track, and we explained why, when the train is at its highest point on the track, ∇f (pointing up the mountain) is a scalar times ∇g (pointing perpendicular to the track).

Why does this work? Why should ∇f be a multiple of ∇g at a maximum (when $\nabla g \neq 0$)?

Answer.

1. The vector ∇g is perpendicular to the level curve $g = c$.
2. At the maximum, ∇f is perpendicular to $g = c$ too.
   (Reason: If not, then for one of the two directions $\mathbf{u}$ along $g = c$, the value of $\nabla f \cdot \mathbf{u}$ would be positive, which says that the directional derivative $D_{\mathbf{u}} f$ would be positive, so one could increase f by moving in that direction along $g = c$, meaning that we weren't really at a maximum of f.)

Since at the maximum both ∇f and ∇g are perpendicular to the level curve, ∇f must be a multiple of ∇g there (if $\nabla g \neq 0$). $\square$

Sample problem: Find the highest point on the intersection of the cylinder $x^2 + y^2 = 13$ and the plane $2x + 3y - z = 8$ in $\mathbb{R}^3$.

Solution.

Steps 1 and 2: Identify the function $f(x, y)$ to be maximized/minimized, and the constraint equation $g(x, y) = c$ and constraint inequalities that define the domain $\mathcal{R}$ of f.

We want to maximize z subject to *two* constraint equations, $x^2+y^2 = 13$ and $2x+3y-z = 8$. But we can use the second equation to eliminate z and express everything in terms of x and y; namely, $z = 2x + 3y - 8$. Now we want to maximize $2x + 3y - 8$ subject to the constraint $x^2 + y^2 = 13$. So take $f(x, y) := 2x + 3y - 8$ and $g(x, y) := x^2 + y^2$ and $c := 13$. No constraint inequalities.

Step 3: Compute ∇f and ∇g.

We get $\nabla f = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\nabla g = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$.

Step 4: Solve the system

$$\begin{aligned} g &= c \\ \nabla f &= \lambda \nabla g \end{aligned}$$

in (x, y, λ) to find the possible pairs (x, y) (we don't care about the value of λ, so a good strategy is to eliminate it).

The system to be solved is

$$\begin{aligned} x^2 + y^2 &= 13 \\ 2 &= \lambda(2x) \\ 3 &= \lambda(2y). \end{aligned}$$

Multiply the second equation by y and the third equation by x and equate to get

$$2y = 3x.$$

Solve for y and substitute into the first equation:

$$\begin{aligned} x^2 + \left(\frac{3}{2}x\right)^2 &= 13 \\ \frac{13}{4}x^2 &= 13 \\ x^2 &= 4 \\ x &= \pm 2. \end{aligned}$$

So we get $(x, y) = (2, 3)$ or $(-2, -3)$.

Step 5: Check also

A. *points on $g = c$ where $\nabla g = 0$*
   No such points, since $\nabla g = 0$ only at $(0, 0)$, which is not on $x^2 + y^2 = 13$.
B. *points on $g = c$ where f or g is not differentiable*
   No such points.
C. *behavior at ∞ (what happens to f as (x, y) approaches ∞ along the constraint curve $g = c$?)*
   Not applicable — the constraint curve is bounded.
D. *boundary points (if there are constraint inequalities)*
   Not applicable.

So the only points to check are $(2, 3)$ and $(-2, -3)$. We have $f(2, 3) = 5$, and $f(-2, -3) = -21$, so the maximum is at $(x, y) = (2, 3)$. There, $z = f(2, 3) = 5$. So the highest point is $(2, 3, 5)$. $\square$

Lagrange multipliers apply also to max/min of functions of more than 2 variables.

(Lagrange multipliers also can be used when there is more than one constraint — this is discussed in the textbook, but we won't study such problems in this course.)

The second derivative test as we stated it cannot be used when there is a constraint equation.

Sample problem 2: Find the point on the surface $x^2 + y^2 = (z - 1)^3$ closest to the origin.

Solution. (We are going to make some mistakes below in red, and then correct them in green.) We want to minimize $f(x, y, z) := x^2 + y^2 + z^2$ subject to the constraint $g = 0$ where $g(x, y, z) := x^2 + y^2 - (z - 1)^3$. We need to solve the Lagrange multiplier system

$$\begin{aligned} x^2 + y^2 - (z - 1)^3 &= 0 \\ \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} &= \lambda \begin{pmatrix} 2x \\ 2y \\ -3(z - 1)^2 \end{pmatrix}, \end{aligned}$$

which, when written out in components, says

$$\begin{aligned} x^2+y^2-(z-1)^3 &= 0 \\ 2x &= \lambda(2x) \\ 2y &= \lambda(2y) \\ 2z &= \lambda(-3(z-1)^2). \end{aligned}$$

Either the second or third equation implies $\lambda = 1$. Substituting $\lambda = 1$ in the last equation leads to

$$\begin{aligned} 2z &= -3(z-1)^2 \\ 3(z-1)^2+2z &= 0 \\ 3z^2-4z+3 &= 0 \end{aligned}$$

but $4^2 - 4(3)(3) < 0$, so there are no solutions.

Oops, we divided by either x or y to obtain $\lambda = 1$, so that was valid only if $x \neq 0$ or $y \neq 0$. Thus to finish finding all solutions to the Lagrange multiplier system, We also need to consider the case in which $x = y = 0$. Then the constraint equation implies $z = 1$, and substituting this into the last of the four equations in the Lagrange multiplier system says

$$2 = \lambda(0)$$

which is impossible. So there is no minimum.

Oops, we forgot to check points where $\nabla g = 0$. This says

$$\begin{pmatrix} 2x \\ 2y \\ -3(z-1)^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which leads to $(x, y, z) = (0, 0, 1)$. (The other parts of Step 5 do not give anything additional.) By the geometry, there has to be a minimum somewhere, and $(0, 0, 1)$ is the only candidate, so $(0, 0, 1)$ is the closest point. □

The surface could be sketched by considering the slices obtained by intersecting with horizontal planes $z = c$. The $z = 1$ slice is a point, and the $z = c$ slice for each $c > 1$ is a circle. The point $(0, 0, 1)$ turns out to be what is called a singularity (it turns out to be a sharp point on the surface).

Friday, October 15

In general, if you see something that is affecting the climate at MIT and you don't feel comfortable discussing it directly with the people involved, there are resources at MIT that you can reach out to. Here are some of them:

- Student Support Services (S^3), `studentlife.mit.edu/s3`
- Office of Minority Education (OME), `ome.mit.edu`
- Institute Discrimination & Harassment Response (IDHR), `idhr.mit.edu`

12.7. **Review: gradient and directional derivatives.**

Geometric interpretation of ∇f:

- direction of ∇f = direction in which f is increasing the fastest (perpendicular to level curve/surface)
- length of ∇f = directional derivative of f in that direction

Formula for directional derivative of $f(\mathbf{x})$, starting at $\mathbf{x}$ and moving in the direction of the unit vector $\mathbf{u}$:

$$D_{\mathbf{u}} f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{u}.$$

Just as the value of a partial derivative $\partial f/\partial x$ at a point is a scalar, the value of a directional derivative is a scalar.

What does $D_{\mathbf{u}} f$ mean? If you travel a distance s in the direction of $\mathbf{u}$, then the value of f increases by approximately $(D_{\mathbf{u}} f)s$.

Example 12.12. Suppose that $f(x, y) = x^2 + 5y$.

What is $(\nabla f)(6, 1)$ *and what does it say geometrically?* Well, $\nabla f = \langle 2x, 5\rangle$, so $(\nabla f)(6, 1) = \langle 12, 5\rangle$. This has length $\sqrt{12^2 + 5^2} = 13$ and direction $\left\langle \frac{12}{13}, \frac{5}{13} \right\rangle$. This says that f is increasing the fastest in the direction of $\left\langle \frac{12}{13}, \frac{5}{13} \right\rangle$, and that the directional derivative (rate of change) in that direction is 13.

We have $f(6, 1) = 41$. *What is the nearest point to* $(6, 1)$ *where* f *takes the value* 41.26*, approximately?* If from $(6, 1)$ we move a distance ϵ in the direction of fastest increase, the value of f increases by 13ϵ, so we need $13\epsilon = 0.26$, so $\epsilon = 0.02$. The point reached from $(6, 1)$ by moving 0.02 units in the direction of $\left\langle \frac{12}{13}, \frac{5}{13} \right\rangle$ is

$$\langle 6, 1\rangle + 0.02 \left\langle \frac{12}{13}, \frac{5}{13} \right\rangle \approx \langle 6, 1\rangle + 0.02 \langle 1, 1/2\rangle = \langle 6.02, 1.01\rangle.$$

(The approximation of $5/13$ by $1/2$ was pretty rough, but good enough for blackboard work.)

What is the equation of the level curve through $(6, 1)$*?*

The value of f at $(6, 1)$ is 41, so the level curve has equation $x^2 + 5y = 41$.

What is the slope of the level curve at $(6, 1)$*?*

Solution 1: It is perpendicular to ∇f, so the slope is $-12/5$.

Solution 2: The level curve is $y = -\frac{1}{5}x^2 + \frac{41}{5}$, i.e., the graph of $\ell(x) := -\frac{1}{5}x^2 + \frac{41}{5}$, and $\ell'(x) = -\frac{2}{5}x$, so the slope is $\ell'(6) = -\frac{12}{5}$.

In which direction(s) $\mathbf{u}$ *is the directional derivative of* f *at* $(6,1)$ *equal to* $56/5$*?*

If $\mathbf{u} = \langle a, b\rangle$, then we need $a^2+b^2 = 1$ (so that $\mathbf{u}$ is a unit vector), and $(\nabla f)(6,1)\cdot\mathbf{u} = 56/5$, which says $12a + 5b = 56/5$. Solving this system (many algebra details skipped) leads to two possibilities

$$\left\langle \frac{3}{5}, \frac{4}{5} \right\rangle, \qquad \left\langle \frac{837}{845}, -\frac{116}{845} \right\rangle.$$

In which direction(s) $\mathbf{u}$ *is the directional derivative of* f *at* $(6,1)$ *equal to* 14*?*

Solution: None, because the maximum directional derivative was 13!

In which direction(s) $\mathbf{u}$ *is the directional derivative of* f *at* $(6,1)$ *equal to* -13*?*

Solution: Since $(\nabla f)(6,1)$ had length 13, the only unit vector $\mathbf{u}$ that dots with it to give -13 is the unit vector in the opposite direction, namely $\begin{pmatrix} -12/13 \\ -5/13 \end{pmatrix}$.

What does the graph of f *look like?*

The part above the horizontal line $y = 0$ is the parabola $z = x^2$.

The part above the horizontal line $y = 1$ is the higher parabola $z = x^2 + 5$.

All these parabolas together form the biggest slide ever.

Example 12.13. Let $f(x,y) = x^2 + 2y^2$. What is $D_{\mathbf{u}}f$ at $(1,3)$, where $\mathbf{u}$ is the direction that is $45°$ counterclockwise from ∇f at $(1,3)$?

Solution. At $(1,3)$,

$$\nabla f = \begin{pmatrix} 2x \\ 4y \end{pmatrix} = \begin{pmatrix} 2 \\ 12 \end{pmatrix},$$

so

$$\begin{aligned} D_{\mathbf{u}}f &= (\nabla f)\cdot\mathbf{u} \\ &= |\nabla f|\ |\mathbf{u}|\ \cos 45° \\ &= \sqrt{2^2+12^2}\ (1)\ (\sqrt{2}/2) \\ &= \sqrt{1^2+6^2}\ \sqrt{2} \\ &= \sqrt{74}. \end{aligned}$$

□

12.8. **Review: level curves and graph.**

Problem 12.14. Draw the level curves of $f(x,y) := x^2 - y^2$. What does the graph look like?

Solution. The hyperbola $x^2 - y^2 = 4$ is the level curve corresponding to the value 4, the union of two lines $x^2 - y^2 = 0$ is the level curve corresponding to the value 0, and so on.

To draw the graph, by visualizing a point above (or below) each point on the xy-plane, with height given by the value of the function there: the result is a saddle, with critical point at $(0,0)$. The point $(0,0)$ is not a local max since there are larger values in any neighborhood of $(0,0)$ (for example, just to the right). It is also not a local min since there are larger values in any neighborhood (for example, just above). A critical point that is neither a local max nor a local min is called a saddle point, so that is what we have. $\square$

12.9. **Review: complex exponential.**

Problem 12.15. Sketch the trajectory of

$$\begin{aligned} f\colon \mathbb{R} &\longrightarrow \mathbb{C} \\ t &\longmapsto e^{(-3+2i)t} \end{aligned}$$

in the complex plane.

Solution. To understand $f(t)$ geometrically, write it in polar form:

$$f(t) = e^{(-3+2i)t} = e^{-3t} e^{i(2t)}.$$

Thus the radius is the exponentially decreasing function e^{-3t}, and the angle is the steadily increasing function $2t$. This means that the trajectory is spiralling in counterclockwise towards 0, going around infinitely many times but never reaching 0. Going backwards in time, the trajectory spirals outward clockwise to ∞. $\square$

12.10. **Review: roots of unity.**

Problem 12.16. Find all complex solutions to $z^8 = 1$, and plot them in the complex plane. (These are called the 8th roots of unity.)

Solution. Write both sides in polar form. Thus $z = re^{i\theta}$, with the unknowns r and θ to be solved for instead of z, and $1 = 1e^{i(0)}$. Now we can rewrite the equation:

$$\begin{aligned} z^8 &= 1 \\ (re^{i\theta})^8 &= 1e^{i(0)} \\ r^8 e^{i(8\theta)} &= 1e^{i(0)}. \end{aligned}$$

Does this mean that $r^8 = 1$ and $8\theta = 0$? Not quite, since it could be that $8\theta = 0 + 2\pi k$ for some integer k, and we need to consider all values of k in order to get all solutions.

The equation $r^8 = 1$ implies $r = 1$, since the distance r in polar form is always a nonnegative real number.

The equation $8\theta = 0 + 2\pi k$ implies $\theta = k\pi/4$, so the possibilities for θ are

$$\ldots, \quad -2\pi/4, \quad -\pi/4, \quad 0, \quad \pi/4, \quad 2\pi/4, \quad \ldots.$$

Reassembling r and θ into z shows that $z = e^{i(k\pi/4)}$, so the possibilities for z are

$$\ldots, \quad e^{i(-2\pi/4)}, \quad e^{i(-\pi/4)}, \quad e^{i(0)}, \quad e^{i(\pi/4)}, \quad e^{i(2\pi/4)}, \quad \ldots.$$

These are points along the unit circle in $\mathbb{C}$, equally separated by an angle $\pi/4$. But they are not all different! They repeat after every 8 terms, so there are only 8 distinct solutions. In other words, adding 8 to k does not change $e^{i(k\pi/4)}$, so one gets all the solutions by taking the ones with $k = 0, 1, 2, \ldots, 7$, say. These are

$$\begin{aligned}
e^{i(0)} &= \cos 0 + i \sin 0 = 1 \\
e^{i(\pi/4)} &= \cos(\pi/4) + i \sin(\pi/4) = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} \\
e^{i(2\pi/4)} &= \cos(2\pi/4) + i \sin(2\pi/4) = i \\
e^{i(3\pi/4)} &= \cos(3\pi/4) + i \sin(3\pi/4) = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} \\
e^{i(4\pi/4)} &= \cos(4\pi/4) + i \sin(4\pi/4) = -1 \\
e^{i(5\pi/4)} &= \cos(5\pi/4) + i \sin(5\pi/4) = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} \\
e^{i(6\pi/4)} &= \cos(6\pi/4) + i \sin(6\pi/4) = -i \\
e^{i(7\pi/4)} &= \cos(7\pi/4) + i \sin(7\pi/4) = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}.
\end{aligned}$$ □

Remark 12.17. One could have predicted in advance that $z^8 = 1$ would have 8 solutions, since

1. they are the roots of the degree 8 polynomial $z^8 - 1$, and
2. none of the roots are multiple roots, because at a multiple root the derivative would have to be 0, but the derivative is $8z^7$ which is nonzero at every root (0 is not a root).

Midterm 2 on Tuesday, October 19.

Thursday, October 21

13. Integrals of multivariable functions

13.1. **Double integrals.** Let R be a region in $\mathbb{R}^2$, cut into tiny regions $R_1, \ldots, R_n$. Choose (x_1, y_1) in R_1, ..., (x_n, y_n) in R_n. Then

$$\iint_R f(x,y)\, dA \approx f(x_1, y_1) \operatorname{Area}(R_1) + \cdots + f(x_n, y_n) \operatorname{Area}(R_n).$$

It's a number. (The actual definition involves a limit as the maximum size of the R_i goes to 0.)

If $f \geq 0$ everywhere on R, then $\iint_R f(x,y)\, dA$ can be interpreted geometrically as the volume under the graph of f (above R).

13.1.1. *Computing double integrals as iterated integrals.*

Suppose that R is a rectangle $[a,b] \times [c,d]$. Partition $[a,b]$ into tiny subintervals, so R gets sliced into thin rectangles, and the volume above it gets sliced like cheese, into slabs. Then

$$\iint_R f(x,y)\,dA = \int_{x=a}^{x=b} \left(\int_{y=c}^{y=d} f(x,y)\,dy \right) dx$$
$$=: \int_a^b \int_c^d f(x,y)\,dy\,dx.$$

Think of the inner integral (with x treated as a constant) as the area of a slab; multiplying it by the width "dx" of a slab gives the volume of the slab and we sum these ("integrate") to get the total volume.

Similarly,

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

13.1.2. *Non-rectangular regions.*

Problem 13.1. Let R be the bounded region between $y - x = 2$ and $y = x^2$. Find $\iint_R (2x+4y)\,dA$.

Solution: We'll compute it as $\iint (2x+4y)\,dy\,dx$. What are the limits of integration?

Step 1: Sketch the region. Solving $y - x = 2$ and $y = x^2$ shows that the line and the parabola intersect at $(-1,1)$ and $(2,4)$.

Step 2: The outer integral goes from the smallest x-coordinate of a point in R to the largest x-coordinate.

The smallest x-coordinate is -1 and the largest x-coordinate is 2. So the outer integral will look like

$$\int_{x=-1}^{x=2}$$

Step 3: Hold x fixed, and increase y; look at the y-values where the line enters and leaves R — usually these depend on x. The inner integral goes from the smaller y-value to the larger y-value.

It enters at $y = x^2$ and leaves at $y = x+2$, so the iterated integral is

$$\int_{x=-1}^{x=2} \int_{y=x^2}^{y=x+2} (2x+4y)\,dy\,dx.$$

Then how do you evaluate the iterated integral?

Step 4: Evaluate the inner *integral first, treating x as constant. The result will be a function of x.*

It's

$$\int_{y=x^2}^{y=x+2} (2x+4y)\,dy = 2xy + 2y^2\Big|_{y=x^2}^{y=x+2}$$
$$= (2x(x+2) + 2(x+2)^2) - (2x^3 + 2x^4)$$
$$= -2x^4 - 2x^3 + 4x^2 + 12x + 8.$$

Step 5: Then evaluate the outer integral, which is just a definite integral of a 1-variable function of x.

Answer:

$$\int_{x=-1}^{x=2}\int_{y=x^2}^{y=x+2} (2x+4y)\,dy\,dx = \int_{-1}^{2}(-2x^4 - 2x^3 + 4x^2 + 12x + 8)\,dx$$
$$= \frac{333}{10}.$$

13.1.3. *Dividing the region into pieces.* Sometimes to compute a double integral, the region needs to divided into two or more pieces. The integral is then the sum of the integrals over the pieces.

Example: rectangle with a smaller rectangle removed from its center.

13.1.4. *Volume between two surfaces.* If $f(x,y)$ and $g(x,y)$ are functions on a region R and $f(x,y) \geq g(x,y)$ for all (x,y) in R, then the volume of the solid between the graphs of f and g is $\iint_R (f(x,y) - g(x,y))\,dA$.

Problem (p. 960, #44): Set up an iterated integral that computes the volume of the region bounded by the surfaces $z = x^2 + 3y^2$ and $z = 4 - y^2$.

Solution: The intersection of the two surfaces is defined by the system

$$z = x^2 + 3y^2$$
$$z = 4 - y^2.$$

The projection to the xy-plane of this is obtained by eliminating z:

$$x^2 + 3y^2 = 4 - y^2.$$

So R is the region bounded by the ellipse

$$x^2 + 4y^2 = 4.$$

Inside this ellipse the function $4 - y^2$ is larger than $x^2 + 3y^2$ (as you can see by comparing their values at $(0,0)$). So we want

$$\iint_R ((4 - y^2) - (x^2 + 3y^2))\,dA = \int_{-1}^{1}\int_{-\sqrt{4-4y^2}}^{\sqrt{4-4y^2}} (4 - x^2 - 4y^2)\,dx\,dy.$$

(Note: it is OK to write $\int_{-1}^{1}$ instead of $\int_{y=-1}^{y=1}$, since the variable is determined by the matching differential: the integrals are *nested*, so the leftmost $\int$ corresponds to the rightmost differential dy, and so on.)

Example 13.2. What's wrong with the following?

$$\int_2^5 \int_{x+3}^{x+7} (x^2 + 2y)\, dx\, dy$$

Since the outer differential on the right is dy, this would mean

$$\int_{y=2}^{y=5} \int_{x=x+3}^{x=x+7} (x^2 + 2y)\, dx\, dy,$$

but the inner limits make no sense. In the inner integral, y is constant, and the range for x could depend on y, but the range for x cannot be expressed in terms of x!

13.2. **Area in polar coordinates.** Question: What is the area of the region described in polar coordinates by $r \in [3, 3.2]$, $\theta \in [\pi/4, \pi/4 + 0.1]$, to the nearest hundredth?

Answer: 0.06.

Why? In general, the region with polar coordinates in $[r, r + \Delta r]$ and $[\theta, \theta + \Delta\theta]$ is approximately a rectangle with sides Δr and $r\Delta\theta$ (the latter is the length of an arc of radius r and measure $\Delta\theta$, by definition of radian). So its area is approximately $r\,\Delta r\,\Delta\theta$, which in our case is $3(0.2)(0.1) = 0.06$. (To be more precise, the area is between the areas of two rectangles, between $0.2(3(0.1)) = 0.06$ and $0.2(3.2(0.1)) = 0.064$.)

To remember:

$$\boxed{dA = dx\, dy = r\, dr\, d\theta.}$$

Friday, October 22

13.3. **Double integrals in polar coordinates.**

Problem 13.3. Re-express

$$\int_1^{\sqrt{2}} \int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}} x\sqrt{x^2+y^2}\, dx\, dy$$

as an iterated integral in polar coordinates.

Solution. Substitute

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$dx\, dy = r\, dr\, d\theta$$

but also re-compute the limits of integration.

The inequalities

$$-\sqrt{2-y^2} \le x \le \sqrt{2-y^2}$$

are equivalent to

$$x^2 \le 2 - y^2,$$

and to

$$x^2 + y^2 \le 2.$$

Thus the region is defined by

$$\begin{aligned} x^2 + y^2 &\le 2 \\ 1 \le y &\le \sqrt{2}. \end{aligned}$$

In the circle of radius $\sqrt{2}$ centered at $(0,0)$, our region is the upper segment obtained by cutting the circle with the chord from $(-1,1)$ to $(1,1)$. (Draw it!)

So $\theta \in [\pi/4, 3\pi/4]$, and the upper limit for r is $\sqrt{2}$, but what is the lower limit for r? The inequality $y \ge 1$ becomes $r \sin\theta \ge 1$ in polar coordinates, so $r \ge 1/\sin\theta$.

Final answer:

$$\int_{\pi/4}^{3\pi/4} \int_{1/\sin\theta}^{\sqrt{2}} (r\cos\theta) r \, r \, dr \, d\theta. \qquad \square$$

13.4. **Applications of double integrals.**

13.4.1. *Average value.* Warm-up: The average of numbers $x_1, \ldots, x_n$ is

$$\frac{x_1 + \cdots + x_n}{n}.$$

Definition 13.4. The average value of $f(x,y)$ on a region R is

$$\frac{\iint_R f \, dA}{\text{Area}(R)}$$

13.4.2. *Mass and centroid.* For a metal plate in the xy-plane,

$$\text{mass} = (\underbrace{\text{mass per unit area}}_{\text{density}})(\text{area})$$

But if its density is not constant, $\delta = \delta(x,y)$ (in g/cm^2, say), then each piece of area must be multiplied by the density *there*.

Definition 13.5. The mass of a 2-dimensional object is

$$m := \iint_R \underbrace{\delta(x,y)\, dA}_{dm}.$$

The x-coordinate of the centroid is the average of the x-coordinates of the points in the region, *weighted by density*. Same for the y-coordinate. So:

Definition 13.6. The centroid of a 2-dimensional object is the point $(\bar{x}, \bar{y})$ where

$$\bar{x} := \frac{\iint_R x\,dm}{m} = \frac{\iint_R x\,\delta(x,y)\,dA}{\iint_R \delta(x,y)\,dA}$$
$$\bar{y} := \frac{\iint_R y\,dm}{m} = \frac{\iint_R y\,\delta(x,y)\,dA}{\iint_R \delta(x,y)\,dA}.$$

(If density is not specified, assume $\delta(x,y) \equiv 1$.) The centroid is also called the center of mass or the center of gravity.

Sometimes symmetry gives a shortcut. For example, the centroid of an equilateral triangle (of constant density) has reflectional symmetry in each of its three altitudes, so the centroid must lie on all three altitudes, hence at the center. Another example: A parallelogram has $180°$ rotational symmetry around the point where the two diagonals meet, so that point must be the centroid.

13.4.3. *Moment of inertia.* The moment of inertia of an object with respect to an axis measures how difficult it is to rotate it (around that axis).

For a point mass:

$$I = (\text{distance to axis})^2\, m$$

In general (since different pieces are at different distances to the axis):

$$I = \iint\limits_R (\text{distance to axis})^2\, dm.$$

Special case: The polar moment of inertia of an object in the xy-plane is its moment of inertia around the z-axis. The distance from (x,y,z) to the nearest point on the z-axis (namely, $(0,0,z)$) is $\sqrt{x^2+y^2}$, so

$$(\text{distance to axis})^2 = x^2 + y^2$$

in this special case.

Example 13.7. Polar moment of inertia of a triangle R of constant density 1 with vertices at $(\pm 3, 0)$ and $(0,2)$?

Answer: Since density is 1, we have $dm = dA$. The polar moment of inertia is

$$\iint\limits_R (x^2+y^2)\,dA = \int_0^2 \int_{-(3-3y/2)}^{3-3y/2} (x^2+y^2)\,dx\,dy = \cdots$$

13.4.4. *Volume of revolution.*

Theorem 13.8 (First theorem of Pappus)**.** *A plane region R lies on one side of an axis, and is then rotated $360°$ around the axis. Let $A = \text{Area}(R)$, and let D be the distance traveled by the centroid of R. Then the solid of revolution has volume $V = AD$.*

Why is this true? Set up a coordinate system so that the axis is the y-axis, and R is in the half-plane $x \geq 0$. A little rectangle of area dA sweeps out a ring of volume approximately $2\pi x\, \Delta A$ since the ring can be cut into approximate rectangular parallelepipeds with base dA and heights summing to the length of the ring ($2\pi x$). Summing over all rectangles and taking a limit as their size goes to 0 yields

$$V = \iint_R 2\pi x\, dA = (2\pi A)\frac{\iint_R x\, dA}{A} = A(2\pi\bar{x}) = AD.$$

Tuesday, October 26

13.5. **Change of variables in double integrals.**

13.5.1. *Review of change of variables in one-variable integrals.* For example,

$$\int_{1/2}^{1} \frac{1}{\sqrt{1-x^2}}\, dx = \int_{\pi/6}^{\pi/2} \frac{1}{\sqrt{1-\sin^2 u}}\, (\cos u\, du)\,.$$

where

$$\begin{aligned} x &= \sin u \\ dx &= \cos u\, du. \end{aligned}$$

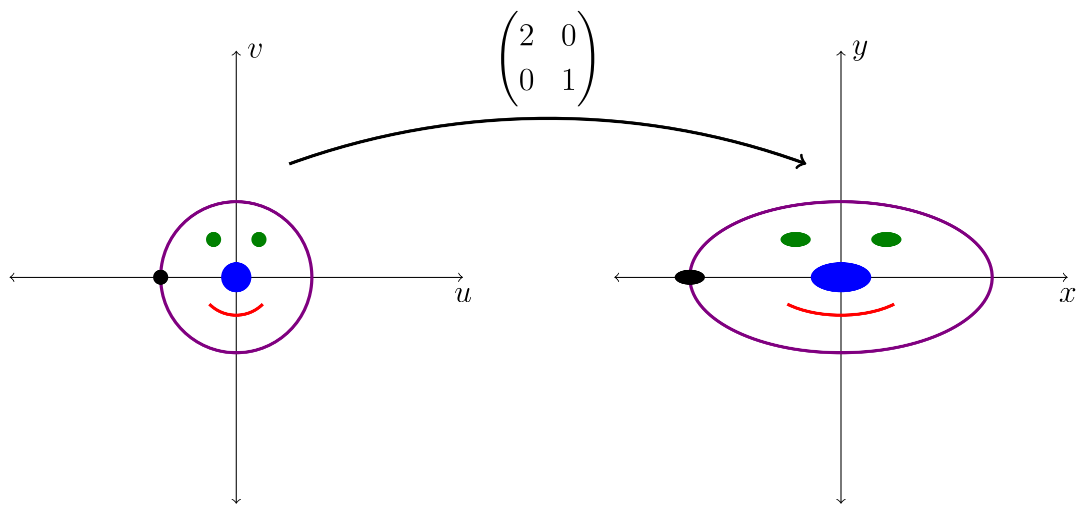
The boundary points were re-expressed in terms of u:

$$\begin{aligned} x = 1 &\quad \text{corresponds to} \quad u = \pi/2 \\ x = 1/2 &\quad \text{corresponds to} \quad u = \pi/6. \end{aligned}$$

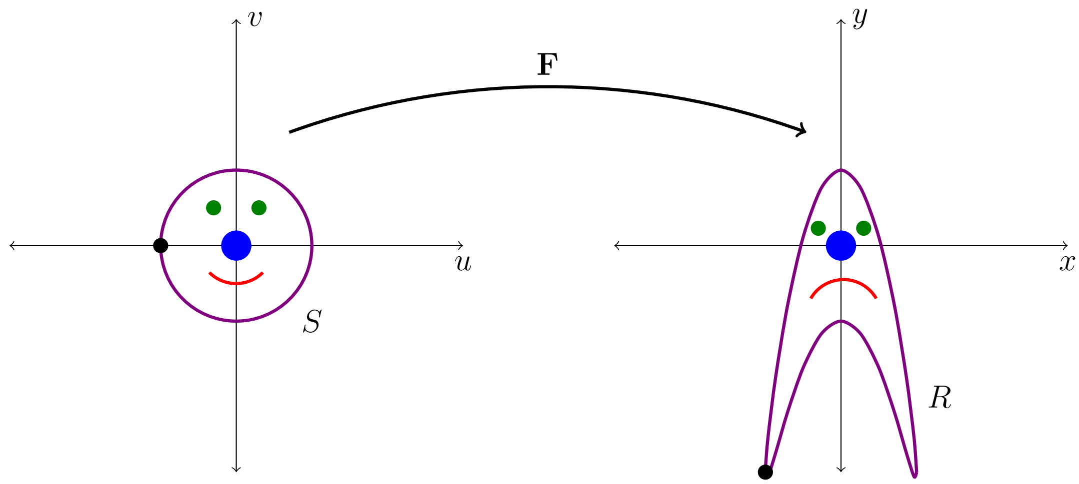
Today: The 2-variable analogue.

13.5.2. *Transformations.* View (u, v) as input variables, and (x, y) as output variables: $(x, y) = \mathbf{F}(u, v)$.

Example 13.9 (from long ago)**.** The linear transformation $\mathbf{F}(u, v) := (2u, v)$ (that is, $x = 2u$, $y = v$) stretches in the horizontal direction. The smiley becomes wider.



Example 13.10. The nonlinear transformation $\mathbf{F}(u,v) := (u, v - 3u^2)$ (that is, $x = u$, $y = v - 3u^2$) does something worse. One way to visualize $\mathbf{F}$: Plug in various input points (u,v) and see where they get mapped to. Another way: Draw images of the lines $u = c$ and $v = c$. The smiley is not smiling anymore. But if you had to do an integral over the deformed smiley, you could convert it into an integral over the original smiley, as we'll explain.



Important: From now on, $\mathbf{F}$ should give a bijective transformation from a region S in the uv-plane to a region R in the xy-plane. Bijective means that $\mathbf{F}$ matches the points in S with the points in R perfectly: each point in R comes from exactly one point in S. In particular, different points in S are mapped to different points inside R (this last condition is called one-to-one in Edwards & Penney). Without this condition, the uv-integral might double-count some parts of the xy-integral. If the condition is violated only the boundary, it's still OK.

13.5.3. *Jacobian determinant and area scaling factor.* The Jacobian determinant of the transformation $\mathbf{F}$ is the determinant of the total derivative:

$$\begin{aligned}\frac{\partial(x,y)}{\partial(u,v)} &:= \det \mathbf{F}'(\mathbf{u}) \\ &= \det \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}.\end{aligned}$$

It is a scalar-valued function of (u,v). The area scaling factor is the *absolute value* of the Jacobian determinant:

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\det \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}\right|.$$

This too is a scalar-valued function of (u,v). A *linear* transformation has a constant area scaling factor, but a nonlinear transformation can stretch different parts of S differently.

Remark 13.11. If for some reason you instead have u,v as functions of x,y, and don't feel like solving for x,y in terms of u,v, then first compute the upside-down Jacobian determinant

$$\frac{\partial(u,v)}{\partial(x,y)} = \det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

and then use the identity

$$\frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1}.$$

The exponent $^{-1}$ here denotes the reciprocal of the function (1 divided by the function), not the inverse function.

13.5.4. *Steps for changing variables.*

Goal: Re-express

$$\iint_R f(x,y)\,dx\,dy$$

as a double integral in new variables u,v.

(1) Choose a transformation $x = x(u,v)$, $y = y(u,v)$ in order to make the integrand simpler or the region simpler.
(2) Find the equations of the boundary curves of R in terms of x,y.
(3) Rewrite these in terms of u,v to find the corresponding region S in the uv-plane.
(4) Compute the Jacobian determinant

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$$

(5) *Take the absolute value* of the Jacobian determinant to get the area scaling factor:

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right|.$$

Area is always nonnegative!

(6) Substitute

$$x = x(u,v)$$
$$y = y(u,v)$$
$$dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,du\,dv.$$

into the integral, and remember to use the equations of the boundary curves in terms of u, v to describe the new region S in the uv-plane, and hence determine the new limits of integration.

Remark 13.12. For the change of variable

$$x = r\cos\theta, \qquad y = r\sin\theta$$

(with r, θ playing the roles of u, v), the area scaling factor $\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|$ turns out to be r. (Check this yourself!) This gives another explanation of the conversion formula

$$dx\,dy = r\,dr\,d\theta.$$

Problem 13.13. Let R be the square with corners $(\pm 1, 0)$ and $(0, \pm 1)$. Evaluate

$$I := \iint\limits_R \frac{\sin^2(x-y)}{x+y+2}\,dx\,dy.$$

Solution. It's often a good idea to choose u, v so that the sides of R correspond to $u =$ constant, $v =$ constant.

In this problem: Try $u = x + y$, $v = x - y$, so $x = (u+v)/2$, $y = (u-v)/2$.

Boundary curves: $u = 1$, $u = -1$, $v = 1$, $v = -1$.

Jacobian:

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} = -1/2$$

so

$$dx\,dy = \left|-\frac{1}{2}\right|\,du\,dv = \frac{1}{2}\,du\,dv.$$

So

$$I = \int_{-1}^{1}\int_{-1}^{1} \frac{\sin^2 v}{u+2}\,\frac{1}{2}\,du\,dv.$$

To finish, one should evaluate the inside integral (with variable u), and then the outside integral (with variable v). □

13.5.5. *Why is the area scaling factor equal to* $\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right|$*?* The rectangle $[u, u+du] \times [v, v+dv]$ maps to (approximately) a parallelogram whose sides are given by the vectors $\begin{pmatrix} x_u\,du \\ y_u\,du \end{pmatrix}$ and $\begin{pmatrix} x_v\,dv \\ y_v\,dv \end{pmatrix}$ In other words, as the input moves du in the u-direction, the x-coordinate of the output changes by approximately $x_u\,du$, and so on, because x_u is the rate of change of x with respect to u (as v is held constant).

The area of the parallelogram formed by those two vectors is the absolute value of the determinant:

$$\left|\det\begin{pmatrix} x_u\,du & x_v\,dv \\ y_u\,du & y_v\,dv \end{pmatrix}\right| = \left|\det\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}\right| du\,dv.$$

(Here we used twice that if one multiplies an entire column of a matrix by a "number", the determinant gets multiplied by that number.) This explains why

$$dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv.$$

Problem 13.14. Let R be the region in the first quadrant inside the ellipse $x^2 + 9y^2 = 12$ but outside the circle $x^2 + y^2 = 4$. Evaluate

$$\iint_R \frac{xye^{y^2}}{1-y^2}\,dx\,dy.$$

Solution. Since x^2 and y^2 appear in the equations of the boundary curves, and since y^2 appears in e^{y^2} in the integrand, let's use the change of variables $u = x^2$ and $v = y^2$, so $x = \sqrt{u}$ and $y = \sqrt{v}$. The original region R is described by $x^2 + 9y^2 \leq 12$ and $x^2 + y^2 \geq 4$ and $x, y \geq 0$. The corresponding region S in the uv-plane is described by $u + 9v \leq 12$ and $u + v \geq 4$ and $u, v \geq 0$; this is a triangle with vertices at $(4, 0)$, $(12, 0)$, and $(3, 1)$. We have

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \det\begin{pmatrix} 2x & 0 \\ 0 & 2y \end{pmatrix} = 4xy,$$

so

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{4xy},$$

which is positive, so its absolute value is also $\frac{1}{4xy}$, which is the area scaling factor:

$$dx\,dy = \frac{1}{4xy}\,du\,dv.$$

Thus the integral becomes

$$\iint_S \frac{xye^{y^2}}{1-y^2}\frac{1}{4xy}\,du\,dv = \frac{1}{4}\iint_S \frac{e^v}{1-v}\,du\,dv.$$

(We re-expressed everything in terms of u and v.) At this point, we have the option of doing the integration in the order $dv\,du$ instead of $du\,dv$, but the order $dv\,du$ would require dividing the triangle into two triangles, so it is easier to keep v as the outer variable. The range for v is $[0,1]$. For each v, we get the lower and upper limits for u by solving $u+v=4$ and $u+9v=12$ for u in terms of v. This leads to

$$\begin{aligned}\frac{1}{4}\int_{v=0}^{v=1}\int_{u=4-v}^{u=12-9v}\frac{e^v}{1-v}\,du\,dv &= \frac{1}{4}\int_{v=0}^{v=1}\frac{e^v}{1-v}u\bigg|_{u=4-v}^{u=12-9v}\,dv \\ &= \frac{1}{4}\int_{v=0}^{v=1}\frac{e^v}{1-v}(8-8v)\,dv \\ &= 2\int_{v=0}^{v=1}e^v\,dv \\ &= 2\,e^v\big|_{v=0}^{v=1} \\ &= 2e-2.\end{aligned}$$

□

Thursday, October 28

13.6. **Triple integrals.** Triple integral:

$$\iiint\limits_T f(x,y,z)\,dV$$

where $f(x,y,z)$ is continuous on a 3-dimensional region T. (Think of dividing T into tiny blocks, multiply the value of f at a sample point in each block by the volume of the block, and add the results to get an approximation.)

Problem 13.15. Let T be the region in $\mathbb{R}^3$ where $x^2+y^2+z^2\le 1$ and $z\ge 0$. So T is the upper half of the unit ball in $\mathbb{R}^3$. Find the centroid of T. (Assume constant density.)

Solution. By symmetry, it must be of the form $(0,0,\bar{z})$, where $\bar{z}$ is the average value of z on the half-ball:

$$\bar{z} := \frac{\iiint_T z\,dV}{\frac{1}{2}\left(\frac{4}{3}\pi 1^3\right)}.$$

At a given height z, the cross-section is the disk D_z defined by $x^2+y^2\le 1-z^2$ (with z viewed as constant), a disk of radius $\sqrt{1-z^2}$. So

$$\iiint\limits_T z\,dV = \int_{z=0}^{z=1}\left(\iint\limits_{D_z} z\,dx\,dy\right)dz.$$

At this point we *could* write the inner double integral as an iterated integral and obtain the following:

$$\int_{z=0}^{z=1} \int_{y=-\sqrt{1-z^2}}^{y=\sqrt{1-z^2}} \int_{x=-\sqrt{1-y^2-z^2}}^{x=\sqrt{1-y^2-z^2}} z\,dx\,dy\,dz.$$

But it is easier instead to factor the z out of the inner double integral (z is a constant for the inner integral, in which x and y are the variables), and to recognize what remains as an area:

$$\begin{aligned}
\iiint_T z\,dV &= \int_{z=0}^{z=1} z \ \mathrm{Area}(D_z)\,dz \\
&= \int_{z=0}^{z=1} z\left(\pi\sqrt{1-z^2}^{\,2}\right)\,dz \\
&= \pi\int_0^1 (z-z^3)\,dz \\
&= \frac{\pi}{4},
\end{aligned}$$

so

$$\begin{aligned}
\bar{z} &= \frac{\pi/4}{\frac{1}{2}\left(\frac{4}{3}\pi 1^3\right)} \\
&= \frac{3}{8}.
\end{aligned}$$

□

13.7. **Cylindrical coordinates.** In $\mathbb{R}^3$, replace x, y by polar coordinates, but keep z: this leads to cylindrical coordinates.

To convert from cylindrical coordinates (r, θ, z) to rectangular coordinates (x, y, z):

$$\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
z &= z.
\end{aligned}$$

The other way: Given (x, y, z), compute

$$\begin{aligned}
r &= \sqrt{x^2+y^2} \\
\tan\theta &= \frac{y}{x} \qquad \text{(check quadrant)} \\
z &= z.
\end{aligned}$$

What happens to volume?

$$dV = dx\,dy\,dz = dz\,r\,dr\,d\theta.$$

This can also be visualized geometrically.

Let's redo the half-ball integral: In cylindrical coordinates, the half-ball is given by $r^2 + z^2 \le 1$ and $z \ge 0$. For each fixed point (r, θ) in the plane, look at the vertical segment above/below it contained in T to get the range of integration for z:

$$\begin{aligned}
\iiint_T z\, dV &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{z=0}^{z=\sqrt{1-r^2}} z\, dz\, r\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^1 \frac{1-r^2}{2}\, r\, dr\, d\theta \\
&= \int_0^{2\pi} \frac{1}{8}\, d\theta \\
&= \frac{\pi}{4},
\end{aligned}$$

which agrees with what we got before.

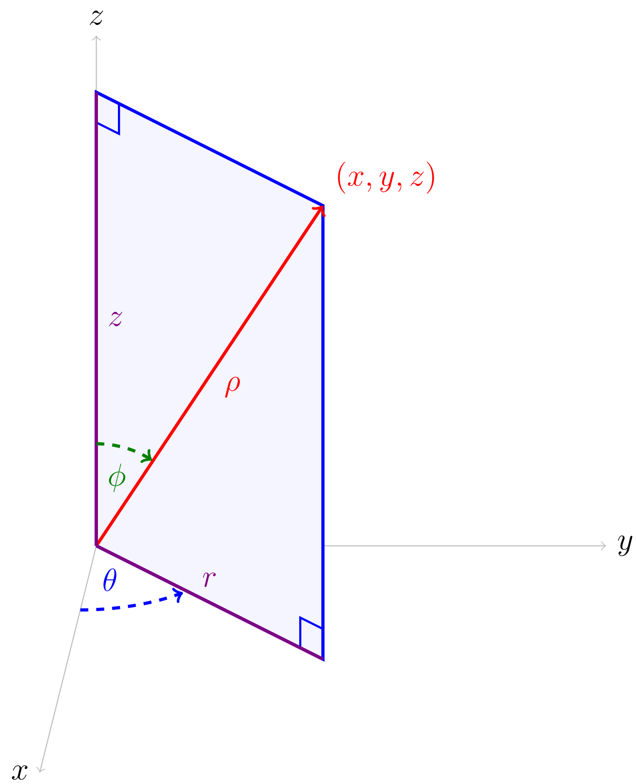
13.8. **Spherical coordinates.**

Spherical coordinates are (ρ, ϕ, θ) (pronounced "rho, phi, theta"), where

$$\begin{aligned}
\rho &:= \text{distance to } (0,0,0) \\
\phi &:= \text{angle down from } z\text{-axis (angle between position vector and positive } z\text{-axis)} \\
\theta &:= \text{same } \theta \text{ as in cylindrical coordinates, "longitude", depends only on } x, y.
\end{aligned}$$

Warning: Some books use different Greek letters here, so when reading another book, check the definitions of the variables.

Imagine an $r \times z$ swinging door hinged along the z-axis, with one corner at $(0,0,0)$ and opposite corner at (x, y, z), so ρ is the length of the diagonal of the door, and ϕ is the angle the diagonal forms with the positive z-axis. By trigonometry in the upper half of the door (above the diagonal), $r = \rho \sin \phi$ is the width, and $z = \rho \cos \phi$ is the height (or $-$height if $\phi > \pi/2$). Finally, θ controls how far counterclockwise the door has swung.



Range of possible values:

$$\rho \in [0, \infty)$$
$$\phi \in [0, \pi]$$
$$\theta \in [0, 2\pi] \qquad (\text{or } [-\pi, \pi] \text{ or } \dots)$$

(Along the z-axis, θ is indeterminate — it can be any value. At the origin, ϕ is indeterminate too.)

13.8.1. *Conversion between spherical coordinates and rectangular coordinates.*

- Given (ρ, ϕ, θ), we have $r = \rho \sin \phi$, so

$$x = r \cos \theta = \rho \sin \phi \cos \theta$$
$$y = r \sin \theta = \rho \sin \phi \sin \theta$$
$$z = \qquad\qquad \rho \cos \phi.$$

- Given (x, y, z), compute

$$\rho := \sqrt{x^2 + y^2 + z^2},$$

then convert (x, y) to polar coordinates (r, θ) to get θ, and finally get ϕ from

$$\cos \phi = \frac{z}{\rho}.$$

(That is better than using
$$\sin\phi = \frac{r}{\rho}$$
since the latter cannot distinguish between ϕ and $\pi - \phi$.)

13.8.2. *Relationship to latitude and longitude.* A great circle on a sphere S is the intersection of S with a plane through the center. If P, Q are in S, the shortest path *along the sphere* connecting P and Q is an arc of a great circle. (To visualize this, rotate the sphere so that P and Q are on the equator.)

Set up an xyz-coordinate system with the origin at the center of the earth, with the positive z-axis passing through the North Pole, with the xy-plane containing the Equator, and with Greenwich (near London) contained in the xz-plane. Then $\theta°$ (i.e., θ measured in degrees) is longitude (east if $\theta > 0$). And $90° - \phi°$ is latitude (number of degrees up from the Equator).

Friday, October 29

13.8.3. *Volume in spherical coordinates.* The region with spherical coordinates in the intervals $[\rho, \rho+d\rho]$, $[\phi, \phi+d\phi]$, $[\theta, \theta+d\theta]$ defines a tiny box of volume approximately $d\rho$ times the surface area of a little "rectangle" on the sphere of sides $\rho\, d\phi$ and $r\, d\theta$ (these "sides" are actually tiny arcs of circles, and the formula for the length of such an arc is (radius)(arc measure)). The volume of the box is approximately
$$(d\rho)(\rho\, d\phi)(r\, d\theta) = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta,$$
so
$$\boxed{dV = dx\, dy\, dz = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.}$$
Another way to find this formula: use the change of variable formula
$$dx\, dy\, dz = \left| \frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} \right| d\rho\, d\phi\, d\theta.$$
(The volume scaling factor is the absolute value of the 3×3 Jacobian determinant.)

13.9. **Gravitation.**

13.9.1. *Gravitational force exerted by a point mass.* Let $\mathbf{F}$ be the gravitational force of a point of mass M with position vector $\mathbf{r}$ acting on a point of mass m at $(0,0,0)$, so $\mathbf{F}$ is in the same direction as $\mathbf{r}$, which points from m to M. Let $\rho = |\mathbf{r}|$, which is the distance between the two masses. Then Newton says
$$\mathbf{F} = \underbrace{\frac{GmM}{\rho^2}}_{\text{magnitude}} \underbrace{\frac{\mathbf{r}}{\rho}}_{\text{direction}} . \tag{2}$$

13.9.2. *Gravitational force exerted by a solid body.* Now suppose that instead of a point of mass M, there is a solid body T whose density is given by $\delta = \delta(x, y, z)$. Now what is the total force $\mathbf{F}$ that T exerts on a point mass m at $(0, 0, 0)$? Each little piece of T will be pulling m, in different directions, and we need to add up (integrate) these little force vectors to compute the total force. A little piece of volume dV at position $\mathbf{r}$ has mass $dM = \delta\, dV$, so by formula (2), the force it exerts on m is

$$\frac{Gm\,dM}{\rho^2}\,\frac{\mathbf{r}}{\rho}.$$

To integrate this would involve a triple integral of a vector-valued function but we've only talked about triple integrals of scalar-valued functions in lecture so far, so let's instead just compute the z-*component* of $\mathbf{F}$:

$$\begin{aligned}
(z\text{-component of } \mathbf{F}) &= \operatorname{comp}_{\mathbf{e}_3} \mathbf{F} \\
&= \mathbf{F} \cdot \mathbf{e}_3 \\
&= \iiint_T \frac{Gm\,dM}{\rho^2}\,\frac{\mathbf{r}}{\rho} \cdot \mathbf{e}_3 \\
&= \iiint_T \frac{Gm\,dM}{\rho^2}\,\frac{z}{\rho} \\
&= Gm \iiint_T \frac{\cos\phi}{\rho^2} dM \qquad (\text{since } z = \rho\cos\phi) \\
&= Gm \iiint_T \frac{\cos\phi}{\rho^2} \delta\, dV \\
&= Gm \iiint_T \delta \cos\phi \sin\phi\, d\rho\, d\phi\, d\theta \qquad (\text{since } dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta).
\end{aligned}$$

Example 13.16. Suppose that T is a solid sphere of radius a centered at $(0, 0, a)$, of constant density $\delta = 1$, and suppose that $m = 1$. Warning: The $(0, 0, 0)$ for our spherical coordinates is not at the center of T, but at its south pole, where m is! This means that if $\mathbf{r}$ is the position vector of a point P inside T, then $\mathbf{r}$ is pointing from the *south pole* to P, and the angle ϕ that $\mathbf{r}$ and the positive z-axis form will be in the range $[0, \pi/2]$, not $[0, \pi]$.

By symmetry, the gravitational force $\mathbf{F}$ is pointing straight up, so

$$\begin{aligned}
|\mathbf{F}| &= (z\text{-component of } \mathbf{F}) \\
&= G \iiint_T \cos\phi \sin\phi \, d\rho \, d\phi \, d\theta \qquad \text{(by the previous calculation)} \\
&= G \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} \int_{\rho=0}^{2a\cos\phi} \cos\phi \sin\phi \, d\rho \, d\phi \, d\theta \\
&= G \int_0^{2\pi} \int_0^{\pi/2} 2a \cos^2\phi \sin\phi \, d\phi \, d\theta \\
&= 2Ga \int_0^{2\pi} -\frac{1}{3} \cos^3\phi \Big|_0^{\pi/2} \, d\theta \\
&= 2Ga \int_0^{2\pi} \frac{1}{3} \, d\theta \\
&= \frac{4\pi Ga}{3}.
\end{aligned}$$

On the other hand, since $\delta = 1$, the total mass of T is $M = \text{Volume}(T) = \frac{4}{3}\pi a^3$, so *if all the mass of T were concentrated at its center*, then Newton's law of gravitation for a point mass would give

$$|\mathbf{F}| = \frac{GM}{a^2} = \frac{4\pi Ga}{3}.$$

Same answer!

Remark 13.17. Newton proved more generally that a solid sphere of constant density exerts the same force on *any* external mass m (not necessarily on the surface) as would a point mass M placed at its center.

Tuesday, November 2

14. Vector fields

Definition 14.1. A vector field is a function whose value at each point of a region is a vector.

It could be showing, for example,

- the wind velocity at each point,
- the velocity of a fluid at each point,
- the strength and direction of an electric field or magnetic field,
- or the gradient of a function at each point.

Mathematically, a 2-dimensional vector field has the form

$$\begin{aligned}\mathbf{F}\colon \mathbb{R}^2 &\longrightarrow \mathbb{R}^2\\ (x,y) &\longmapsto \begin{pmatrix} P(x,y) \\ Q(x,y) \end{pmatrix} = P(x,y)\,\mathbf{e}_1 + Q(x,y)\,\mathbf{e}_2,\end{aligned}$$

where $P\colon \mathbb{R}^2 \to \mathbb{R}$ and $Q\colon \mathbb{R}^2 \to \mathbb{R}$ are functions.

Example 14.2. Consider the vector field $\mathbf{F}(x,y) := \begin{pmatrix} -y \\ 0 \end{pmatrix}$. Its values in the upper half plane are vectors pointing to the left, and its values in the lower half plane are vectors pointing to the right. (The students in class demonstrated this by forming one big vector field with their hands.)

Example 14.3. The gradient field of $f(x,y) := x^2 + y^2$ is $\mathbf{F}(x,y) := \begin{pmatrix} 2x \\ 2y \end{pmatrix}$, which is always pointing outward. Think of the direction of fastest increase, looking at a top view of a paraboloid. The gradient field is everywhere perpendicular to the level curves, which are circles.

$$\begin{aligned}
\mathbf{F} \text{ is continuous} &\iff P \text{ and } Q \text{ are continuous.}\\
\mathbf{F} \text{ is differentiable} &\iff P \text{ and } Q \text{ are differentiable.}\\
\mathbf{F} \text{ is continuously differentiable} &\iff P \text{ and } Q \text{ are differentiable,}\\
&\qquad \text{and their partial derivatives are continuous.}
\end{aligned}$$

14.1. **Line integrals.** Let C be an oriented curve in $\mathbb{R}^2$. (Oriented means that one of the two directions along the curve has been chosen.)

Three kinds of line integrals:

- Line integral with respect to arc length: $\int_C f(x,y)\,ds$.
  (Remember: s = distance traveled = arc length.)
- Line integral with respect to coordinate variables: $\int_C f(x,y)\,dx$ or $\int_C f(x,y)\,dy$ or $\int_C P(x,y)\,dx + Q(x,y)\,dy$.
- Line integral of a vector field: $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Which are vectors and which are scalars? They are all scalars!

14.1.1. *What do these mean?* Divide C into n pieces by labelling points $P_0, P_1, \ldots, P_n$ in order along C where P_0 and P_n are the endpoints. Choose a sample point P_i^* on the arc from

P_{i-1} to P_i. Let Δs_i be the arc length of that arc. Then

$$\int_C f(x,y)\,ds := \lim_{\max \Delta s_i \to 0} \left(f(P_1^*)\Delta s_1 + \cdots + f(P_n^*)\Delta s_n\right)$$
$$\int_C f(x,y)\,dx := \lim_{\max \Delta s_i \to 0} \left(f(P_1^*)(x(P_1) - x(P_0)) + \cdots + f(P_n^*)(x(P_n) - x(P_{n-1}))\right)$$
$$\int_C \mathbf{F} \cdot d\mathbf{r} := \lim_{\max \Delta s_i \to 0} \left(\mathbf{F}(P_1^*) \cdot \overrightarrow{P_0 P_1} + \cdots + \mathbf{F}(P_n^*) \cdot \overrightarrow{P_{n-1} P_n}\right).$$

Example 14.4.

$$\int_C 1\,ds = \text{ arc length of } C.$$

14.1.2. *How do you compute these?*

(1) Choose a parametrization of C, say $\mathbf{r}(t) := \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ for $t \in [a,b]$. Now $x, y, s, \mathbf{r}$ are all functions of t.
(2) Substitute $x = x(t)$ and $y = y(t)$ in the integrand.
(3) Also substitute

$$\begin{aligned} dx &= x'(t)\,dt \\ dy &= y'(t)\,dt \\ ds &= \sqrt{dx^2 + dy^2} = \sqrt{x'(t)^2 + y'(t)^2}\,dt \\ d\mathbf{r} &= \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} dt. \end{aligned}$$

(4) Change $\int_C$ to $\int_a^b$.
(5) Evaluate the resulting 1-variable integral in t.

Problem 14.5. Let C be the upper half of the circle $x^2 + y^2 = 4$, oriented counterclockwise. Let $\mathbf{F}(x,y) = \begin{pmatrix} -y \\ 0 \end{pmatrix}$. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Is the answer going to be positive or negative? Positive, because everywhere along the curve, $\mathbf{F}$ and $d\mathbf{r}$ form an angle less than $90°$.

Solution. Choose the parametrization $\begin{pmatrix} 2\cos t \\ 2\sin t \end{pmatrix}$ for $t \in [0, \pi]$. Then

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C \begin{pmatrix} -y \\ 0 \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \end{pmatrix} \\
&= \int_0^\pi \begin{pmatrix} -2\sin t \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -2\sin t \\ 2\cos t \end{pmatrix} dt \\
&= \int_0^\pi 4\sin^2 t \, dt \\
&= \int_0^\pi (2 - 2\cos 2t) \, dt \\
&= 2\pi.
\end{aligned}$$

□

14.1.3. *Geometric interpretation.* Recall that $d\mathbf{r} = \mathbf{T}\, ds = \begin{pmatrix} dx \\ dy \end{pmatrix}$, where $\mathbf{T}$ is the unit tangent vector. So if $\mathbf{F} = \begin{pmatrix} P \\ Q \end{pmatrix}$, then

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C \underbrace{\mathbf{F} \cdot \mathbf{T}}_{\text{tangential component of } \mathbf{F}} \, ds \\
&= \int_C P\, dx + Q\, dy.
\end{aligned}$$

14.2. **Applications of line integrals.**

14.2.1. *Applications of line integrals with respect to arc length.* If C is a wire whose density (mass per unit length) is $\delta(x, y)$, then $dm = \delta\, ds$, so the quantities

$$\begin{aligned}
\text{mass } m &:= \int_C dm \\
\text{mass-weighted average } \bar{f} &:= \frac{\int_C f\, dm}{m} \\
\text{centroid} &:= (\bar{x}, \bar{y}) \\
\text{moment of inertia } I &:= \int_C (\text{distance to axis})^2 \, dm
\end{aligned}$$

all involve line integrals with respect to *arc length*.

14.2.2. *Work.* Given a force field that is constant (independent of position),

$$\begin{aligned}
\text{work} &:= \overrightarrow{\text{force}} \cdot \overrightarrow{\text{displacement}} \\
&:= \mathbf{F} \cdot \Delta\mathbf{r},
\end{aligned}$$

More generally, for an object moving along C, the work done by a not necessarily constant force field $\mathbf{F}$ is

$$W := \int_C \mathbf{F} \cdot d\mathbf{r}.$$

(Think of adding up the work done over each little piece of C.)

14.3. **Adding and reversing paths.** If the endpoint of path C_1 equals the start point of C_2, then concatenating gives a path $C_1 + C_2$, and

$$\int_{C_1+C_2} \cdots \quad = \quad \int_{C_1} \cdots \quad + \quad \int_{C_2} \cdots .$$

Let C be a piecewise differentiable oriented curve. Then $-C$ denotes the same curve with the reverse orientation. For each piece of C, the "tiny change in position" $d\mathbf{r}$ changes sign, so its components dx and dy change sign too, but the "tiny length" $ds = |d\mathbf{r}|$ stays the same. Thus

$$\int_{-C} f\, ds = \int_C f\, ds$$

$$\int_{-C} P\, dx + Q\, dy = -\int_C P\, dx + Q\, dy$$

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}.$$

Thursday, November 4

14.4. **Fundamental theorem of calculus for line integrals.**

1st FTC: If $f(x)$ is a continuous function on $[a, b]$, and

$$F(x) := \int_a^x f(t)\, dt$$

then $F'(x) = f(x)$.

Main purpose: Construction of an antiderivative of $f(x)$.

2nd FTC: If $G(x)$ is such that $G'(x)$ is continuous on an interval $[a, b]$, then

$$\int_a^b G'(t)\, dt = G(b) - G(a).$$

Main purpose: Evaluation of a definite integral $\int_a^b G'(t)\, dt$.

FTC for line integrals: If C starts at A and ends at B, and f is a function that is continuously differentiable at each point of C, then

$$\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A).$$

The left side is adding up the change in f along each piece of the curve, and the right side is the total change in f. The formula is the same for a function f in any number of variables.

Proof. Let $\mathbf{r}(t)$ for $t \in [a, b]$ be a parametrization of C. Let $G(t) = f(\mathbf{r}(t))$. Then

$$\begin{aligned}
\int_C \nabla f \cdot d\mathbf{r} &= \int_a^b \nabla f \cdot \mathbf{r}'(t)\, dt \\
&= \int_a^b G'(t)\, dt \qquad \text{(by the gradient form of the chain rule)} \\
&= G(b) - G(a) \qquad \text{(by the 2}^{\text{nd}}\text{ FTC)} \\
&= f(\mathbf{r}(b)) - f(\mathbf{r}(a)) \\
&= f(B) - f(A).
\end{aligned}$$

□

Consequences for *gradient* fields:

- It is very easy to compute a line integral of a gradient field. There's no need to parametrize the curve — the FTC gives the answer directly!
- $\int_A^B \nabla f \cdot d\mathbf{r}$ is path independent for each pair of points A and B: this says that the value of $\int_C \nabla f \cdot d\mathbf{r}$ is the same for every curve C from A to B!
- If C is a closed curve (starts and ends at the same point), then $\oint_C \nabla f \cdot d\mathbf{r} = 0$. (There is no difference between $\oint$ and $\int$: the former is just a notation to help remind you that the curve is closed.)

These are properties special to *gradient* vector fields.

Example 14.6. Let $\mathbf{F} = \begin{pmatrix} y \\ 0 \end{pmatrix}$. Let C be the upper half of the unit circle, and let C' be the lower half, both oriented from $(1, 0)$ to $(-1, 0)$. The geometry shows that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is negative, but $\int_{C'} \mathbf{F} \cdot d\mathbf{r}$ is positive. So line integrals of $\mathbf{F}$ are *not* path independent. Thus $\mathbf{F}$ cannot be ∇f for any function f.

14.5. **The operator $\boldsymbol{\nabla}$.** Introduce the notation

$$\nabla := \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{pmatrix}.$$

The symbol ∇ is sometimes pronounced "nabla" (ancient Greek word for harp). It is not an actual vector, since its entries are not numbers (or functions). But it helps us remember the definition of gradient of a function $f(x,y,z)$,

$$\nabla f := \begin{pmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \\ \dfrac{\partial f}{\partial z} \end{pmatrix},$$

which is a kind of derivative of f (it is the transpose of the total derivative $f'(\mathbf{x})$).

14.6. **Divergence of a 3D vector field.** For a differentiable 3D vector field

$$\mathbf{F}(x,y,z) = \begin{pmatrix} P(x,y,z) \\ Q(x,y,z) \\ R(x,y,z) \end{pmatrix},$$

we can now use ∇ to define two new fancy kinds of derivative, called *divergence* and *curl*.

Definition 14.7. The divergence of $\mathbf{F}$ is the function

$$\begin{aligned} \operatorname{div} \mathbf{F} &:= \nabla \cdot \mathbf{F} \\ &:= \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} P \\ \\ Q \\ \\ R \end{pmatrix} \\ &:= P_x + Q_y + R_z. \end{aligned}$$

Vector or scalar? The divergence of $\mathbf{F}$ is a scalar (at each point), just like any dot product. So don't make the mistake of thinking that $\operatorname{div} \mathbf{F}$ is $\begin{pmatrix} P_x \\ Q_y \\ R_z \end{pmatrix}$!

Problem 14.8. Let $\mathbf{F}(x,y,z) := \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, a 3D vector field that is pointing radially outward at each point. What is $\operatorname{div} \mathbf{F}$?

Solution. By definition,

$$\begin{aligned}
\operatorname{div} \mathbf{F} &= \nabla \cdot \mathbf{F} \\
&= \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\
&= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) \\
&= 3.
\end{aligned}$$

(Usually $\operatorname{div} \mathbf{F}$ is a function taking different values at different points in $\mathbb{R}^3$, but in this problem, $\operatorname{div} \mathbf{F}$ turned out to be a constant function.) □

Physical interpretation of divergence:

Let $\mathbf{F}$ be the velocity field of a fluid of density 1. Then, at each point, think of $\operatorname{div} \mathbf{F}$ as measuring

the net amount of fluid exiting a tiny cube per unit volume and per unit time.

(If more fluid is entering than exiting, then the net amount will be negative.)

It is not obvious from the formula for $\operatorname{div} \mathbf{F}$ why $\operatorname{div} \mathbf{F}$ measures this. We'll explain this later, using the concept of *flux*.

Example 14.9. Suppose that the velocity field $\mathbf{F}$ is the vector field of Problem 14.8: $\mathbf{F}(x, y, z) := \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. For a tiny cube centered at $(0, 0, 0)$, the net flow is outward. For a tiny cube centered at $(0, 0, 1)$, the net flow is again outward, since the rate of flow out the top is slightly greater than the rate of flow into the bottom (and there is also fluid flowing out the sides). These observations are consistent with $\operatorname{div} \mathbf{F}$ being positive at these points.

14.7. **Curl of a 3D vector field.** Again assume that

$$\mathbf{F}(x, y, z) = \begin{pmatrix} P(x, y, z) \\ Q(x, y, z) \\ R(x, y, z) \end{pmatrix}.$$

Definition 14.10. The curl of $\mathbf{F}$ is the function

$$\begin{aligned}
\operatorname{curl} \mathbf{F} &:= \nabla \times \mathbf{F} \\
&:= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\
&:= (R_y - Q_z)\mathbf{e}_1 + (P_z - R_x)\mathbf{e}_2 + (Q_x - P_y)\mathbf{e}_3.
\end{aligned}$$

Vector or scalar? The curl of $\mathbf{F}$ is a vector (at each point), just like any cross product.

Theorem 14.11 (Curl of a gradient field is zero)**.** *For any function $f\colon \mathbb{R}^3 \to \mathbb{R}$ with continuous second partial derivatives,*

$$\operatorname{curl}(\nabla f) = \mathbf{0}.$$

Proof. By definition,

$$\begin{aligned}
\operatorname{curl}(\nabla f) &= \nabla \times \nabla f \\
&= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} \\
&= (f_{zy} - f_{yz})\mathbf{e}_1 + (f_{xz} - f_{zx})\mathbf{e}_2 + (f_{yx} - f_{xy})\mathbf{e}_3 \\
&= \mathbf{0}.
\end{aligned}$$

□

Physical interpretation of curl:

Let $\mathbf{F}$ be the velocity field of a fluid again. Through any given point, there is an axis around which a tiny paddlewheel in the fluid would rotate the fastest. Let ω be the angular speed in radians per second.

- The direction of $\operatorname{curl} \mathbf{F}$ is along this axis, with direction given by the right hand rule (if your fingers are in the direction of rotation, your thumb gives the direction of $\operatorname{curl} \mathbf{F}$).
- The length of $\operatorname{curl} \mathbf{F}$ is 2ω.

If instead $\mathbf{F}$ is a *force* field (measuring force exerted per unit mass), then $\operatorname{curl} \mathbf{F}$ measures the torque per unit of moment of inertia exerted on a tiny dumbbell. An analogy:

$$\frac{\overrightarrow{\text{force}}}{\text{mass}} = \overrightarrow{\text{acceleration}} := \frac{d}{dt}(\overrightarrow{\text{velocity}})$$

$$\frac{\overrightarrow{\text{torque}}}{\text{moment of inertia}} = \overrightarrow{\text{angular acceleration}} := \frac{d}{dt}(\overrightarrow{\text{angular velocity}}).$$

(The direction of rotational quantities like torque or angular velocity is *along the axis of rotation*, given by the right hand rule.)

Friday, November 5

14.8. **Simply connected regions.** Suppose that T is a connected region in $\mathbb{R}^2$ or $\mathbb{R}^3$. Let's say that a closed curve C inside T is shrinkable in T if it is possible to continuously shrink C to a point *within* T.

Example 14.12. If T is $\mathbb{R}^2$ with the unit disk removed, then a circle of radius 2 centered at the origin is contained in T but is not shrinkable in T. (To shrink it, the curve would have to enter the unit disk, at least temporarily, and that is not allowed.)

Definition 14.13. Call T simply connected if every closed curve C in T is shrinkable in T.

Question 14.14. Which of the following are simply connected?

- The annulus in $\mathbb{R}^2$ defined by $1 < x^2 + y^2 < 9$? No, because the circle $x^2 + y^2 = 4$ is not shrinkable inside the annulus.
- $\mathbb{R}^2$ with a point removed? No, for the same reason.
- $\mathbb{R}^3$? Yes.
- a solid ball? Yes.
- a solid torus (donut)? No, because a curve going all the way around the donut is not shrinkable.
- $\mathbb{R}^3$ with a solid ball removed? Yes, since a curve going around the ball can be slipped around the ball.
- $\mathbb{R}^3$ with a point removed? Yes, for the same reason.
- $\mathbb{R}^3$ with an infinite cylinder removed? No, because a curve going around the cylinder is not shrinkable.

15. Conservative vector fields

All vector fields $\mathbf{F}$ today are assumed to be continuous in a 3D region T. If considering $\operatorname{curl} \mathbf{F}$, then $\mathbf{F}$ is assumed to be continuously differentiable.

15.1. **Equivalent conditions.** Here are four conditions that $\mathbf{F}$ might or might not satisfy:

(1) $\mathbf{F}$ is a gradient field (equal to ∇f for some $f = f(x, y, z)$ on T).
(2) $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is path independent for every A and B in T (for paths inside T).
(3) $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed curve C inside T.
(4) $\operatorname{curl} \mathbf{F} = \mathbf{0}$ at every point of T.

Theorem 15.1 (Equivalence of conditions)**.**

- *Conditions* (1), (2), (3) *are equivalent.*
- *Any of* (1), (2), (3) *implies* (4).
- *If T is simply connected, then all four conditions are equivalent.*

Example 15.2. Suppose that $\mathbf{F}$ is a vector field that satisfies condition (2), the path independence condition. Then the equivalence of (1), (2), (3) means that $\mathbf{F}$ also is a gradient field, and $\mathbf{F}$ also satisfies the closed curve condition. Moreover, any of those conditions implies (4), so $\operatorname{curl} \mathbf{F} = \mathbf{0}$ too.

Example 15.3. Suppose that $\mathbf{F}$ is a vector field that satisfies condition (4), that $\operatorname{curl} \mathbf{F} = \mathbf{0}$ at every point of T. There is no guarantee that $\mathbf{F}$ satisfies conditions (1), (2), (3) too (the implication goes the wrong direction). But if we also know that T is simply connected, then $\mathbf{F}$ satisfies all four conditions.

Definition 15.4. Say that $\mathbf{F}$ is conservative if it $\mathbf{F}$ satisfies any of conditions (1), (2), (3) (it doesn't matter which, since they are all equivalent!). Any differentiable function f such that $\nabla f = \mathbf{F}$ is called a potential function for $\mathbf{F}$ — the multivariable analogue of an antiderivative.

Problem 15.5. Consider a solid spherical planet of mass M. Let the region T be all space *outside* the planet. The gravitational field $\mathbf{F}$ at each point of T represents the force that the planet would exert on a unit mass at that point. In terms of spherical coordinates (ρ, ϕ, θ) with origin at the planet's center, Newton's inverse square law says

$$\mathbf{F} = -\frac{GM}{\rho^2}\,\widehat{\boldsymbol{\rho}}.$$

Is $\mathbf{F}$ conservative?

Solution. It is enough to check any of conditions (1), (2), (3). In fact, since T is simply connected, we can check any of (1), (2), (3), (4). Usually (2) and (3) are hard to check, because they involve the integrals along many paths. Usually (4) is the easiest to check since it just involves calculating a curl, but we didn't ever work out the formula for curl in spherical coordinates.

So let's instead check (1), that $\mathbf{F}$ is a gradient field. To do this, we'll guess f and then verify that $\mathbf{F} = \nabla f$. Let

$$f = \frac{GM}{\rho}.$$

Is it true that $\nabla f = \mathbf{F}$? Yes, because

- ∇f is radially inward, in the direction of $-\hat{\boldsymbol{\rho}}$, just like $\mathbf{F}$, by symmetry; and
- ∇f has the same $\hat{\boldsymbol{\rho}}$-component as $\mathbf{F}$:

$$\nabla f \cdot \hat{\boldsymbol{\rho}} = D_{\hat{\boldsymbol{\rho}}} f = \frac{\partial}{\partial \rho}\frac{GM}{\rho} = -\frac{GM}{\rho^2} = \mathbf{F} \cdot \hat{\boldsymbol{\rho}}.$$

Thus $\mathbf{F}$ is a gradient field. In other words, $\mathbf{F}$ is conservative. $\square$

Remark 15.6. Another way to verify that $\nabla f = \mathbf{F}$ would have been to use the general formula for ∇f in spherical coordinates:

$$\begin{aligned}\nabla f &= \frac{\partial f}{\partial \rho}\,\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\,\hat{\boldsymbol{\phi}} + \frac{1}{\rho \sin\phi}\frac{\partial f}{\partial \theta}\,\hat{\boldsymbol{\theta}} \\ &= -\frac{GM}{\rho^2}\,\hat{\boldsymbol{\rho}} + 0\hat{\boldsymbol{\phi}} + 0\hat{\boldsymbol{\theta}} \\ &= \mathbf{F}.\end{aligned}$$

15.2. **Proof of equivalence.** Why are conditions (1), (2), (3) equivalent?

Proof.

(1) $\Longrightarrow$ (3): We already showed that a line integral of a gradient field along a closed curve is 0.

(3) $\Longrightarrow$ (2): Suppose that $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed curve C. We need to prove path independence. So suppose that C_1 and C_2 are two paths from A to B. Then $C_1 + (-C_2)$ is a closed loop, so

$$0 = \oint_{C_1+(-C_2)} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

So $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$.

(2) $\Longrightarrow$ (1): Suppose that line integrals of $\mathbf{F}$ are independent of path. Choose a start point $\mathbf{a}$ in T. Define

$$f(\mathbf{x}) := \int_{\mathbf{a}}^{\mathbf{x}} \mathbf{F} \cdot d\mathbf{r}.$$

(This makes sense since the value is independent of the path.) We will check $\nabla f = \mathbf{F}$ coordinate by coordinate. To calculate the first coordinate f_x at a point $\mathbf{b}$, we need the values of f along the horizontal line through $\mathbf{b}$, so define

$$g(x) := f(x, b_2, b_3) = \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{r} + \int_{\mathbf{b}}^{(x,b_2,b_3)} \mathbf{F} \cdot d\mathbf{r}.$$

Along that line,

$$\begin{aligned}f_x &= \frac{dg}{dx} \\ &= \frac{d}{dx}\int_{\mathbf{b}}^{(x,b_2,b_3)} \mathbf{F} \cdot d\mathbf{r} && \text{(the first term of } g(x) \text{ is a constant)} \\ &= \frac{d}{dx}\int_{b_1}^{x} \begin{pmatrix} P \\ Q \\ R \end{pmatrix} \cdot \begin{pmatrix} dt \\ 0 \\ 0 \end{pmatrix} && \text{(using } \mathbf{r}(t) := (t, b_2, b_3)\text{)} \\ &= \frac{d}{dx}\int_{b_1}^{x} P\,dt \\ &= P && \text{by the one-variable 1}^{\text{st}} \text{ FTC).}\end{aligned}$$

Similarly, $f_y = Q$ and $f_z = R$. Thus $\nabla f = \mathbf{F}$. $\square$

Also, Theorem 14.11 shows that if $\mathbf{F}$ is a gradient field, then $\operatorname{curl}\mathbf{F} = \mathbf{0}$, so (1) $\Longrightarrow$ (4).

To finish the proof of Theorem 15.1, we need to explain why if T is simply connected, (4) implies one of the first three conditions. This requires Stokes' theorem, so we'll postpone this.

15.3. **Finding the potential.** We had time only to sketch the first solution to Problem 2 below. The details of this section will be covered in recitation on Monday, November 8.

Consider the vector field

$$\mathbf{F} = \begin{pmatrix} 3x^2y + 5yz \\ x^3 + 7z + 5xz \\ 7y + 5xy + e^z \end{pmatrix}$$

on $\mathbb{R}^3$.

Problem 1: Is there a function f such that $\nabla f = \mathbf{F}$?

Solution: Since $\mathbb{R}^3$ is simply connected, we can check any of the four conditions. Let's test (4):

$$\begin{aligned}
\operatorname{curl}\mathbf{F} &:= \nabla \times \mathbf{F} \\
&= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2y + 5yz & x^3 + 7z + 5xz & 7y + 5xy + e^z \end{vmatrix} \\
&= \mathbf{e}_1((7+5x) - (7+5x)) - \mathbf{e}_2(5y - 5y) + \mathbf{e}_3((3x^2+5z) - (3x^2+5z)) \\
&= \mathbf{0}.
\end{aligned}$$

So the answer is yes.

Problem 2: Can you *find* such a potential function f?

First solution (FTC in reverse): If $\nabla f = \mathbf{F}$, then

$$\begin{aligned}
f(a,b,c) - f(0,0,0) &= \int_C \nabla f \cdot d\mathbf{r} \\
&= \int_C \mathbf{F} \cdot d\mathbf{r} \\
&= \int_{(0,0,0)}^{(a,b,c)} (3x^2y + 5yz)\, dx + (x^3 + 7z + 5xz)\, dy + (7y + 5xy + e^z)\, dz.
\end{aligned}$$

Choose the path $C_1 + C_2 + C_3$ where C_1 goes from $(0,0,0)$ to $(a,0,0)$, C_2 goes from $(a,0,0)$ to $(a,b,0)$, and C_3 goes from $(a,b,0)$ to (a,b,c). Add up the three integrals. For C_1, use the parametrization $\mathbf{r}(t) := \langle t, 0, 0\rangle$ for $t \in [0, a]$; then $x = t$, $y = 0$, $z = 0$, $dx = dt$, $dy = 0$, $dz = 0$, so

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^a 0\, dt = 0.$$

Similarly, for C_2 use $\langle a, t, 0\rangle$ for $t \in [0, b]$; then $x = a$, $y = t$, $z = 0$, $dx = 0$, $dy = dt$, $dz = 0$, so

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^b a^3\, dt = a^3 b.$$

Finally, for C_3, use the parametrization $\langle a, b, t\rangle$ for $t \in [0, c]$, so $x = a$, $y = b$, $z = t$, $dx = 0$, $dy = 0$, $dz = dt$, so

$$\begin{aligned}\int_{C_3} \mathbf{F} \cdot d\mathbf{r} &= \int_0^c (7b + 5ab + e^t)\, dt \\ &= 7bc + 5abc + e^c - 1.\end{aligned}$$

Summing yields

$$\int_C \mathbf{F} \cdot d\mathbf{r} = a^3 b + 7bc + 5abc + e^c - 1.$$

Thus

$$f(x, y, z) = x^3 y + 7yz + 5xyz + e^z - 1$$

is one possibility. (The others are obtained by adding any constant.)

Second solution (antiderivative method): We know

$$\begin{aligned} f_x &= 3x^2 y + 5yz \\ f_y &= x^3 + 7z + 5xz \\ f_z &= 7y + 5xy + e^z. \end{aligned}$$

The f_x equation implies

$$f = x^3 y + 5xyz + g(y, z) \tag{3}$$

for some $g(y, z)$. Taking $\frac{\partial}{\partial y}$ gives $f_y = x^3 + 5xz + g_y$ and comparing with the given f_y equation shows that $g_y = 7z$, so $g = 7yz + h(z)$ for some function $h(z)$. Substituting back into equation (3) gives

$$f = x^3 y + 5xyz + 7yz + h(z). \tag{4}$$

Taking $\frac{\partial}{\partial z}$ gives $f_z = 5xy + 7y + h_z$ and comparing with the given f_z equation shows that $h_z = e^z + c$ for some constant c. Substituting back into equation (4) gives

$$f = x^3 y + 5xyz + 7yz + e^z + c.$$

We check that this really has the right gradient.

Third solution (guess and check): Just guess one possible potential function f, and check that it has the right gradient; then the complete set of solutions is the set of functions of the form $f + c$, where c is any number.

Midterm 3 covers up to here.

Tuesday, November 9

15.4. **Conservative force fields in physics.** In physics, kinetic energy is $\frac{1}{2}m|\mathbf{v}|^2 = \frac{1}{2}m\mathbf{v}\cdot\mathbf{v}$, and if $\mathbf{F}$ is a force field (e.g., gravitational force) and $\mathbf{F} = -\nabla V$, then the scalar-valued function V is called potential energy. If an object moves from A to B along a path C,

$$\begin{aligned}
\underbrace{\frac{1}{2}m\mathbf{v}\cdot\mathbf{v}\Big|_A^B}_{\text{increase in kinetic energy}} &\overset{\text{FTC}}{=} \int m\mathbf{v}'(t)\cdot\mathbf{v}(t)\,dt \qquad \text{(by the product rule for the dot product)} \\
&= \int m\mathbf{a}(t)\cdot\mathbf{r}'(t)\,dt \\
&= \underbrace{\int_C \mathbf{F}\cdot d\mathbf{r}}_{\text{work done by } \mathbf{F} \text{ on the object}} \\
&\overset{\text{FTC}}{=} \underbrace{-V\big|_A^B}_{\textit{decrease} \text{ in potential energy}},
\end{aligned}$$

so

$$\text{kinetic energy} + \text{potential energy} \qquad \text{is } \textit{constant}:$$

conservation of energy! That's why $\mathbf{F}$ is called conservative. And this also explains why physicists like to write $\mathbf{F}$ as $-\nabla V$ instead of just ∇V.

16. Review

16.1. **Equivalent conditions for a vector field to be conservative.** A *gradient field* (one that is ∇f for some differentiable function $f(x,y,z)$) is a special kind of vector field. What is another name for gradient fields? *Conservative* vector fields. Most 3D vector fields are not conservative, not gradient fields. If $\mathbf{F}$ is a gradient field, any function f such that $\nabla f = \mathbf{F}$ is called a *potential function*.

Question 16.1. Can a vector field $\mathbf{F}$ have more than one potential function?

Answer. Yes, if there is any potential function at all, say f, then $f + c$ for any number c is another one, since $\nabla(f + c) = \nabla f$. In fact, these are all the potential functions for $\mathbf{F}$: if f_1, f_2 are both potential functions for $\mathbf{F}$, then the function $g := f_1 - f_2$ satisfies $\nabla g = \nabla f_1 - \nabla f_2 = \mathbf{F} - \mathbf{F} = \mathbf{0}$ everywhere, so all partial derivatives of g are 0, which means that g is constant. $\square$

Given a 3D vector field $\mathbf{F}$, why is it helpful to know whether $\mathbf{F}$ is conservative?

- It makes computing line integrals $\int_C \mathbf{F} \cdot d\mathbf{r}$ easy, at least if one can find a potential function f, because of the FTC for line integrals.
- It has other nice consequences, such as path independence and the closed curve property.

Here are the four conditions that $\mathbf{F}$ might or might not satisfy:

(1) $\mathbf{F}$ is a gradient field (equal to ∇f for some $f = f(x, y, z)$ on T).
(2) $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is path independent for every A and B in T (for paths inside T).
(3) $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed curve C inside T.
(4) $\operatorname{curl} \mathbf{F} = \mathbf{0}$ at every point of T.

Case 1: T is simply connected. Then all four conditions are equivalent. Does this mean that every $\mathbf{F}$ satisfies the the four conditions? No. What it means is that if $\mathbf{F}$ satisfies any one of the conditions, then $\mathbf{F}$ satisfies all four conditions. The easiest one to check is usually (4) (because one just has to compute $\operatorname{curl} \mathbf{F}$) or sometimes (1) (if one can find a potential function f).

Case 2: T is not simply connected. Then the first three conditions are equivalent, and they imply the fourth. If one knows only that the fourth condition holds, then the first three *might* hold, or they might fail.

16.2. **Example in which equivalence fails.** This example was rewritten from what was said in class, to hopefully make it clearer.

Consider $\mathbb{R}^2 - \{(x, 0) : x \geq 0\}$, which is the plane with a slit along the nonnegative x-axis. Let $\theta_1(x, y)$ be the scalar-valued function with domain $\mathbb{R}^2 - \{(x, 0) : x \geq 0\}$ that gives the value of the polar coordinate θ in the range $(0, 2\pi)$. It cannot be extended to a continuous function on the punctured plane $\mathbb{R}^2 - \{(0, 0)\}$ because its values jump as one crosses the positive x-axis. Let $\theta_2(x, y)$ be the function with domain $\mathbb{R}^2 - \{(x, 0) : x \leq 0\}$ that gives the value of θ in the range $(-\pi, \pi)$. The functions θ_1 and θ_2 do not come from a single function defined on $\mathbb{R}^2 - \{(0, 0)\}$ because they do not agree in the lower half-plane.

On the other hand, they differ there only by a constant (namely, 2π), so $\nabla\theta_1$ and $\nabla\theta_2$ *do* agree at every point where both are defined. Thus there is a 2D vector field $\mathbf{F}$ defined on $\mathbb{R}^2 - \{(0, 0)\}$ that agrees with each of these. We drew $\mathbf{F}$: it is pointing in the counterclockwise direction, and is stronger at points closer to origin, like a whirlpool.

What is the formula for $\mathbf{F}$? At a point in $\mathbb{R}^2 - \{(0, 0)\}$ with position vector $\mathbf{r}$,

- the *direction* of $\mathbf{F}$ is the direction in which θ increases the fastest, which is 90° counterclockwise from $\mathbf{r}$, and
- the *length* of $\mathbf{F}$ is the rate of increase in that direction, which is the directional derivative $\dfrac{d\theta}{ds}$, which equals $\dfrac{1}{r}$, since the little piece of arc length ds is $r\, d\theta$.

Thus

$$\begin{aligned}
\mathbf{F} &= \frac{1}{r}\left(90^\circ \text{ counterclockwise rotation of the outward unit vector } \frac{\mathbf{r}}{r}\right) \\
&= \frac{1}{r^2}\left(90^\circ \text{ counterclockwise rotation of } \mathbf{r}\right) \\
&= \frac{1}{x^2+y^2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\
&= \frac{1}{x^2+y^2}\begin{pmatrix} -y \\ x \end{pmatrix} \\
&= \begin{pmatrix} -\dfrac{y}{x^2+y^2} \\ \dfrac{x}{x^2+y^2} \end{pmatrix}.
\end{aligned}$$

(Another way to get the formula for $\mathbf{F}$ would be to take the gradient of $\tan^{-1}(y/x) + \text{constant}$ or $\pi/2 - \tan^{-1}(x/y) + \text{constant}$; at least one of these formulas makes sense at each point $(x, y) \neq (0, 0)$.)

Now let us add one dimension, and think of θ_1, θ_2, $\mathbf{F}$ as functions of (x, y, z) that happen not to depend on z, with $\mathbf{F}$ having a horizontal value at each point. The new $\mathbf{F}$ is a 3D vector field defined on the region $T := \mathbb{R}^3 - (z\text{-axis})$, given by

$$\mathbf{F}(x, y, z) := \begin{pmatrix} -\dfrac{y}{x^2+y^2} \\ \dfrac{x}{x^2+y^2} \\ 0 \end{pmatrix}.$$

Question 16.2. Is T simply connected?

Answer. No, because a closed curve that goes around the z-axis is not shrinkable inside T. □

Question 16.3. Does $\operatorname{curl} \mathbf{F} = \mathbf{0}$ hold at every point of T?

Answer. Yes, because in a neigborhood of each point of T, the vector field $\mathbf{F}$ agrees with either $\nabla\theta_1$ or $\nabla\theta_2$, and the curl of any gradient field is $\mathbf{0}$. □

Question 16.4. Is $\mathbf{F}$ conservative on T?

This does not follow automatically from $\operatorname{curl} \mathbf{F} = \mathbf{0}$ since (4) does not necessarily imply (1), (2), (3) on a region that is not simply connected.

Answer. No. Let us check that the closed curve condition (3) fails. If C is the counterclockwise unit circle, then $\mathbf{F}$ and $d\mathbf{r}$ point in the same direction along C, so the geometric interpretation of the dot product shows that $\int_C \mathbf{F} \cdot d\mathbf{r} > 0$. (In fact, the value of the integral is 2π, because it is adding up the change in θ along C.) Thus it is not true that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed curve C in T. This means that (3) fails. Since (1), (2), (3) are equivalent, they all fail, and $\mathbf{F}$ is not conservative. □

Conclusion: $\mathbf{F}\colon T \to \mathbb{R}^3$ is a vector field for which conditions (1), (2), (3) fail, but (4) holds.

16.3. **Cylinder in spherical coordinates.**

Problem 16.5. Let T be the 3D region defined by $x^2 + y^2 \le 3$ and $0 \le z \le 1$. Find limits of integration for an iterated integral over T in *spherical coordinates*.

Solution: The iterated integral should have the shape

$$\int_{\theta=} \int_{\phi=} \int_{\rho=} \cdots \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

The range for θ is $[0, 2\pi]$. For each θ, the range for ϕ is $[0, \pi/2]$ (from straight up to horizontal). For each θ and ϕ, the range for ρ is from 0 up to *something*, but the upper limit has a different formula depending on whether the ray outward from the origin hits the top surface or the lateral surface of the cylinder, which depends on how ϕ compares to $\pi/3$ (an angle of a right triangle of width $\sqrt{3}$ and height 1). This means the integral must be broken into two. If $\phi \le \pi/3$, then the upper limit for ρ satisfies $z = 1$, which is $\rho\cos\phi = 1$, so $\rho = 1/\cos\phi$. If $\phi \ge \pi/3$, then the upper limit for ρ satisfies $r = \sqrt{3}$, which is $\rho\sin\phi = \sqrt{3}$, so $\rho = \sqrt{3}/\sin\phi$. Thus the answer is

$$\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/3} \int_{\rho=0}^{1/\cos\phi} \cdots \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta + \int_{\theta=0}^{2\pi} \int_{\phi=\pi/3}^{\pi/2} \int_{\rho=0}^{\sqrt{3}/\sin\phi} \cdots \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

Tuesday, November 16

17. Surfaces

17.1. **Parametrized curves again.**

Problem 17.1. Parametrize the upper half of the circle $x^2 + y^2 = 9$, counterclockwise.

To parametrize a curve in $\mathbb{R}^2$, the parameter needs to be one variable t that has a different value at each point of the curve, so that the different values of the parameter correspond to the different points on the curve. Then express the x-coordinate and y-coordinate of the point on the curve in terms of t to get $\mathbf{r}(t) := \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$.

Solution 1. Let t be the counterclockwise angle from the x-axis, so t is the θ of polar coordinates. The range for t is the interval $[0, \pi]$, and each number t corresponds to the point

$$\mathbf{r}(t) := \begin{pmatrix} 3\cos t \\ 3\sin t \end{pmatrix}.$$

The function $\mathbf{r}$ maps $[0, \pi]$ in the t-line to the semicircle in the xy-plane. □

Solution 2. Let $t = -x$; we use $-x$ instead of x so that as the parameter increases, the point moves counterclockwise. The range for t is the interval $[-3, 3]$. What is the point on the curve with a given t-value? Well, it has $x = -t$ and then solving $x^2 + y^2 = 9$ for y gives $y = \sqrt{9 - x^2} = \sqrt{9 - t^2}$; we use the positive square root since we are parametrizing the upper half of the circle. Thus the parametrization is

$$\mathbf{r}(t) := \begin{pmatrix} -t \\ \sqrt{9 - t^2} \end{pmatrix}$$

for $t \in [-3, 3]$. □

17.2. **Examples of parametrized surfaces.** To parametrize a surface S in $\mathbb{R}^3$, one needs two parameters that together specify a point on the surface, and then one needs to express the x-, y-, and z-coordinates of the point in terms of the two parameters.

Problem 17.2. Let S be the lateral surface of the cylinder $x^2 + y^2 = 9$ for $0 \le z \le 5$. (Lateral means "on the side". In cylindrical coordinates, the equation would be $r = 3$; that's why it's a cylinder.) What is a parametrization of S?

Solution. We can try to use cylindrical coordinates (r, θ, z), but r is not helping to distinguish different points on the cylinder because it has the same value 3 at every point of S. So use θ and z as the parameters. What are x, y, z of a point on S in terms of the parameters? Answer: $x = 3\cos\theta$, $y = 3\sin\theta$, and $z = z$! So the parametrization is

$$\mathbf{r}(\theta, z) := \begin{pmatrix} 3\cos\theta \\ 3\sin\theta \\ z \end{pmatrix}.$$

The range for θ is $[0, 2\pi]$ and the range for z is $[0, 5]$. The function $\mathbf{r}$ maps the rectangle $R := [0, 2\pi] \times [0, 5]$ onto S. (Strictly speaking, the points with $\theta = 0$ duplicate the points with $\theta = 2\pi$, so the vertical segment on the right of the cylinder has been doubly parametrized, but we won't worry about this, because we are interested in surface area, and the doubly-counted segment has zero area.) □

Problem 17.3. Parametrize the surface S of the Earth, assuming that it is perfectly spherical.

Solution. One could use latitude and longitude as the two parameters, but let's use ϕ and θ. One would not include ρ because it does not help specify the position of a point already known to be on the surface. The x-, y-, and z-coordinates can be expressed in terms of ϕ and θ, using formulas we have seen before:

$$\begin{aligned} x &= r\cos\theta = \rho\sin\phi\cos\theta \\ y &= r\sin\theta = \rho\sin\phi\sin\theta \\ z &= \qquad\qquad\ \rho\cos\phi \end{aligned}$$

so

$$\mathbf{r}(\phi,\theta) := \begin{pmatrix} \rho\sin\phi\cos\theta \\ \rho\sin\phi\sin\theta \\ \rho\cos\phi, \end{pmatrix}$$

for $\phi \in [0,\pi]$ and $\theta \in [0,2\pi]$. (In these formulas, ρ is not a variable: ρ is a constant, namely the radius of the Earth.) The function $\mathbf{r}$ maps the rectangle $R := [0,\pi] \times [0,2\pi]$ onto S. $\square$

Problem 17.4. Let S be the lateral surface of the cone $x^2 + y^2 = z^2$ for $0 \le z \le 5$. (In cylindrical coordinates, the equation would be $r = z$; that's why it's a cone.) What is a parametrization of S?

Solution. Use the r and θ of cylindrical coordinates. (Using θ and z would also work.) The parametrization is

$$\mathbf{r}(r,\theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ r \end{pmatrix},$$

for $r \in [0,3]$ and $\theta \in [0,2\pi]$. Thus $\mathbf{r}$ maps the rectangle $R := [0,3] \times [0,2\pi]$ onto S. $\square$

17.3. **Parametrized surfaces in general.** A region R in $\mathbb{R}^2$ can be mapped to a curved surface S in $\mathbb{R}^3$. Example: A disk can be mapped to a Salvador Dalí watch.

In general, a parametrized surface S is the image of a region R in the uv-plane under a function

$$\begin{aligned} \mathbf{r}\colon R &\longrightarrow \mathbb{R}^3 \\ (u,v) &\longmapsto \mathbf{r}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}. \end{aligned}$$

(One could use any variable names in place of u and v.) Form

$$\mathbf{r}_u := \begin{pmatrix} x_u \\ y_u \\ z_u \end{pmatrix}, \qquad \mathbf{r}_v := \begin{pmatrix} x_v \\ y_v \\ z_v \end{pmatrix}$$

(these are the two columns of the total derivative of $\mathbf{r}$). At each point (u, v) in R, one can evaluate $\mathbf{r}_u$ and $\mathbf{r}_v$ there to get two vectors in $\mathbb{R}^3$. The parametrized surface S is smooth at the point corresponding to (u, v) if these two vectors are linearly independent (i.e., nonzero and nonparallel). This condition guarantees that tiny rectangles in R get mapped to tiny "parallelograms" in S (instead of being compressed into a 1-dimensional object, for example). The quotation marks are there because the "parallelograms" in S could be slightly warped and hence are not actual parallelograms.

Problem 17.5. Parametrize the lateral surface S of the cone $x^2 + y^2 = z^2$ for $0 \leq z \leq 5$ as before. Is S smooth?

Solution. Our parametrization was

$$\mathbf{r}(r, \theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ r \end{pmatrix},$$

Then

$$\mathbf{r}_r = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 1 \end{pmatrix}, \qquad \mathbf{r}_\theta = \begin{pmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{pmatrix}.$$

At a point where $r = 0$, we get $\mathbf{r}_\theta = \mathbf{0}$. But at a point where $r \neq 0$, we have that $\mathbf{r}_\theta$ is a nonzero vector in the xy-plane while $\mathbf{r}_r$ has a nonzero vertical component so $\mathbf{r}_r$ and $\mathbf{r}_\theta$ are nonzero and nonparallel. Thus S is smooth except at the points where $r = 0$, which corresponds to $\mathbf{r} = \mathbf{0}$. This makes sense: the cone is smooth except at its vertex. $\square$

17.4. **Surface area.** Suppose that S is a surface parametrized by $\mathbf{r}\colon R \to S$. To approximate the surface area of S, we cut R into tiny rectangles, look at their images under $\mathbf{r}$, and add up the areas of these tiny "parallelograms" covering S.

Imagine a tiny rectangle $[u, u + du] \times [v, v + dv]$ in the uv-plane. It will be mapped to a "parallelogram" whose sides are the vectors $\mathbf{r}_u\, du$ and $\mathbf{r}_v\, dv$. Define

$$\begin{aligned} d\mathbf{S} &:= (\mathbf{r}_u\, du) \times (\mathbf{r}_v\, dv) \\ &= \mathbf{r}_u \times \mathbf{r}_v\; du\, dv, \end{aligned}$$

which we imagine as a tiny vector. What is the geometric meaning of $d\mathbf{S}$?

- The length of $d\mathbf{S}$ is the *area* of the parallelogram, which should be thought of as a piece of surface area, denoted

$$dS := |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv.$$

- The direction of $d\mathbf{S}$ is a unit normal vector $\mathbf{n}$ to the surface.

So

$$d\mathbf{S} = \mathbf{n}\, dS.$$

Define the surface area of S to be

$$\text{Area}(S) := \iint_R dS = \iint_R |\mathbf{r}_u \times \mathbf{r}_v|\ du\, dv.$$

Thursday, November 18

17.4.1. *Surface area of a hemisphere.*

Problem 17.6. Let S be the upper half of the unit sphere centered at the origin. What is its surface area?

(There are many ways to do this; our purpose here is to use the general method for calculating surface area via a parametrization.)

Solution. We already know a parametrization of S: the function

$$\mathbf{r}(\phi, \theta) := \begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix}$$

maps the rectangle $R := [0, \pi/2] \times [0, 2\pi]$ in the $\phi\theta$-plane onto S. Then

$$\mathbf{r}_\phi = \begin{pmatrix} \cos\phi\cos\theta \\ \cos\phi\sin\theta \\ -\sin\phi \end{pmatrix}, \qquad \mathbf{r}_\theta = \begin{pmatrix} -\sin\phi\sin\theta \\ \sin\phi\cos\theta \\ 0 \end{pmatrix},$$

so

$$
\begin{aligned}
\mathbf{r}_\phi \times \mathbf{r}_\theta &= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \end{vmatrix} \\
&= (\sin^2\phi\cos\theta)\mathbf{e}_1 + (\sin^2\phi\sin\theta)\mathbf{e}_2 + (\sin\phi\cos\phi\cos^2\theta + \sin\phi\cos\phi\sin^2\theta)\mathbf{e}_3 \\
&= (\sin^2\phi\cos\theta)\mathbf{e}_1 + (\sin^2\phi\sin\theta)\mathbf{e}_2 + (\sin\phi\cos\phi)\mathbf{e}_3 \\
&= (\sin\phi)\mathbf{r} \\
|\mathbf{r}_\phi \times \mathbf{r}_\theta| &= \sin\phi \\
dS &= \sin\phi\, d\phi\, d\theta \\
\text{Area}(S) &= \int_0^{2\pi}\int_0^{\pi/2} \sin\phi\, d\phi\, d\theta \\
&= \int_0^{2\pi} d\theta \\
&= 2\pi. \qquad \square
\end{aligned}
$$

Note: The geometric interpretation of the cross product shows that the vector $\mathbf{r}_\phi \times \mathbf{r}_\theta$ is perpendicular to the sphere at each point; this explains why it turned out to be a scalar multiple of $\mathbf{r}$.

17.5. **Surface integrals.**

Recall: For a curve C,

$$\int_C f\, ds = \int_a^b f(\mathbf{r}(t))\, |\mathbf{r}'(t)|\, dt.$$

(That is, we compute the left side by choosing a parametrization $\mathbf{r}(t)$ of C, for $t \in [a, b]$.)

Similarly: If S is a surface, and $f = f(x, y, z)$, then

$$\iint_S f\, dS = \iint_R f(\mathbf{r}(u, v))\, |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv,$$

where $\mathbf{r}(u, v)$ is a parametrization of S, mapping the points $(u, v) \in R$ to the points of S.

To be remembered:

1. To compute a 1-dimensional integral on a curve C, you must choose a parametrization $\mathbf{r}(t)$ to convert it to an integral on a straight interval $[a, b]$ in the real line.
2. Generally, to compute a 2-dimensional integral on a curved surface S, you must choose a parametrization $\mathbf{r}(u, v)$ to convert it to an integral over a flat region R in the uv-plane.

The only exceptions to this are:

- If the integrand is a constant c, then

$$\int_C c\,ds = c\,\text{Length}(C)$$
$$\iint_S c\,dS = c\,\text{Area}(S),$$

and you might know Length(C) or Area(S) already by geometry.
- Sometimes you can use the fundamental theorem of calculus to calculate a line integral. Similarly, sometimes you can use fancy versions of the fundamental theorem of calculus called the *divergence theorem* and *Stokes' theorem* (coming soon), to convert the integral to some other kind of integral.

17.6. **Applications of surface integrals.**

Definition 17.7. The average value of f on a surface S is

$$\bar{f} := \frac{\iint_S f\,dS}{\text{Area}(S)}$$

If S is a metal surface whose density (mass per unit area) at each point is given by a function $\delta\colon S \to \mathbb{R}$, then $dm = \delta\,dS$, so the quantities

$$\text{mass } m := \iint_S dm$$
$$\text{mass-weighted average } \bar{f} := \frac{\iint_S f\,dm}{m}$$
$$\text{centroid} := (\bar{x}, \bar{y}, \bar{z})$$
$$\text{moment of inertia } I := \iint_S (\text{distance to axis})^2\,dm$$

all involve surface integrals.

17.7. **Flux.** Suppose that $\mathbf{F}$ is a continuous 3D vector field, and S is a parametrized surface. Then one gets $d\mathbf{S} = \mathbf{n}\,dS$, where $\mathbf{n}$ is a unit normal at each point of S. The flux of $\mathbf{F}$ across S is a special kind of surface integral:

$$\boxed{\text{flux of } \mathbf{F} \text{ across } S \;:= \iint_S \underbrace{\mathbf{F}\cdot\mathbf{n}}_{\text{normal component of } \mathbf{F}}\,dS = \iint_S \mathbf{F}\cdot d\mathbf{S}.}$$

There are actually two choices for $\mathbf{n}$ at each point, and the parametrization is specifying one of them; if one wants flux for the opposite direction of flow, negate the integral. If S bounds a 3D region (like a sphere enclosing a ball), then one usually chooses $\mathbf{n}$ to be the *outward* unit normal.

17.7.1. *Physical meaning of flux.* Intuitive explanation: Imagine a 3D fluid with *constant* velocity field $\mathbf{F}$. The amount of fluid that flows across a tiny parallelogram of area dS in unit time is the fluid in a parallelepiped of base dS and height $\mathbf{F} \cdot \mathbf{n}$. Summing over all tiny parallelepipeds comprising a surface S gives

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS.$$

That's flux!

This intuition works more generally for any continuous $\mathbf{F}$ and smooth S, since $\mathbf{F}$ is almost constant on small regions, and S is well approximated by tiny parallelograms. Flux measures the *rate* of flow (volume per unit time, measured in m^3/s, say).

17.7.2. *Computing 3D flux.* To compute a flux integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S} :$$

(1) Choose a parametrization of S, say $\mathbf{r}(u, v)$ for (u, v) in the flat $2D$ region R. (The whole point is to convert the surface integral over the curved surface S into a double integral over the flat region R.)
(2) Compute the partial derivatives $\mathbf{r}_u$, $\mathbf{r}_v$, and their cross product $\mathbf{r}_u \times \mathbf{r}_v$.
(3) Make sure that $\mathbf{r}_u \times \mathbf{r}_v$ is in the same direction as the desired unit normal $\mathbf{n}$. (If it is in the opposite direction, negate your answer at the end.)
(4) Substitute

$$d\mathbf{S} = \mathbf{r}_u \times \mathbf{r}_v \, du \, dv.$$

to get

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_R \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, du \, dv.$$

(5) Evaluate the double integral over the flat 2-dimensional region R; usually this will be done by converting it to an iterated integral.

Example 17.8. Let S in $\mathbb{R}^3$ be defined by $x^2 + y^2 = 4$ and $0 \le z \le 3$. (So S is the lateral surface of a cylinder.) Let $\mathbf{F}(x, y, z) = \begin{pmatrix} yz \\ x^2 + 2 \\ 5 \end{pmatrix}$. What is the outward flux of $\mathbf{F}$ across S?

The solution below was finished on the next day.

Solution. Use the θ and z of cylindrical coordinates as parameters (r is the constant 2, hence not usable as a parameter). The parametrization is

$$\mathbf{r}(\theta, z) := \begin{pmatrix} 2\cos\theta \\ 2\sin\theta \\ z \end{pmatrix},$$

for $0 \le \theta \le 2\pi$ and $0 \le z \le 3$. Then

$$\mathbf{r}_\theta = \begin{pmatrix} -2\sin\theta \\ 2\cos\theta \\ 0 \end{pmatrix}, \qquad \mathbf{r}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

so

$$\begin{aligned} \mathbf{r}_\theta \times \mathbf{r}_z &= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ -2\sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &= \begin{pmatrix} 2\cos\theta \\ 2\sin\theta \\ 0 \end{pmatrix}. \end{aligned}$$

Notice that $\mathbf{r}_\theta \times \mathbf{r}_z$ is in the same direction as the outward unit normal $\mathbf{n}$. Next

$$\begin{aligned} d\mathbf{S} &= \mathbf{r}_\theta \times \mathbf{r}_z \, d\theta \, dz \\ &= \begin{pmatrix} 2\cos\theta \\ 2\sin\theta \\ 0 \end{pmatrix} d\theta \, dz. \end{aligned}$$

So the flux is

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^3 \int_0^{2\pi} \begin{pmatrix} yz \\ x^2+2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 2\cos\theta \\ 2\sin\theta \\ 0 \end{pmatrix} d\theta \, dz \\ &= \int_0^3 \int_0^{2\pi} \big(yz(2\cos\theta) + (x^2+2)(2\sin\theta) \big) \, d\theta \, dz \\ &= \int_0^3 \int_0^{2\pi} \big((2\sin\theta)z(2\cos\theta) + ((2\cos\theta)^2+2)(2\sin\theta) \big) \, d\theta \, dz \\ &= \int_0^3 \int_0^{2\pi} \big(2z\sin 2\theta + 8\cos^2\theta\sin\theta + 4\sin\theta \big) \, d\theta \, dz \\ &= 0, \end{aligned}$$

because $\int_0^{2\pi} \sin 2\theta \, d\theta$ is the integral over two complete cycles, which is 0, and the last two terms in the integrand are negated by $\theta \mapsto \theta + \pi$ so their integrals over $[0, \pi]$ cancel with the integrals over $[\pi, 2\pi]$. □

Friday, November 19

18.1. **Flux across the faces of a box.** Let us explain the connection between flux and divergence.

Imagine a rectangular box $[x_0, x_1] \times [y_0, y_1] \times [z_0, z_1]$ whose dimensions

$$\begin{aligned} \Delta x &:= x_1 - x_0 \\ \Delta y &:= y_1 - y_0 \\ \Delta z &:= z_1 - z_0 \end{aligned}$$

are very small. Let

$$\mathbf{F}(x, y, z) := \begin{pmatrix} P(x, y, z) \\ Q(x, y, z) \\ R(x, y, z) \end{pmatrix}$$

be a continuously differentiable 3D vector field; this means that all nine partial derivatives, like P_y and R_z, are continuous functions.

Question 18.1. What is the flux of $\mathbf{F}$ across the top face of the box?

Solution. On the top face, z takes the constant value z_1, and x and y can be used as parameters. The geometry shows that $dS = dx\, dy$ and $\mathbf{n} = \mathbf{e}_3$ (no need to calculate a cross product). Thus the flux across the top face is

$$\begin{aligned} \int_{\text{top face}} \mathbf{F} \cdot \mathbf{n}\, dS &= \int_{y=y_0}^{y_1} \int_{x=x_0}^{x_1} \mathbf{F}(x, y, z_1) \cdot \mathbf{e}_3\, dx\, dy \\ &= \int_{y=y_0}^{y_1} \int_{x=x_0}^{x_1} R(x, y, z_1)\, dx\, dy. \end{aligned}$$ □

Question 18.2. What is the flux of $\mathbf{F}$ across the bottom face of the box?

Solution. On the bottom face, z takes the constant value z_0, and x and y can again be used as parameters. The geometry shows that $dS = dx\, dy$ and $\mathbf{n} = -\mathbf{e}_3$. Thus the flux across the bottom face is

$$\begin{aligned} \int_{\text{bottom face}} \mathbf{F} \cdot \mathbf{n}\, dS &= \int_{y=y_0}^{y_1} \int_{x=x_0}^{x_1} \mathbf{F}(x, y, z_0) \cdot (-\mathbf{e}_3)\, dx\, dy \\ &= \int_{y=y_0}^{y_1} \int_{x=x_0}^{x_1} -R(x, y, z_0)\, dx\, dy. \end{aligned}$$ □

Question 18.3. If the box is very small, what is an estimate for the net flux through both the top and bottom faces?

Solution. Adding the previous two answers gives

$$\int_{y=y_0}^{y_1} \int_{x=x_0}^{x_1} (R(x, y, z_1) - R(x, y, z_0))\, dx\, dy.$$

If the height Δz is very small, then we can use the relative change formula

$$R(x, y, z_1) - R(x, y, z_0) \approx R_z \, \Delta z$$

(there is only one term instead of three, since x and y are not changing). Moreover, if the box is very small, then the continuous function R_z is approximately constant on the box, so

$$\begin{aligned}
\int_{y=y_0}^{y_1} \int_{x=x_0}^{x_1} (R(x, y, z_1) - R(x, y, z_0)) \, dx \, dy &\approx \int_{y=y_0}^{y_1} \int_{x=x_0}^{x_1} R_z \, \Delta z \, dx \, dy \\
&\approx R_z \, \Delta z \int_{y=y_0}^{y_1} \int_{x=x_0}^{x_1} dx \, dy \\
&= R_z \, \Delta z \, (\Delta x \, \Delta y) \\
&= R_z \, \mathrm{Vol(box)}.
\end{aligned}$$

□

Question 18.4. If the box is very small, what is an estimate for the net outward flux across all six faces of the box?

Solution. Using similar arguments for each opposite pair of faces gives

Faces	Approximate flux through them
top and bottom	R_z Vol(box)
front and back	P_x Vol(box)
left and right	Q_y Vol(box)
total	$\operatorname{div} \mathbf{F}$ Vol(box)

□

Conclusion:

$$\boxed{\operatorname{div} \mathbf{F} \approx \frac{\text{net outward flux across the sides the box}}{\text{volume of the box}}.}$$

Thus, at each point, $\operatorname{div} \mathbf{F}$ measures the flux out of a tiny box, per unit volume.

18.2. **Divergence as a source rate.** Suppose that $\mathbf{F}$ is the velocity field of an incompressible 3D fluid. Suppose also that fluid is being created at each point of the region (maybe there are tiny pipes pumping fluid into each location). The source rate at a point is the rate at which fluid is being created per unit volume.

Incompressible means that the amount of fluid within any region is constant, so if fluid is being created inside a tiny box at some rate, then fluid must overflow out of the box at the *same* rate. In other words,

$$\text{source rate} = \text{outward flux per unit volume} = \operatorname{div} \mathbf{F}$$

at each point.

18.3. **Bounded regions.** What about the outward flux across the boundary of a larger region? That is what the divergence theorem is about. But first we have to discuss what kind of regions are allowed.

Definition 18.5. A region T in $\mathbb{R}^3$ is called bounded if there is a ball that contains it.

The *big mouth test*: Does eating T in one bite require just a big mouth, or an *infinitely big* mouth? If a big mouth is enough, then T is bounded.

Question 18.6. Which of the following regions are bounded?

- a solid cube? Yes.
- the orthant defined by $x, y, z \geq 0$? No, because it has points arbitrarily far from the origin — no ball can contain them all.
- a solid torus? Yes.
- the solid infinite cylinder in $\mathbb{R}^3$ defined by $x^2 + y^2 \leq 4$? No.
- $\mathbb{R}^3$ itself? No!

18.4. **Closed surfaces.**

Definition 18.7. A closed surface S is a (piecewise smooth) surface that is the entire boundary of a bounded region T in $\mathbb{R}^3$. To say that it is positively-oriented means that at each (smooth) point of S we choose the *outward* unit normal $\mathbf{n}$.

Question 18.8. Which of the following are closed surfaces?

(1) the sphere $x^2 + y^2 + z^2 = 9$? Yes, it is the boundary of a ball.
(2) (the outside of) a cube? Yes, it is the boundary of the solid cube.
(3) (the outside of) a torus? Yes, it is the boundary of the solid torus.
(4) the lateral part of a cylinder? No, since the boundary of a solid cylinder would have to include the top and bottom disks too.
(5) the infinite cylinder $x^2 + y^2 = 4$ in $\mathbb{R}^3$? No — it is the boundary of a solid infinite cylinder, but the solid infinite cylinder is not bounded.

An integral over a closed surface S is sometimes written using the notation $\oiint_S$, although $\iint_S$ is also correct.

18.5. **Divergence theorem.**

Theorem 18.9 (Divergence theorem, also known as Gauss's theorem or Ostrogradsky's theorem — actually discovered earlier by Lagrange)**.** *Let S be a positively-oriented closed surface bounding a region T in $\mathbb{R}^3$. Let $\mathbf{F}$ be a vector field that is continuously differentiable not only on S but also on T. Then*

$$\oiint_S \mathbf{F} \cdot d\mathbf{S} \;=\; \iiint_T \operatorname{div} \mathbf{F} \, dV.$$

18.6. **Physical interpretation of the divergence theorem.** Suppose that $\mathbf{F}$ is the velocity field of an incompressible 3D fluid. As we saw before, $\operatorname{div} \mathbf{F}$ is the source rate (the rate at which fluid is being created per unit volume). Thus $\operatorname{div} \mathbf{F}\, dV$ is the rate of fluid creation inside the little piece of volume dV. Integrating shows that the right side of the divergence theorem is the rate of fluid creation inside all of T.

But the fluid is *incompressible*, so fluid must be flowing out of T at the same rate, to keep the amount of fluid inside T constant. The rate at which fluid is flowing out of T is the outward flux of $\mathbf{F}$ across the boundary of T, which is the left side of the divergence theorem.

Tuesday, November 23

18.7. **Another explanation of the divergence theorem.** Suppose that two tiny boxes are stacked next to each other to form a larger box. Then the flux out of box 1 plus the flux out of box 2 equals the flux out of the combined box, because the fluxes across the now-interior face are computed in opposite directions and hence cancel.

The same principle holds for stacking many tiny boxes: adding up the fluxes out of each box gives the total flux out of their union U.

Now T can be approximated by such a union U of tiny boxes. So

$$\text{flux out of } U \quad \approx \quad \text{the sum of the fluxes out of the tiny boxes} \tag{5}$$

What happens in the limit as the boxes get smaller, so that U approximates T better and better? The left side of (5) tends to the flux out of T, which is $\oiint_S \mathbf{F} \cdot d\mathbf{S}$, the outward flux across the boundary of T. The flux out of each tiny box of volume dV is $\operatorname{div} \mathbf{F}\, dV$, and we are adding these up. Thus in the limit, (5) becomes

$$\oiint_S \mathbf{F} \cdot d\mathbf{S} \;=\; \iiint_T \operatorname{div} \mathbf{F}\, dV.$$

This should make the divergence theorem believable even if we did not justify the details. Good enough for physics!

18.8. **The extended divergence theorem.** Sometimes it happens that the boundary of a region T has more than one connected component.

Example 18.10. If T is the solid spherical shell defined by $4 \le x^2 + y^2 + z^2 \le 9$, then the boundary of T with outward unit normal consists of an outer sphere S of radius 3 with outward unit normal and an inner sphere S' of radius 2 with *inward* unit normal (because outward from the point of view of T is towards the center).

The extended divergence theorem says that if the boundary of T consists of positively-oriented closed surfaces $S_1, \ldots, S_n$, then

$$\oiint_{S_1} \mathbf{F} \cdot d\mathbf{S} \; + \; \cdots \; + \; \oiint_{S_n} \mathbf{F} \cdot d\mathbf{S} \; = \; \iiint_T \operatorname{div} \mathbf{F} \, dV.$$

In the example above, the extended divergence theorem becomes

$$\oiint_{S} \mathbf{F} \cdot d\mathbf{S} \; - \oiint_{S'} \mathbf{F} \cdot d\mathbf{S} \; = \; \iiint_T \operatorname{div} \mathbf{F} \, dV \tag{6}$$

if each sphere is oriented so that $\mathbf{n}$ points *away* from the origin. The minus sign is there because the orientation of S' in the extended divergence theorem (from the point of view of T) is given by the unit normal pointing *towards* the origin, the opposite of what we are using in (6).

18.9. **Divergence and gravitation.**

Lemma 18.11. *Let* $\mathbf{F}$ *be a 3D vector field pointing radially outward and whose magnitude is* $1/\rho^2$ *(defined everywhere except the origin). Then* $\operatorname{div} \mathbf{F} = 0$ *at every point except the origin.*

Proof. Since the radially outward *unit* vector is $\dfrac{\mathbf{r}}{|\mathbf{r}|} = \dfrac{\mathbf{r}}{\rho}$, an explicit formula for $\mathbf{F}$ is

$$\mathbf{F} = \frac{1}{\rho^2} \frac{\mathbf{r}}{\rho} = \rho^{-3} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \rho^{-3} x \\ \rho^{-3} y \\ \rho^{-3} z \end{pmatrix}.$$

To calculate $\operatorname{div} \mathbf{F}$, we need to calculate partial derivatives of the three coordinate functions, while remembering that ρ is really a function of x, y, z. We could just substitute $\rho = \sqrt{x^2 + y^2 + z^2}$ and calculate the partial derivatives explicitly. Alternatively, taking $\frac{\partial}{\partial x}$ of

$$\rho^2 = x^2 + y^2 + z^2$$

gives

$$2\rho \frac{\partial \rho}{\partial x} = 2x,$$

so

$$\frac{\partial \rho}{\partial x} = \frac{x}{\rho}.$$

Now

$$\begin{aligned} \frac{\partial}{\partial x}(\rho^{-3} x) &= \rho^{-3} + (-3\rho^{-4}) \frac{\partial \rho}{\partial x} x \\ &= \rho^{-3} - 3\rho^{-5} x^2. \end{aligned}$$

Summing this with the corresponding equations for y and z gives

$$\begin{aligned}
\operatorname{div} \mathbf{F} &= 3\rho^{-3} - 3\rho^{-5}(x^2+y^2+z^2) \\
&= 3\rho^{-3} - 3\rho^{-5}\rho^2 \\
&= 0.
\end{aligned}$$

□

Let $\mathbf{F} = \mathbf{F}(x,y,z)$ be the gravitational field of a point mass M at $(0,0,0)$ (i.e., the force that it would exert on a unit mass at (x,y,z)). By Newton's inverse square law,

$$\mathbf{F} = -\frac{GM}{\rho^2}\,\frac{\mathbf{r}}{\rho}.$$

This is just a constant times the vector field in Lemma 18.11, so

$$\operatorname{div} \mathbf{F} = 0$$

at every point in $\mathbb{R}^3$ except $(0,0,0)$.

Question 18.12. Let $\mathbf{F}$ be the gravitational field of a point of mass M at the origin. Let S_a be the sphere of radius a centered at $(0,0,0)$. Let S_{2a} be the sphere of twice the radius. Let's compare the flux across S_a with the flux across S_{2a}. Which of the following is correct?

(1) The flux across S_{2a} is 4 times as much, because the integral is over a surface area that is 4 times bigger.
(2) The flux across S_{2a} is 1/4 as much, because the gravitational field is 1/4 as strong.
(3) The fluxes are equal and nonzero.
(4) The fluxes are both 0, because the divergence theorem says

$$\iint_{S_a} \mathbf{F} \cdot d\mathbf{S} = \iiint_T \operatorname{div} \mathbf{F}\, dV = 0,$$

since $\operatorname{div} \mathbf{F} = 0$ everywhere.

Answer: The fluxes are equal and nonzero. (The two effects in the first two answers cancel each other out. The application of the divergence theorem in the last answer is wrong: $\mathbf{F}$ is not defined at $(0,0,0)$, so the right side of the divergence theorem does not even make sense.)

Question 18.13. What *is* the flux across the sphere S_a of radius a centered at $(0,0,0)$?

Answer: At every point, $\mathbf{F}$ and $\mathbf{n}$ are in *opposite* directions, so

$$\begin{aligned}
\text{flux} &= \oiint_{S_a} \mathbf{F} \cdot \mathbf{n}\, dS \\
&= \oiint_{S_a} -|\mathbf{F}|\, dS \\
&= \oiint_{S_a} -\frac{GM}{a^2}\, dS \\
&= -\frac{GM}{a^2} \text{ Area}(S_a) \\
&= -\frac{GM}{a^2}(4\pi a^2) \\
&= -4\pi GM.
\end{aligned}$$

It is independent of the radius a!

Even better, if S is *any* closed surface enclosing the point mass, and T is the 3D region between S and a small sphere S_a centered at the point mass (so T has a bubble inside), then the extended divergence theorem shows that

$$\begin{aligned}
\oiint_S \mathbf{F} \cdot d\mathbf{S} - \oiint_{S_a} \mathbf{F} \cdot d\mathbf{S} &= \iiint_T \operatorname{div} \mathbf{F}\, dV \\
&= 0,
\end{aligned}$$

so

$$\begin{aligned}
\oiint_S \mathbf{F} \cdot d\mathbf{S} &= \oiint_{S_a} \mathbf{F} \cdot d\mathbf{S} \\
&= -4\pi GM.
\end{aligned}$$

The same claim is true if there are many point masses inside S, or even some planets inside S, because the force fields add. This proves Gauss's law:

$$\boxed{\text{gravitational flux across } S \ = -4\pi GM,}$$

where M is the mass enclosed by S.

Problem 18.14. What is the gravitational field inside a hollow spherical planet?

Solution. Inside a centered sphere S of radius r inside the hollow part, symmetry implies that $\mathbf{F} = c\mathbf{n}$ for some c depending only on r. Then the flux across S is $4\pi r^2 c$, but Gauss's law says that it equals 0, so $c = 0$. Thus the gravitational field is $\mathbf{0}$ everywhere inside the hollow part. □

Challenge question 18.15. For a donut-shaped planet, if you are standing on the inner circle of the planet, is gravity pulling you towards the center of mass or is it pulling you towards the planet under your feet?

Hint: Imagine filling in the donut hole with a cylinder of very small height. The total flux through the cylinder is 0 by Gauss's law. On the other hand, is the outward flux through the top and bottom disks positive or negative? If you figure that out, that can help answer the question, because it must be cancelled by the flux through the lateral surface of the cylinder.

Tuesday, November 30

18.10. **Application of the divergence theorem to an electric field.** Place a point charge Q at $(0,0,0)$. Let $\mathbf{E} = \mathbf{E}(x,y,z)$ be the electric field it creates. Coulomb's law:

$$\boxed{\mathbf{E} = \frac{Q/4\pi\epsilon_0}{\rho^2}\,\frac{\mathbf{r}}{\rho},}$$

where ϵ_0 is a constant.

The physics is different, but the math is the same as for gravitation, with the constant $Q/4\pi\epsilon_0$ in place of $-GM$. The gravitational flux was $-4\pi GM$, which is 4π times the constant appearing in the inverse square law for gravitation. Similarly, the electric flux is

$$4\pi(Q/4\pi\epsilon_0) = \frac{Q}{\epsilon_0}.$$

Summary:

	field	flux
gravitational	$\mathbf{F} = -\frac{GM}{\rho^2}\,\frac{\mathbf{r}}{\rho}$	$-4\pi GM$
electric	$\mathbf{E} = \frac{Q/4\pi\epsilon_0}{\rho^2}\,\frac{\mathbf{r}}{\rho}$	Q/ϵ_0

So we get Gauss's law for an electric field (also called the Gauss–Coulomb law):

> The electric flux across a closed surface S equals the charge enclosed divided by ϵ_0,
>
> $$\oiint_S \mathbf{E}\cdot d\mathbf{S} = \frac{Q}{\epsilon_0}.$$

19.1. Curves bounding surfaces. In $\mathbb{R}^3$, let C be a closed curve, and let S be a bounded surface whose boundary is C. Roughly speaking, S "fills in" C. (Here S is usually not a closed surface; it usually does not enclose a 3-dimensional region.) For example, C could be the Equator on the Earth, and S could be the Northern Hemisphere.

1. An orientation of C is a choice of direction along C.
2. An orientation of S is a (continuously varying) choice of unit normal vector $\mathbf{n}$ at each point of S.

For Stokes' theorem to hold, these two choices must be compatible. There are two equivalent ways to say what compatible means:

- If you walk along C in the chosen direction, with S to your left, then $\mathbf{n}$ is pointing up.
- If your right thumb is pointing in the direction of $\mathbf{n}$, then your fingers point in the chosen direction along C.

In the example above, if the orientation of the Equator C is east, then the compatible orientation of the Northern Hemisphere S is the one that has $\mathbf{n}$ pointing up into space at each point.

19.2. The theorem.

Theorem 19.1 (Stokes' theorem[5])**.** *In $\mathbb{R}^3$, let C be an oriented piecewise smooth curve, and let S be an oriented, bounded, piecewise smooth surface with boundary C. Assume that the orientations of C and S are compatible. Let $\mathbf{F}$ be a 3D vector field that is continuously differentiable everywhere on S. Then*

$$\boxed{\oint_C \mathbf{F}\cdot d\mathbf{r} \;=\; \iint_S (\operatorname{curl}\mathbf{F})\cdot d\mathbf{S}.}$$

Remark 19.2. The orientation of C is needed to define $\oint_C \mathbf{F}\cdot d\mathbf{r}$, and the orientation of S is needed to define $d\mathbf{S} = \mathbf{n}\,dS$ in $\iint_S (\operatorname{curl}\mathbf{F})\cdot d\mathbf{S}$. If these orientations are not chosen compatibly, the two sides in Stokes' theorem will differ by a sign.

19.3. Example.

Problem 19.3. Let S be the boundary of the cylinder $x^2+y^2 \le 9$, $0 \le z \le 2$ excluding the base in the xy-plane. Let $\mathbf{F} = \begin{pmatrix} y \\ -x \\ y^3 \end{pmatrix}$. Compute the outward flux of $\operatorname{curl}\mathbf{F}$ across S in as many ways as you can.

[5] Actually first proved by Lord Kelvin, who mailed it to Stokes, who included it as a question in a physics competition for students.

Solution 1: Use Stokes' theorem to convert it to a line integral on the circle C given by $x^2 + y^2 = 9$ in the xy-plane.

$$\begin{aligned} \text{flux of curl}\,\mathbf{F} \text{ across } S &= \iint_S (\operatorname{curl}\mathbf{F}) \cdot d\mathbf{S} \\ &\overset{\text{Stokes'}}{=} \oint_C \mathbf{F} \cdot d\mathbf{r}. \end{aligned}$$

To compute the latter, choose a parametrization of C (counterclockwise, so as to agree with the orientation of S). Let's use

$$\mathbf{r}(t) = \begin{pmatrix} 3\cos t \\ 3\sin t \\ 0 \end{pmatrix} \qquad \text{for } 0 \le t \le 2\pi,$$

so

$$\begin{aligned} d\mathbf{r} &= \begin{pmatrix} -3\sin t \\ 3\cos t \\ 0 \end{pmatrix} dt \\ \oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \begin{pmatrix} 3\sin t \\ -3\cos t \\ (3\sin t)^3 \end{pmatrix} \cdot \begin{pmatrix} -3\sin t \\ 3\cos t \\ 0 \end{pmatrix} dt \\ &= \int_0^{2\pi} \left(-9\sin^2 t - 9\cos^2 t\right) dt \\ &= \int_0^{2\pi} -9\,dt \\ &= -18\pi. \end{aligned}$$

Solution 2: Compute the flux directly from the definition. The surface S consists of the top disk S_1 and the lateral surface S_2 of the cylinder. We have

$$\begin{aligned} \operatorname{curl}\mathbf{F} &= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & y^3 \end{vmatrix} \\ &= 3y^2\mathbf{e}_1 - 2\mathbf{e}_3. \end{aligned}$$

On S_1, the unit normal is $\mathbf{n} = \mathbf{e}_3$, so

$$\begin{aligned} (\operatorname{curl}\mathbf{F}) \cdot \mathbf{n} &= -2 \\ \iint_{S_1} (\operatorname{curl}\mathbf{F}) \cdot \mathbf{n}\,dS &= -2\,\text{Area}(S_1) \\ &= -18\pi. \end{aligned}$$

For S_2, we use the z and θ of cylindrical coordinates as parameters:

$$\mathbf{r}(z,\theta) := \begin{pmatrix} 3\cos\theta \\ 3\sin\theta \\ z \end{pmatrix}$$

for $z \in [0,2]$ and $\theta \in [0,2\pi]$. Then

$$\begin{aligned}
\mathbf{r}_z &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\
\mathbf{r}_\theta &= \begin{pmatrix} -3\sin\theta \\ 3\cos\theta \\ 0 \end{pmatrix} \\
\mathbf{r}_z \times \mathbf{r}_\theta &= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 0 & 0 & 1 \\ -3\sin\theta & 3\cos\theta & 0 \end{vmatrix} \\
&= (-3\cos\theta)\,\mathbf{e}_1 + (-3\sin\theta)\,\mathbf{e}_2.
\end{aligned}$$

This gives the *opposite* of the desired orientation for $\mathbf{n}$ (the vector $\mathbf{r}_z \times \mathbf{r}_\theta$ is horizontally inward instead of outward), so insert a minus sign to get

$$\mathbf{n}\,dS = -\mathbf{r}_z \times \mathbf{r}_\theta \; dz\,d\theta,$$

so that the outward flux is

$$\begin{aligned}
\iint_{S_1} (\operatorname{curl}\mathbf{F}) \cdot \mathbf{n}\,dS &= -\int_0^{2\pi}\int_0^2 (\operatorname{curl}\mathbf{F}) \cdot (\mathbf{r}_z \times \mathbf{r}_\theta)\,dz\,d\theta \\
&= -\int_0^{2\pi}\int_0^2 \begin{pmatrix} 3(3\sin\theta)^2 \\ 0 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} -3\cos\theta \\ -3\sin\theta \\ 0 \end{pmatrix} dz\,d\theta \\
&= -\int_0^{2\pi}\int_0^2 -81\sin^2\theta\cos\theta\,dz\,d\theta \\
&= \int_0^{2\pi} 162\sin^2\theta\cos\theta\,d\theta \\
&= 54\sin^3\theta\Big|_0^{2\pi} \\
&= 0.
\end{aligned}$$

So the total flux of $\operatorname{curl}\mathbf{F}$ across S is

$$-18\pi + 0 = -18\pi.$$

Solution 3: Use the divergence theorem to compute the flux of $\operatorname{curl}\mathbf{F}$ *across the* entire *boundary of the cylinder, and then subtract the flux across the bottom disk.* We already calculated that

$$\operatorname{curl}\mathbf{F} = 3y^2\,\mathbf{e}_1 - 2\,\mathbf{e}_3$$

so

$$\operatorname{div}(\operatorname{curl}\mathbf{F}) = \frac{\partial}{\partial x}3y^2 + \frac{\partial}{\partial y}0 + \frac{\partial}{\partial z}(-2) = 0 + 0 + 0 = 0.$$

(In fact, it turns out that $\operatorname{div}(\operatorname{curl}\mathbf{F}) = 0$ holds for any vector field whose component functions have continuous second partial derivatives.) By the divergence theorem, the outward flux of $\operatorname{curl}\mathbf{F}$ across the entire boundary of the cylinder is

$$\begin{aligned}\oiint\limits_{\text{entire boundary}} (\operatorname{curl}\mathbf{F}) \cdot d\mathbf{S} &= \iiint\limits_{\text{solid cylinder}} \operatorname{div}(\operatorname{curl}\mathbf{F})\, dV \\ &= \iiint\limits_{\text{solid cylinder}} 0\, dV \\ &= 0.\end{aligned}$$

On the other hand, the outward flux across the bottom disk of the cylinder is

$$\begin{aligned}\iint\limits_{\text{bottom disk}} (\operatorname{curl}\mathbf{F}) \cdot d\mathbf{S} &= \iint\limits_{\text{bottom disk}} (\operatorname{curl}\mathbf{F}) \cdot (-\mathbf{e}_3)\, dS \quad \text{(since the outward unit normal is downward)} \\ &= \iint\limits_{\text{bottom disk}} (3y^2\mathbf{e}_1 - 2\mathbf{e}_3) \cdot (-\mathbf{e}_3)\, dS \\ &= \iint\limits_{\text{bottom disk}} 2\, dS \\ &= 2\,\text{Area(bottom disk)} \\ &= 2(\pi \cdot 3^2) \\ &= 18\pi.\end{aligned}$$

Subtracting gives the flux of $\operatorname{curl}\mathbf{F}$ through S only:

$$\iint\limits_S (\operatorname{curl}\mathbf{F}) \cdot d\mathbf{S} = 0 - 18\pi = -18\pi.$$

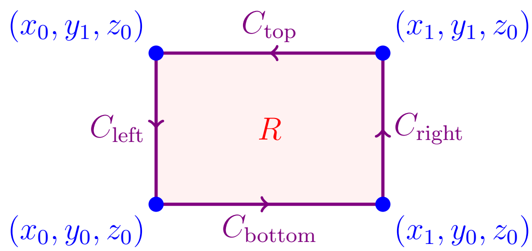
19.4. **Line integral around a tiny rectangle.** Remember how we explained the divergence theorem by approximating the 3D region T by a union of little boxes and by showing that the flux out of each little box was approximated by $\operatorname{div}\mathbf{F}\,dV$?

We are now going to explain Stokes' theorem in a similar way by approximating the surface S by a union of tiny rectangles hinged together (like a disco ball) and by showing that the line integral around each little rectangle of area dS is approximated by $(\operatorname{curl}\mathbf{F}) \cdot \mathbf{n}\, dS$.

Imagine a horizontal rectangle $R = [x_0, x_1] \times [y_0, y_1] \times \{z_0\}$ whose dimensions

$$\Delta x := x_1 - x_0$$
$$\Delta y := y_1 - y_0$$

are very small. Its area, which we'll call dS, is $\Delta x\,\Delta y$. Its boundary C_{tiny} is a closed curve consisting of four line segments, oriented as shown:



Question 19.4. What is the line integral $\int_{C_{\text{bottom}}} \mathbf{F} \cdot d\mathbf{r}$ along the bottom?

Answer. The path C_{bottom} is parametrized by (x, y_0, z_0) for $x \in [x_0, x_1]$, so

$$\begin{aligned}\int_{C_{\text{bottom}}} \mathbf{F} \cdot d\mathbf{r} &= \int_{x=x_0}^{x_1} \mathbf{F}(x, y_0, z_0) \cdot \mathbf{e}_1 \, dx \\ &= \int_{x=x_0}^{x_1} P(x, y_0, z_0) \, dx. \end{aligned}$$

□

Question 19.5. What is the line integral $\int_{C_{\text{top}}} \mathbf{F} \cdot d\mathbf{r}$ along the top?

Answer. The path C_{top} is parametrized *backwards* by (x, y_1, z_0) for $x \in [x_0, x_1]$, so

$$\begin{aligned}\int_{C_{\text{top}}} \mathbf{F} \cdot d\mathbf{r} &= -\int_{x=x_0}^{x_1} \mathbf{F}(x, y_1, z_0) \cdot \mathbf{e}_1 \, dx \\ &= -\int_{x=x_0}^{x_1} P(x, y_1, z_0) \, dx. \end{aligned}$$

□

Question 19.6. What is an estimate for the line integrals along the top and bottom *combined*? (They almost cancel.)

Answer.

$$\begin{aligned}\int_{C_{\text{bottom}}} \mathbf{F} \cdot d\mathbf{r} + \int_{C_{\text{top}}} \mathbf{F} \cdot d\mathbf{r} &= \int_{x=x_0}^{x_1} (P(x, y_0, z_0) - P(x, y_1, z_0)) \, dx \\ &\approx -\int_{x=x_0}^{x_1} P_y(x, y_0, z_0) \, \Delta y \, dx \qquad \text{(linear approximation)} \\ &\approx -P_y \, \Delta y \int_{x=x_0}^{x_1} dx \qquad (P_y \text{ is nearly constant on the rectangle}) \\ &= -P_y \, \Delta y \, \Delta x \\ &= -P_y \, dS. \end{aligned}$$

□

Question 19.7. What is an estimate for the line integral $\oint_{C_{\text{tiny}}} \mathbf{F} \cdot d\mathbf{r}$ around the whole tiny rectangle?

Answer. A similar calculation shows that

$$\int_{C_{\text{right}}} \mathbf{F} \cdot d\mathbf{r} + \int_{C_{\text{left}}} \mathbf{F} \cdot d\mathbf{r} \quad \approx \quad Q_x \, dS. \tag{7}$$

Combining and answer to Question 19.6 and (7) gives

$$\oint_{C_{\text{tiny}}} \mathbf{F} \cdot d\mathbf{r} \quad \approx \quad (Q_x - P_y) \, dS = (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n} \, dS, \tag{8}$$

since $\operatorname{curl} \mathbf{F} = \begin{pmatrix} R_y - Q_z \\ P_z - R_x \\ Q_x - P_y \end{pmatrix}$ and $\mathbf{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ since R is horizontal. Moreover, the smaller R is, the better the approximation. □

The same will hold for a tiny rectangle in any position in space, because of the geometric invariance of work and curl under rotations of space (they have a meaning that is independent of the coordinate system you are working in).

19.5. **Explaining Stokes' theorem for a general surface.** Suppose that two tiny rectangles are hinged along one side. Then adding the line integrals around each gives the line integral around the combined surface, since the line integrals along the hinge segment cancel. The same holds for a surface built out of more than two rectangles. Therefore, when equation (8) is summed over all these rectangles, the left side sum becomes the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ around the boundary of the whole surface; meanwhile, in the limit as the rectangles' sizes tend to 0, the right side sum tends to a surface integral and the approximation becomes an equality:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n} \, dS.$$

This should make Stokes' theorem believable even if we did not justify the details.

19.6. **Conservative vector fields and Stokes' theorem.**

Do you remember the four conditions

(1) $\mathbf{F}$ is a gradient field (equal to ∇f for some $f = f(x, y, z)$ on T).
(2) $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is path independent for every A and B in T (for paths inside T).
(3) $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed curve C inside T.
(4) $\operatorname{curl} \mathbf{F} = \mathbf{0}$ at every point of T.

on a continuously differentiable 3D vector field $\mathbf{F}$? We explained why (1), (2), (3) are equivalent, and why any of (1), (2), (3) implies (4). We also *claimed* that when T is simply connected, then all four conditions are equivalent, but we never explained why (4) implied any of (1), (2), (3). Now that we have Stokes' theorem, we can finally explain this.

Proposition 19.8. *When T is simply connected,* (4) *implies* (3).

Sketch of proof. Suppose that T is simply connected and that (4) holds, so $\operatorname{curl} \mathbf{F} = \mathbf{0}$ at every point of T. We need to prove that (3) holds. Since (3) is a statement about every closed curve in T, the proof begins by letting C be any closed curve in T, and we need to show that $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$.

Because T is simply connected, the curve C is shrinkable to a point in T. A time lapse photograph of the shrinking would show C tracing out a surface S as it shrinks, and S is contained in T. It turns out that if C is piecewise smooth, then S can be arranged to be piecewise smooth too, so that Stokes' theorem applies:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\operatorname{curl} \mathbf{F}) \cdot d\mathbf{S} = 0,$$

since $\operatorname{curl} \mathbf{F} = \mathbf{0}$ everywhere on S by the assumption (4). □

19.7. **Extended Stokes' theorem.** Sometimes the boundary of a surface S in $\mathbb{R}^3$ may consist of *several* closed curves $C_1, \ldots, C_n$. In this situation, the extended Stokes' theorem says

$$\oint_{C_1} \mathbf{F} \cdot d\mathbf{r} + \cdots + \oint_{C_n} \mathbf{F} \cdot d\mathbf{r} = \iint_S (\operatorname{curl} \mathbf{F}) \cdot d\mathbf{S}.$$

It can also be written

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

where ∂S is an abbreviation for the boundary of S. (The ∂ is the same symbol used for partial differentiation, but its meaning here is different!)

Example 19.9. If S is a pair of pants, oriented by the outward unit normal, then ∂S consists of three curves: the waistline, and the two cuffs at the bottom. The waistline is oriented clockwise when viewed from the top.

At this point, all the material on the final exam has been covered. The remaining topics are partly for review, and partly to show how the material is used in physics and elsewhere.

Friday, December 3

20. Green's theorem

Stokes' theorem and the divergence theorem have 2D versions:

	$\oint_C \mathbf{F} \cdot d\mathbf{r}$	flux
2D	Green's theorem	Green's theorem for flux
3D	Stokes' theorem	divergence theorem

The 2D versions are less important for physics, but they come up sometimes.

20.1. Green's theorem is Stokes' theorem in 2D.

20.1.1. *Setup.*

R: a bounded region in the xy-plane

C: the boundary of R, assumed to be a piecewise smooth closed curve

$\mathbf{F}(x,y) = \begin{pmatrix} P(x,y) \\ Q(x,y) \end{pmatrix}$: a continuously differentiable 2D vector field on R.

Viewing R as a surface in $\mathbb{R}^3$, orient R by choosing $\mathbf{n} = \mathbf{e}_3$ (up, out of the xy-plane). What is the compatible orientation of C? Counterclockwise, by the right hand rule.

Stokes' theorem is for 3D vector fields, not 2D vector fields. So build from $\mathbf{F}$ a 3D vector field $\tilde{\mathbf{F}}(x,y,z) := \begin{pmatrix} P(x,y) \\ Q(x,y) \\ 0 \end{pmatrix}$. Then

$$\operatorname{curl} \tilde{\mathbf{F}} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = (Q_x - P_y)\mathbf{e}_3. \tag{9}$$

The expression $Q_x - P_y$ is sometimes called the 2D scalar curl of the 2D vector field $\mathbf{F}$.

20.1.2. *Application of Stokes' theorem.*

Apply Stokes' theorem to $\tilde{\mathbf{F}}$ on R.

Left side of Stokes' theorem:

$$\oint_C \tilde{\mathbf{F}} \cdot d\mathbf{r} = \oint_C \begin{pmatrix} P \\ Q \\ 0 \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$

is the same as

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \boxed{\oint_C P\,dx + Q\,dy}.$$

Right side of Stokes' theorem:

$$\begin{aligned}\iint_R (\operatorname{curl} \tilde{\mathbf{F}}) \cdot d\mathbf{S} &= \iint_R (\operatorname{curl} \tilde{\mathbf{F}}) \cdot \mathbf{n}\,dS \\ &= \iint_R (Q_x - P_y)\mathbf{e}_3 \cdot \mathbf{e}_3\,dA \\ &= \boxed{\iint_R (Q_x - P_y)\,dA}.\end{aligned}$$

So the result of applying Stokes' theorem is

Theorem 20.1 (Green's theorem)**.** *For the setup above,*

$$\oint_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dA.$$

Another way to write the same equation:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \underbrace{\operatorname{curl} \mathbf{F}}_{\text{2D scalar curl}}\,dA.$$

20.1.3. *Computing area with Green's theorem.* Let R and C be as in the setup for Green's theorem. If we choose $P = 0$ and $Q = x$, then $Q_x - P_y = 1$, so Green's theorem (with the two sides reversed) gives

$$\operatorname{Area}(R) = \oint_C x\,dy.$$

If we choose $P = -y$ and $Q = 0$, then $Q_x - P_y = 1$ again, so

$$\operatorname{Area}(R) = \oint_C -y\,dx.$$

These formulas give the area of R in terms of a calculation along the boundary!

In the 1800s, people invented the planimeter, a mechanical device with an arm attached to a pencil such that if the pencil traces out a closed curve, the device calculates the area enclosed. It works by effectively calculating $\oint_C x\,dy$.

20.2. Green's theorem for flux is the divergence theorem in 2D.

20.2.1. *Setup.*

R: a bounded region in the xy-plane

C: the boundary of R, assumed to be a piecewise smooth closed curve

$\mathbf{n}$: the outward unit normal in $\mathbb{R}^2$ at each point of C

$\mathbf{F}(x,y) = \begin{pmatrix} M(x,y) \\ N(x,y) \end{pmatrix}$: a continuously differentiable 2D vector field on R.

The divergence theorem requires a 3D vector field on a 3D region T. So build from $\mathbf{F}$ a 3D vector field $\tilde{\mathbf{F}}(x,y,z) := \begin{pmatrix} M(x,y) \\ N(x,y) \\ 0 \end{pmatrix}$ and thicken R into a slab $T := R \times [0,1]$. The boundary S of T consists of the lateral surface $C \times [0,1]$, the bottom $R \times \{0\}$, and the top $R \times \{1\}$. Equip S with the outward unit normal vector $\tilde{\mathbf{n}}$ at each point.

20.2.2. *Application of the divergence theorem.*

Apply the divergence theorem to $\tilde{\mathbf{F}}$ on T.

Left side of the divergence theorem: First, what is $\tilde{\mathbf{F}} \cdot \tilde{\mathbf{n}}$ at a boundary point (x,y,z)?

- along the lateral surface, $\tilde{\mathbf{F}} \cdot \tilde{\mathbf{n}}$ at (x,y,z) has the same value as $\mathbf{F} \cdot \mathbf{n}$ at (x,y);
- along the top, $\tilde{\mathbf{F}} \cdot \tilde{\mathbf{n}} = 0$ since $\tilde{\mathbf{F}}$ is horizontal while $\tilde{\mathbf{n}}$ is vertical;
- along the bottom, $\tilde{\mathbf{F}} \cdot \tilde{\mathbf{n}} = 0$ since $\tilde{\mathbf{F}}$ is horizontal while $\tilde{\mathbf{n}}$ is vertical.

Thus

$$\begin{aligned}
\text{flux across } S &= \text{flux across the lateral surface } C \times [0,1] \\
&= \iint_{C \times [0,1]} \tilde{\mathbf{F}} \cdot \tilde{\mathbf{n}} \, dS \\
&= \oint_C \int_{z=0}^{1} \mathbf{F} \cdot \mathbf{n} \, dz \, ds \\
&= \oint_C (\mathbf{F} \cdot \mathbf{n}) \left(\int_{z=0}^{1} dz \right) ds \quad (\text{since } \mathbf{F} \cdot \mathbf{n} \text{ does not depend on } z) \\
&= \boxed{\underbrace{\oint_C \mathbf{F} \cdot \mathbf{n} \, ds}_{\text{flux of 2D field } \mathbf{F} \text{ across } C}}.
\end{aligned}$$

Right side of the divergence theorem: First, $\operatorname{div}\tilde{\mathbf{F}} = M_x + N_y$, which we also call the 2D divergence of $\mathbf{F}$, denoted $\operatorname{div}\mathbf{F}$. So the right side is

$$\iiint_T \operatorname{div}\tilde{\mathbf{F}}\,dV = \iint_R \int_{z=0}^1 \operatorname{div}\mathbf{F}\,dz\,dA$$
$$= \iint_R (\operatorname{div}\mathbf{F})\left(\int_{z=0}^1 dz\right)dA$$
$$= \boxed{\iint_R \operatorname{div}\mathbf{F}\,dA}.$$

So the result of applying the divergence theorem is

Theorem 20.2 (Green's theorem for flux, also called Green's theorem in normal form)**.** *For the setup above,*

$$\boxed{\underbrace{\oint_C \mathbf{F}\cdot\mathbf{n}\,ds}_{\text{flux of 2D field }\mathbf{F}\text{ across }C} = \iint_R \underbrace{\operatorname{div}\mathbf{F}}_{\text{2D divergence}}\,dA}. \tag{10}$$

Now $\mathbf{n}\,ds$ is the $90°$ clockwise rotation of $\mathbf{T}\,ds = d\mathbf{r} = \begin{pmatrix} dx \\ dy \end{pmatrix}$ in the plane, so on the left of (10) we have

$$\mathbf{F}\cdot\mathbf{n}\,ds = \begin{pmatrix} M \\ N \end{pmatrix}\cdot\begin{pmatrix} dy \\ -dx \end{pmatrix} = -N\,dx + M\,dy,$$

while on the right of (10) we have $\operatorname{div}\mathbf{F} = M_x + N_y$, so (10) can be rewritten as

$$\boxed{\oint_C -N\,dx + M\,dy = \iint_R (M_x + N_y)\,dA}. \tag{11}$$

Mathematically, Green's theorem and Green's theorem for flux are essentially the same: Taking $P = -N$ and $Q = M$ in Green's theorem gives Green's theorem for flux.

Tuesday, December 7

21. Generalized Stokes' theorem

The fundamental theorem of calculus, the fundamental theorem of calculus for line integrals, the divergence theorem, and Stokes' theorem all fit the template

$$\text{“}\int_{\partial R}\mathbf{F} = \int_R d\mathbf{F}\text{”},$$

where ∂R is the boundary of R and $d\mathbf{F}$ is some kind of derivative/differential of $\mathbf{F}$.

FTC: The boundary of an interval $[a, b]$ is a set of two points $\{a, b\}$.

$$f(b) - f(a) = \int_a^b f'(x)\, dx.$$

FTC for line integrals: The boundary of an oriented curve C from A to B is $\{A, B\}$.

$$f(B) - f(A) = \int_C \nabla f \cdot d\mathbf{r}.$$

Divergence theorem: The boundary of a 3D region T is a closed surface S (or many closed surfaces). Let $\mathbf{F}$ be a 3D vector field.

$$\oiint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_T \operatorname{div} \mathbf{F}\, dV.$$

Stokes' theorem: The boundary of a surface S in $\mathbb{R}^3$ is a closed curve C (or many closed curves). Let $\mathbf{F}$ be a 3D vector field.

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}.$$

Can you make Green's theorem and Green's theorem for flux also fit the template? Yes, try it!

In fact, all six theorems are special cases of a generalized Stokes' theorem that applies to differential forms on n-dimensional manifolds for any n!

22. Maxwell's equations

Maxwell's equations relate electric and magnetic fields. They can be expressed in differential form (involving the derivative-like operators div and curl), or in integrated form (involving line integrals and surface integrals).

Mathematics proves none of them. Instead, the role of mathematics is to show that *the differential form is mathematically equivalent to the integrated form.*

22.1. Definitions of the physical quantities.

Introduce the following quantities (in SI units):

$$\begin{aligned}
\mathbf{E} &:= \text{electric field (newtons/coulomb = volt/m)} \\
\mathbf{B} &:= \text{magnetic field (teslas)} \\
\rho &:= \text{charge density (coulomb/m}^3\text{)} \\
\mathbf{J} &:= \text{current density (amp/m}^2\text{)} \\
t &:= \text{time (s)}
\end{aligned}$$

μ_0, ϵ_0 are constants.

22.2. **The two forms of Maxwell's equations.**

Here are Maxwell's equations in differential form:

$$\begin{aligned}
\operatorname{div} \mathbf{E} &= \frac{\rho}{\epsilon_0} \\
\operatorname{div} \mathbf{B} &= 0 \\
\operatorname{curl} \mathbf{E} &= -\frac{d\mathbf{B}}{dt} \\
\operatorname{curl} \mathbf{B} &= \mu_0\epsilon_0\frac{d\mathbf{E}}{dt} + \mu_0\mathbf{J}.
\end{aligned}$$

And here are Maxwell's equations in integrated form:

$$\begin{aligned}
&\oiint_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\epsilon_0} \qquad (\text{Gauss-Coulomb law}) \\
&\oiint_S \mathbf{B} \cdot d\mathbf{S} = 0 \qquad (\text{Gauss's law for magnetism}) \\
&\oint_C \mathbf{E} \cdot d\mathbf{r} = -\frac{d}{dt}\iint_S \mathbf{B} \cdot d\mathbf{S} \qquad (\text{Faraday's law}) \\
&\oint_C \mathbf{B} \cdot d\mathbf{r} = \mu_0\epsilon_0\frac{d}{dt}\iint_S \mathbf{E} \cdot d\mathbf{S} + \mu_0 I_S \qquad (\text{Ampère's law}).
\end{aligned}$$

In the first two equations, S is a closed surface, and Q is the charge enclosed by S. In the last two equations, C is the boundary curve of a surface S, and I_S is the current flowing through S (that is, the flux of $\mathbf{J}$ across S).

As you can see, the differential form of the equations is a little simpler. But the integrated form tells us about the macroscopic quantities that we ultimately care about. (A similar thing happens with gravitation: Newton's law for gravitation is simplest when expressed as a law about force or acceleration, but integrating it twice gives information about position, which is ultimately what we care about.)

22.3. **Equivalence of the two forms.** Let's prove the equivalence of the two forms of the third Maxwell equation (Faraday's law). The key to the proof will be Stokes' theorem.

Proof that the differential form implies the integrated form. First suppose that

$$\operatorname{curl} \mathbf{E} = -\frac{d\mathbf{B}}{dt}$$

at every point. Take the surface integral of both sides over S:

$$\iint_S \operatorname{curl} \mathbf{E} \cdot d\mathbf{S} = \iint_S -\frac{d\mathbf{B}}{dt} \cdot d\mathbf{S}.$$

Apply Stokes' theorem to the left side, and on the right side convert the integral of a derivative to the derivative of an integral (usually it is OK to do this, just as the sum of derivatives is the derivative of the sum):

$$\oint_C \mathbf{E} \cdot d\mathbf{r} = -\frac{d}{dt} \iint_S \mathbf{B} \cdot d\mathbf{S}. \qquad \square$$

Proof that the integrated form implies the differential form. Suppose that

$$\oint_C \mathbf{E} \cdot d\mathbf{r} = -\frac{d}{dt} \iint_S \mathbf{B} \cdot d\mathbf{S}.$$

holds for every surface S, with C being the boundary of S. Reversing the steps above leads to

$$\iint_S \operatorname{curl} \mathbf{E} \cdot d\mathbf{S} = \iint_S -\frac{d\mathbf{B}}{dt} \cdot d\mathbf{S}$$

for every surface S. By Lemma 22.1 below, this implies that

$$\operatorname{curl} \mathbf{E} = -\frac{d\mathbf{B}}{dt}$$

everywhere. $\square$

Lemma 22.1. *If* $\mathbf{F}$ *and* $\mathbf{G}$ *are continuous 3D vector fields such that*

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{G} \cdot d\mathbf{S}$$

for every surface S*, then* $\mathbf{F} = \mathbf{G}$ *everywhere.*

Proof. Suppose not. Then we can choose a point P where $\mathbf{F} \neq \mathbf{G}$. Let S be a tiny disk perpendicular to $\mathbf{F} - \mathbf{G}$ at P. If S is small enough, then

$$\iint_S (\mathbf{F} - \mathbf{G}) \cdot d\mathbf{S} \neq 0.$$

Thus

$$\iint_S \mathbf{F} \cdot d\mathbf{S} \neq \iint_S \mathbf{G} \cdot d\mathbf{S},$$

a contradiction. $\square$

23. Review

23.1. **Flux.** Flux quantifies how much a vector field $\mathbf{F}$ is "flowing" across a surface S:

$$\text{flux} := \iint_S \underbrace{\mathbf{F} \cdot \mathbf{n}}_{\text{normal component of } \mathbf{F}} \, dS = \iint_S \mathbf{F} \cdot d\mathbf{S}.$$

The part of $\mathbf{F}$ that matters is the part going *across* S — this is $\mathbf{F} \cdot \mathbf{n}$, which is the scalar component of $\mathbf{F}$ in the direction of the unit normal vector to S. Because of the geometric interpretation of the dot product $\mathbf{F} \cdot \mathbf{n}$ in terms of the angle θ between $\mathbf{F}$ and $\mathbf{n}$, the flux through a tiny piece of surface area dS will be

- positive if $\mathbf{F}$ is roughly in the same direction as $\mathbf{n}$ (that is, $\theta < \pi/2$),
- negative if $\mathbf{F}$ and $\mathbf{n}$ are roughly in opposite directions (that is, $\theta > \pi/2$), , and
- zero if $\mathbf{F}$ is perpendicular to $\mathbf{n}$ (that is, $\theta = \pi/2$, so $\mathbf{F}$ is parallel to S, so that $\mathbf{F}$ is not really flowing across S).

Also, the definition of flux is a surface integral because one needs to add up the flux through each tiny piece of S.

Question 23.1. How do you *compute* flux? Is it necessary to parametrize S?

Answer. Parametrizing S will work, but it is not always necessary:

- If $\mathbf{F} \cdot \mathbf{n}$ is constant on S, then pull it out of $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$ so that it remains to compute the surface area of S.
- Otherwise, find a parametrization $\mathbf{r}(u, v)$ of S and use

$$\mathbf{n}\, dS = d\mathbf{S} = \mathbf{r}_u \times \mathbf{r}_v \; du\, dv. \qquad \square$$

Thursday, December 9

23.2. **A parade of differentials.** What is the difference between ds, dS, $d\mathbf{S}$, etc.?

- $d\mathbf{r}$ can be thought of as a tiny vector measuring change of position along a curve.
  Use: Line integrals of vector fields (for example, work) have the shape $\int_C \mathbf{F} \cdot d\mathbf{r}$.

$$d\mathbf{r} = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \mathbf{T}\, ds.$$

- ds can be thought of as the length of a tiny piece of a curve.
  Uses: Length of a curve, line integrals of *scalar* functions.

$$ds = |d\mathbf{r}| = \sqrt{dx^2 + dy^2 + dz^2} = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt.$$

- dA can be thought of as the area of a tiny piece of $\mathbb{R}^2$:

$$\begin{aligned} dA &= dx\, dy \qquad \text{(rectangular)} \\ &= r\, dr\, d\theta \qquad \text{(polar)} \\ &= \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv \qquad \text{(change of variable).} \end{aligned}$$

  (The last formula is for a change of variable $x = x(u, v)$ and $y = y(u, v)$.)

- dS can be thought of as the area of a tiny piece of a surface.
  Uses: Surface area, surface integrals of *scalar* functions.
  If the surface is parametrized by $\mathbf{r}(u, v)$, then

$$dS = |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv$$

  (That's the area of a tiny parallelogram of sides $\mathbf{r}_u \, du$ and $\mathbf{r}_v \, dv$ in $\mathbb{R}^3$.) For a sphere of radius ρ parametrized by ϕ, θ, this becomes

$$dS = \rho^2 \sin\phi \, d\phi \, d\theta.$$

- $d\mathbf{S}$ means $\mathbf{n} \, dS$, where $\mathbf{n}$ is a unit normal to the surface.
  Use: Flux of a 3D vector field $\mathbf{F}$ across a surface S is $\iint_S \mathbf{F} \cdot d\mathbf{S}$.
  In terms of $\mathbf{r}(u, v)$:

$$d\mathbf{S} = \mathbf{r}_u \times \mathbf{r}_v \; du \, dv.$$

  The length of $d\mathbf{S}$ is dS. The direction of $d\mathbf{S}$ is the unit normal $\mathbf{n} = \dfrac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$.
- dV can be thought of as the volume of a tiny region in $\mathbb{R}^3$:

$$\begin{aligned} dV &= dx \, dy \, dz \qquad \text{(rectangular)} \\ &= dz \, r \, dr \, d\theta \qquad \text{(cylindrical)} \\ &= \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \qquad \text{(spherical)} \\ &= \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du \, dv \, dw \qquad \text{(change of variable).} \end{aligned}$$

- dm can be thought of as a tiny bit of mass. It is $\delta \, ds$, or $\delta \, dS$, or $\delta \, dV$, depending on whether the object is of dimension 1, 2, or 3 (such as a wire, a warped metal plate, or a planet). The density δ is mass per unit length, or mass per unit area, or mass per unit volume.
  Uses: Mass, mass-weighted average, centroid, moment of inertia, gravitational field.

23.3. **Table of differentials and their uses.**

n-dim integral	Differential n-forms	Uses
$\int$	ds, $d\mathbf{r}$	arc length, work
$\iint$	dS, $d\mathbf{S}$	surface area, flux across surface
$\iiint$	dV	volume
it depends	dm	mass, average, centroid, moment of inertia

Make sure that the dimension of the integral matches what you are trying to compute!

Example 23.2. Flux of a 3D vector field $\mathbf{F}$ is flux across a *surface*, so it should be a 2D integral. (In fact, it is the integral of the normal component $\mathbf{F}\cdot\mathbf{n}$ with respect to surface area, so it is $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$.)

23.4. **Ice cream in rectangular, cylindrical, and spherical coordinates.**

Problem 23.3. Let T be an ice cream cone with vertex $(0,0,0)$, height 4, and flat top of radius 3. (It's the disappointing kind of ice cream cone in which the ice cream does not bulge at the top.) Find triple integrals expressing the volume in rectangular, cylindrical, and spherical coordinates.

Solution.

Cylindrical: Above a point in the xy-plane with polar coordinates (r,θ) (with $r \le 3$) the height of the bottom of the cone is $\frac{4}{3}r$ by similar triangles, and the height of the top of the cone is 4. In other words, the cone is described by $\frac{4}{3}r \le z \le 4$ (which implies $r \le 3$). So the volume is

$$\int_{\theta=0}^{2\pi}\int_{r=0}^{3}\int_{z=\frac{4}{3}r}^{z=4} dz\, r\, dr\, d\theta.$$

We did not have time for the rest of this solution, but it will be discussed in my review session.

Rectangular: The inequality $r \le 3$ corresponds to $x^2+y^2 \le 9$, and the range for z above the point (x,y) in this disk is $\frac{4}{3}\sqrt{x^2+y^2} \le z \le 4$. So the volume is

$$\int_{x=-3}^{3}\int_{y=-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{z=\frac{4}{3}\sqrt{x^2+y^2}}^{4} dz\, dy\, dx.$$

Spherical: Either argue geometrically to find the ranges for ρ,ϕ,θ, or just substitute the change of coordinates into the inequalities. Let's do the latter: $\frac{4}{3}r \le z \le 4$ becomes

$$\frac{4}{3}\rho\sin\phi \le \rho\cos\phi \le 4.$$

The first inequality says $\tan\phi \le 3/4$, so $\phi \le \tan^{-1}(3/4)$. The second inequality says $\rho \le 4/\cos\phi$. So the volume is

$$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\tan^{-1}(3/4)}\int_{\rho=0}^{4/\cos\phi} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta.$$ □

Problem 23.4. Same cone without the ice cream. What is a parametrization for the lateral surface of the cone?

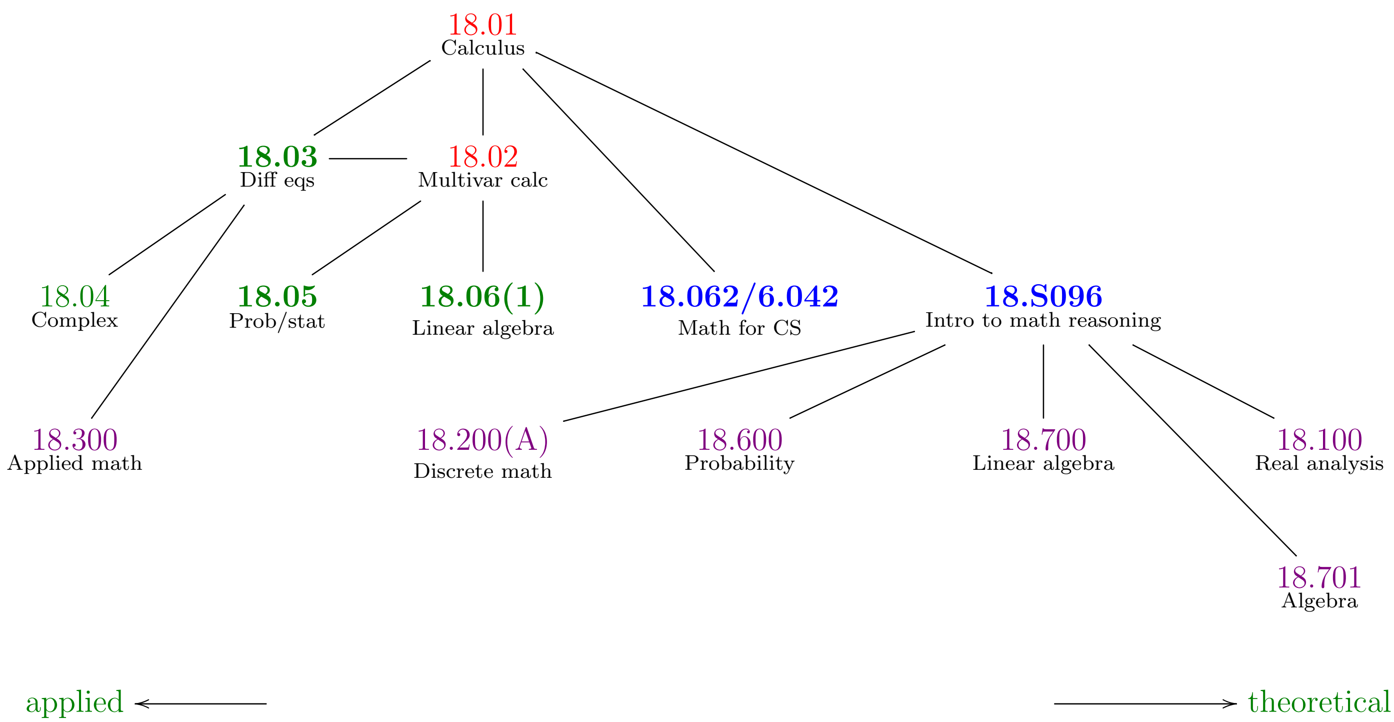
Solution. First, choose your parameters. How do you describe a position on the cone with two numbers? One possible answer: give the angle θ and the height z, i.e., the θ and z of cylindrical coordinates. The radius at height z is $\frac{3}{4}z$, so

$$x = \frac{3}{4} z \cos\theta$$
$$y = \frac{3}{4} z \sin\theta$$
$$z = z.$$

In other words,

$$\mathbf{r}(\theta, z) = \begin{pmatrix} \frac{3}{4} z \cos\theta \\ \frac{3}{4} z \sin\theta \\ z \end{pmatrix}. \qquad \square$$

24. Life after 18.02



For sciences/engineering, 18.03, 18.05, 18.06 are useful.

For computer science, 18.06 and 18.062/6.042.

To prepare for more advanced math subjects (18.100 and greater), or if considering a math major (or double major with math), take 18.S096. This is a new subject developed by me and Profs. Dyatlov and Seidel that uses selected topics from combinatorics, algebra, and analysis (including fun topics like permutations and different sizes of infinity) to teach how to understand abstract concepts and prove theorems. The subject 18.062/6.042 is similar, but it uses topics aimed towards computer science subjects.

Many 18.02 students become math majors, or do a double major with math. If you are considering this, taking 18.S096 is a great way to get a sense of what it is like to be a math major — it is designed for students with your background.

25. Thank you

Thank you to

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- the professors I consulted in other departments whose advice helped me modernize the 18.02 content this fall to make it more relevant to their subjects;
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And thank you to all the 18.02 students for making this class fun to teach! I hope that you all ace the final!