Accurate Jordan Structures of Irreducible Totally Nonnegative Matrices

Plamen Koev
Department of Mathematics
San Jose State University

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Definition

- *Totally nonnegative* means all minors are nonnegative
- Examples: Hilbert, Pascal, Vandermonde with increasing nodes

$$\begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$$

- Nonsymmertic in general
- Interested in the *irreducible case*

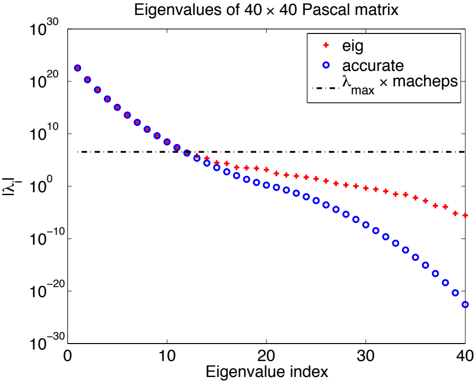
Goal

- Compute the Jordan structure of IrTN matrix in floating point arithmetic to high relative accuracy:
  - all eigenavlues will have correct sign and leading digits (including tiniest ones)
  - the Jordan blocks will be correctly computed

Background

- Spectral structure well understood (Fallat/Johnson/Gekhtman):
  - All eigenvalues are nonnegative
  - Positive eigenvalues are distinct
  - Zero eigenvalues can have Jordan blocks
- Computationally hard: TN matrices can be ill conditioned, so accuracy in tiny eigenvalues lost in floating point

Example: 40 $\times$ 40 Pascal matrix, $P = \binom{n}{k}$



$\mathrm{cond}(P) \approx 10^{45}$; eigenvalues $< \lambda_{\max} * 10^{-16}$ lost

Reason accuracy is lost in floating point arithmetic

- Relative accuracy preserved in $\times, +, /$
  Proof: $(1+\delta)$ factors accumulate multiplicatively
- Subtractions of approximate quantities dangerous:

$$\begin{array}{rl} & .123456789\textit{xxx} \\ - & .123456789\textit{yyy} \\ \hline & .000000000\textit{zzz} \end{array}$$

- Thus, if we avoid subtractions, we get accuracy

Previous results

- All linear algebra with *nonsingular* TN matrices possible accurately (K., '05)
  - Eigenvalues
  - Singular Values
  - Product
  - LU
  - submatrix
  - R factor of QR (still TN)
  - Converse, ...
- Only 3 operations needed and all 3 possible accurately
  - Subtracting a multiple of one row from next to create a zero
  - Add a multiple of one row to the previous
  - Diagonal scaling (trivial)

Bidiagonal decompositions

- Trick: Work on the *bidiagonal decomposition* (BD), not on the matrix!
- That representation reveals the TN structure
- Result of Neville elimination

$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 1 & 1 & \\ & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & \\ & 1 & 3 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & 2 \\ & & 1 \end{bmatrix}$$

- Then operate on those entries, NOT on the matrix entries!
- TN-preserving operations require no subtractions $\Rightarrow$ accuracy

Basic Operations

- Subtracting a multiple of one row from next to create a 0 is equivalent to setting an entry of the BD to 0

$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 1 & 1 & \\ & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & \\ & 1 & 3 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & 2 \\ & & 1 \end{bmatrix}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 0 & 1 & 7 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 1 & 1 & \\ & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & \\ & 1 & 3 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & 2 \\ & & 1 \end{bmatrix}$$

- No subtractions $\Rightarrow$ accuracy
- New matrix still TN

Basic Operations

- Adding a multiple of one row/col to next/previous is done by changing the entries of the BD only

$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 0 & 1 & 7 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 1 & 1 & \\ & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & \\ & 1 & 3 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & 2 \\ & & 1 \end{bmatrix}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\begin{bmatrix} 1 & 6 & 4 \\ 1 & 12 & 9 \\ 0 & 8 & 7 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 2 & 1 & \\ & \frac{3}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 6 & \\ & & 2 \end{bmatrix} \begin{bmatrix} 1 & 6 & \\ & 1 & \frac{4}{3} \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & 2 \\ & & 1 \end{bmatrix}$$

- New entries are rational functions with > 0 coefficients
- Again, no subtractions $\Rightarrow$ accuracy
- New matrix is still TN (Cauchy–Binet)

Eigenvalues of Nonsingular TN matrices

- Reduction to tridiagonal form using above similarities
- To create a 0 in position (3,1) of

$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}$$

- We use similarity

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 0 & 1 & 7 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 4 \\ 1 & 12 & 9 \\ 0 & 8 & 7 \end{bmatrix}$$

Eigenvalues of Nonsingular TN matrices

- Reduction to tridiagonal form possible using standard approach (Cryer '76)

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \to \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \end{bmatrix} \to \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \end{bmatrix} \to \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \end{bmatrix} \to$$

$$\begin{bmatrix} + & + & + & 0 \\ + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \end{bmatrix} \to \begin{bmatrix} + & + & 0 & 0 \\ + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \end{bmatrix} \to \begin{bmatrix} + & + & 0 & 0 \\ + & + & + & 0 \\ 0 & + & + & + \\ 0 & 0 & + & + \end{bmatrix}$$

Irreducible (singular) TN matrices

- At the end we have an (irreducible TN) tridiagonal in factored form

$$\begin{bmatrix} + & + & 0 & 0 \\ + & + & + & 0 \\ 0 & + & + & + \\ 0 & 0 & + & + \end{bmatrix} = \begin{bmatrix} 1 & & & \\ l_1 & 1 & & \\ & l_2 & 1 & \\ & & l_3 & 1 \end{bmatrix} \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & d_4 \end{bmatrix} \begin{bmatrix} 1 & l_1 & & \\ & 1 & l_2 & \\ & & 1 & l_3 \\ & & & 1 \end{bmatrix}$$

- Eigenvalues readily computable accurately as singular values of bidiagonal factor (Demmel–Kahan, 1990)

Irreducible (singular) TN matrices

- Bidiagonal decompositions exist, but not unique:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

Irreducible (singular) TN matrices – 2

- Bidiagonal decompositions exist, but not unique:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

Subtract 2nd row from 3rd

Irreducible (singular) TN matrices – 3

- Bidiagonal decompositions exist, but not unique:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

Irreducible (singular) TN matrices – 4

- Bidiagonal decompositions exist, but not unique:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

Then 1st from 2nd

Irreducible (singular) TN matrices – 5

- Bidiagonal decompositions exist, but not unique:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

Irreducible (singular) TN matrices – 6

- Bidiagonal decompositions exist, but not unique:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

- We want to swap rows 2 and 3 using a TN transformation
- The obvious solution

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

is *not* TN!

Irreducible (singular) TN matrices – 7

- Bidiagonal decompositions exist, but not unique:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

- We want to swap rows 2 and 3 using a TN transformation
- The obvious solution

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

  is *not* TN!
- Dealing with zeros cuts both ways:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{\text{TN}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Irreducible (singular) TN matrices – 8

- The bidiagonal decompositions may have zeros on the diagonals:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 1 & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 1 & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & {\color{red}0} & \\ & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 1 & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & {\color{red}0} & \\ & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & {\color{red}0} \end{bmatrix} \begin{bmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & 3 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{bmatrix}$$

Tridiagonal reduction breaks

- All TN linear algebra still possible accurately (unaffected by the new zeros)
- But tridiagonal reduction can no longer be done with EB matrices only:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & 4 \end{bmatrix}$$

- To kill the (3,1) entry we need to form

$$\begin{bmatrix} 1 & & \\ & 1 & \\ -1 & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ 1 & & 1 \end{bmatrix}$$

—the similarity is not NOT TN!!!

Preserve the nonzero eigenvalues

- We can erase zero rows and columns (TN preserving operations)

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}$$

  which preserves the nonzero eigenvalues
- This yields accurate nonzero eigenvalues; how about the zero ones (and those Jordan blocks)?

Jordan blocks corresponding to zero eigenvalues

- $n - \mathrm{rank}(A) =$# Jordan blocks
- $\mathrm{rank}(A) - \mathrm{rank}(A^2) =$# of Jordan blocks of size ≥ 2
- ...
- $\mathrm{rank}(A), \mathrm{rank}(A^2), \ldots$ readily obtainable from its BD
- A^2 is TN (as a product of TN) and its BD is a TN-preserving op, thus BD accurate
- need to form BD of $A^2, \ldots, A^n$, a potential $O(n^4)$ algorithm

Example

```
A =
      3  3  2  1
      2  2  3  2
      1  1  2  3
      1  1  2  3

>> eig(A)
ans =
      7.828427124746188e+00
      2.171572875253811e+00
      5.247731480861326e-16
     -1.110223024625157e-16

>> TNEigenValues(BD))
ans =
      7.828427124746190e+00
      2.171572875253810e+00
                          0
                          0

>> [TNRank(BD),TNRank(TNProduct(BD,BD))]
ans=
      3       2
```