Accurate and Efficient Matrix Computations with Totally Positive Matrices

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Goals

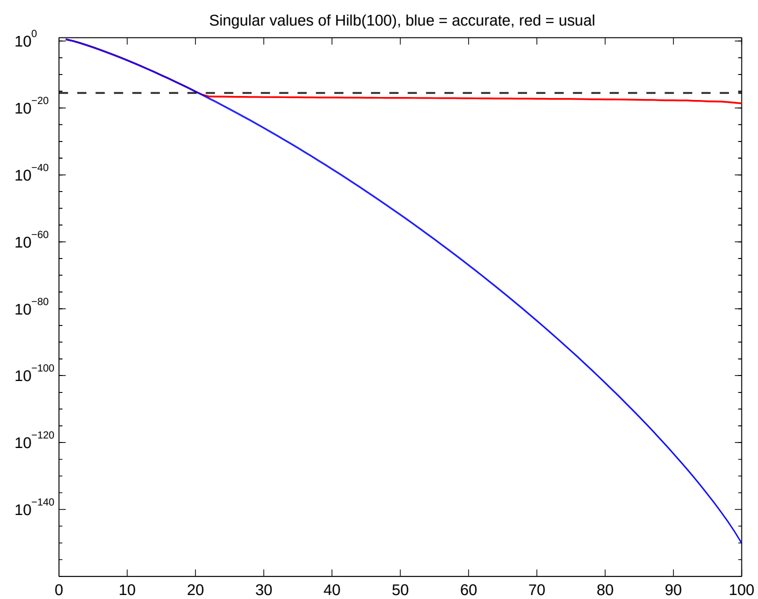
- **Accurate and Efficient Linear Algebra with Totally Positive Matrices**
  - Linear Equation Solving, Inversion
  - Eigenvalues, Eigenvectors
  - Singular Values, Singular Vectors
- **Efficient means $O(n^3)$**
  - Usual arithmetic model $\mathrm{fl}(a \odot b) = (a \odot b)(1+\delta), \; |\delta| \le \epsilon$
- **Accurate means correct sign and leading digits**

$$|\hat{\lambda}_i - \lambda_i| \le O(\epsilon)|\lambda_i|, \qquad |\hat{\sigma}_i - \sigma_i| \le O(\epsilon)\sigma_i,$$

  - Contrast: Traditional algorithms

$$|\hat{\lambda}_i - \lambda_i| \le O(\epsilon)\frac{||A||}{|y^* x|}, \qquad |\hat{\sigma}_i - \sigma_i| \le O(\epsilon)\sigma_1$$

Example: 100 × 100 Hilbert Matrix $H = 1/(i + j)$



How do we lose accuracy in floating point?

- $\mathrm{fl}(a \otimes b) = (a \otimes b)(1+\delta)$, no under/overflow
- How can we lose accuracy in computing in floating point?
  - OK to multiply, divide, add positive numbers
    Proof: $1+\delta$ factors can be factored out
  - $x_i \pm x_j$, where x_i and x_j are initial data (so exact)
  - Cancellation when subtracting approximate results dangerous:

```
   .12345xxx
-  .12345yyy
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   .00000zzz
```

- Trick: We will only compute:
  - $1/(1+xy)$; $1/x$; xy; $1/(x_i - y_j)$; $x, y > 0$
  - never subtract approximate quantities

Bidiagonal Matrices

- Demmel–Kahan '91

$$B = \begin{bmatrix} a_1 & b_1 & & & \\ & a_2 & b_2 & & \\ & & \ddots & \ddots & \\ & & & a_{n-1} & b_{n-1} \\ & & & & a_n \end{bmatrix}$$

- a_i, b_i determine σ_i accurately
- dqds preserves the relative accuracy
- Eigenvalues of SPD tridiagonal matrices: $\lambda_i(B^T B) = \sigma_i^2(B)$
  – computable accurately given the Cholesky factors

Total Positivity (TP)

- Def: A is TP if all minors > 0
  - Hilbert $H = 1/(i+j)$
  - Cauchy $C = 1/(x_i + y_j)$, if $0 < x_i, y_j$-increasing
  - Generalized Vandermonde $V = x_i^{a_j}$, $0 < x_i, a_i$ - increasing
  - TP·TP
  - Schur-complement(TP)
  - diag(±1)·TP·diag(±1)
  - See books by Karlin, Gantmacher, Krein
- Theory extends to Totally Nonnegative, Oscillatory, Sign-regular
- Motivation:
  - Eigenvalues and SVD well determined
  - Right answer costs $O(n^3)$, same as the wrong
- Trick: choose the right parametrization: Bidiagonal

Total Positivity and Neville Elimination – 1

- **Gaussian Elimination**

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{4} & \mathbf{8} \\ \mathbf{1} & \mathbf{3} & \mathbf{9} & \mathbf{27} \\ \mathbf{1} & \mathbf{4} & \mathbf{16} & \mathbf{64} \end{bmatrix}$$

Total Positivity and Neville Elimination – 2

- **Gaussian Elimination**

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{4} & \mathbf{8} \\ \mathbf{1} & \mathbf{3} & \mathbf{9} & \mathbf{27} \\ \mathbf{0} & \mathbf{3} & \mathbf{15} & \mathbf{63} \end{bmatrix}$$

- **Gaussian Elimination**

$$\begin{bmatrix} \color{red}{1} & \color{red}{1} & \color{red}{1} & \color{red}{1} \\ \color{red}{1} & \color{red}{2} & \color{red}{4} & \color{red}{8} \\ 0 & 2 & 8 & 26 \\ 0 & 3 & 15 & 63 \end{bmatrix}$$

Total Positivity and Neville Elimination – 4

- **Gaussian Elimination**

$$\begin{bmatrix} \color{red}{1} & \color{red}{1} & \color{red}{1} & \color{red}{1} \\ 0 & 1 & 3 & 7 \\ 0 & 2 & 8 & 26 \\ 0 & 3 & 15 & 63 \end{bmatrix}$$

- **In Matrix form:**

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 0 & 1 & \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 2 & 8 & 26 \\ 0 & 3 & 15 & 63 \end{bmatrix}$$

- **Neville Elimination**

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{4} & \mathbf{8} \\ \mathbf{1} & \mathbf{3} & \mathbf{9} & \mathbf{27} \\ \mathbf{1} & \mathbf{4} & \mathbf{16} & \mathbf{64} \end{bmatrix}$$

- **Neville Elimination**

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{4} & \mathbf{8} \\ \mathbf{1} & \mathbf{3} & \mathbf{9} & \mathbf{27} \\ \mathbf{0} & \mathbf{1} & \mathbf{7} & \mathbf{37} \end{bmatrix}$$

- **Neville Elimination**

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{4} & \mathbf{8} \\ \mathbf{0} & \mathbf{1} & \mathbf{5} & \mathbf{19} \\ \mathbf{0} & \mathbf{1} & \mathbf{7} & \mathbf{37} \end{bmatrix}$$

- **Neville Elimination**

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{3} & \mathbf{7} \\ \mathbf{0} & \mathbf{1} & \mathbf{5} & \mathbf{19} \\ \mathbf{0} & \mathbf{1} & \mathbf{7} & \mathbf{37} \end{bmatrix}$$

Neville Elimination – We can choose the pattern

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 0 & 1 & 7 & 37 \end{bmatrix}$$

Neville Elimination – We can choose the pattern

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & {\color{red}1} & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 0 & {\color{red}1} & {\color{red}5} & {\color{red}19} \\ 0 & {\color{red}1} & {\color{red}7} & {\color{red}37} \end{bmatrix}$$

Neville Elimination – We can choose the pattern

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & \color{red}{1} & 1 \end{bmatrix} \begin{bmatrix} \color{red}{1} & \color{red}{1} & \color{red}{1} & \color{red}{1} \\ \color{red}{1} & \color{red}{2} & \color{red}{4} & \color{red}{8} \\ 0 & 1 & 5 & 19 \\ 0 & 0 & 2 & 18 \end{bmatrix}$$

Neville Elimination – We can choose the pattern

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 1 & 5 & 19 \\ 0 & 0 & 2 & 18 \end{bmatrix}$$

Neville Elimination – We can choose the pattern

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & \color{red}{1} & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & \color{red}{2} & \color{red}{12} \\ 0 & 0 & \color{red}{2} & \color{red}{18} \end{bmatrix}$$

Neville Elimination – We can choose the pattern

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & 1 & 1 & \\ & & {\color{red}1} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 2 & 12 \\ 0 & 0 & 0 & {\color{red}6} \end{bmatrix}$$

Neville Elimination – We can choose the pattern

$$
\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =
$$

$$
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix}
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix}
\begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix}
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 2 & \\ & & & 6 \end{bmatrix}
\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Neville Elimination – We can choose the pattern

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 2 & \\ & & & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}$$

Neville Elimination – We can choose the pattern

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 2 & \\ & & & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 & & \\ & 1 & 2 & \\ & & 1 & 3 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & 1 & \\ & & 1 & 2 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}$$

Notice: positive bidiagonal decomposition

$$U^{(k)}_{i,i+1} = \frac{\det(A(1:k, i-k+2:i+1))}{\det(A(1:k-1, i-k+2:i))} \cdot \frac{\det(A(1:k-1, i-k+1:i-1))}{\det(A(1:k, i-k+1:i))}.$$

Total Non-Negativity is preserved in Schur complementation

- **Neville Elimination yield Bidiagonal Decomposition of Any Totally Positive Matrix**

$$\mathcal{BD}\left(\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix}\right) =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ + & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{bmatrix} \begin{bmatrix} 1 & + & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix}$$

- **n^2 entries of factors intrinsically parametrize all TP matrices**
- **For numerical computation:**
  - **Discard A**
    Instead: Use $\mathcal{BD}(A)$ above
  - **Let $A \longrightarrow A'$ be a linear transformation**
    - **Subtract a row from another, apply Givens rot, etc.**
  - **We will compute $\mathcal{BD}(A')$ from $\mathcal{BD}(A)$**
    without losing relative accuracy

- **With $\mathcal{BD}(A)$ we can do accurately**
  - **Positive diagonal scaling – trivial**
  - **Subtract a row/column from next to create a zero**
  - **Multiply by a positive bidiagonal matrix**
- **All linear algebra can be "assembled" from above**

Subtract a row/column from next to create a zero – 1

- **Equivalent to setting an entry in $\mathcal{BD}(A)$ to 0 and doing no arithmetic**

$$
\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} =
$$

$$
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}
\underbrace{
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix}
\begin{bmatrix} 1 & & & \\ + & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix}
\begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{bmatrix}
\begin{bmatrix} 1 & + & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix}
\begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix}
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix}
}_{\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix}}
$$

Subtract a row/column from next to create a zero – 2

- **Equivalent to setting an entry in $\mathcal{BD}(A)$ to 0 and doing no arithmetic**

$$
\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} =
$$

$$
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}
\underbrace{
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix}
\begin{bmatrix} 1 & & & \\ + & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix}
\begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{bmatrix}
\begin{bmatrix} 1 & + & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix}
\begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix}
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix}
}_{\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \end{bmatrix}}
$$

$$
\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \end{bmatrix} =
$$

$$
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix}
\begin{bmatrix} 1 & & & \\ + & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix}
\begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{bmatrix}
\begin{bmatrix} 1 & + & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix}
\begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix}
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix}
$$

Accurate TP×BIDIAGONAL

$$\mathcal{BD}\left(\mathcal{BD}(A) \times \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & & 1 \end{bmatrix}\right) = ?$$

"Commuting" of positive bidiagonals

- Assume $x, y, z > 0$, $b_i \geq 0$. Then

$$\begin{bmatrix} 1 & b_1 & & & \\ & 1 & b_2 & & \\ & & \ddots & \ddots & \\ & & & 1 & b_{n-1} \\ & & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & y & & & \\ & & x & z & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & y' & & & \\ & & x' & z' & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} \begin{bmatrix} 1 & b'_1 & & & \\ & 1 & b'_2 & & \\ & & \ddots & \ddots & \\ & & & 1 & b'_{n-1} \\ & & & & 1 \end{bmatrix}$$

where

$$\begin{aligned} x' &= x \\ y' &= y + b_i x \\ z' &= 1/y' \\ b'_{i-1} &= b_{i-1} y \\ b'_i &= b_i z / y_1 \end{aligned}$$

- No loss of relative accuracy
- "Commute accurately".

"Commuting" of positive bidiagonals - 2

- Assume $x > 0$, $b_i \geq 0$. Then

$$\begin{bmatrix} 1 & & & & & \\ b_1 & 1 & & & & \\ & \ddots & \ddots & & & \\ & & b_{n-2} & 1 & \\ & & & b_{n-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & x & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & x' & 1 & \\ & & & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & & \\ b'_1 & 1 & & & \\ & \ddots & \ddots & & \\ & & b'_{n-2} & 1 & \\ & & & b'_{n-1} & 1 \end{bmatrix}$$

where

$$\begin{aligned} x' &= x b_{i+1}/(b_i + x) \\ b'_i &= b_i + x \\ b'_{i+1} &= x b_i / b'_i \end{aligned}$$

- No loss of relative accuracy

Applying similarity to bidiagonal form – 1

$$
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix}
\begin{bmatrix} 1 & & & \\ + & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix}
\begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{bmatrix}
\begin{bmatrix} 1 & + & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix}
\begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix}
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix}
\color{blue}{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}}
$$

Applying similarity to bidiagonal form – 2

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ + & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{bmatrix} \begin{bmatrix} 1 & + & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & y & \\ & & + & z \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix}$$

Applying similarity to bidiagonal form – 3

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ + & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{bmatrix} \begin{bmatrix} 1 & + & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix} {\color{blue}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & y' & \\ & & + & z' \end{bmatrix}} \begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & {\color{red}+} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix}$$

Applying similarity to bidiagonal form – 4

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ + & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{bmatrix} {\color{blue}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & y'' & \\ & & + & z'' \end{bmatrix}} \begin{bmatrix} 1 & + & & \\ & 1 & + & \\ & & 1 & {\color{red}+} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix}$$

Applying similarity to bidiagonal form – 5

$$
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix}
\begin{bmatrix} 1 & & & \\ + & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix}
{\color{blue}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}}
\begin{bmatrix} + & & & \\ & + & & \\ & & {\color{red}+} & \\ & & & {\color{red}+} \end{bmatrix}
\begin{bmatrix} 1 & + & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix}
\begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix}
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix}
$$

Applying similarity to bidiagonal form – 6

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ + & 1 & & \\ & + & 1 & \\ & & \color{red}{+} & 1 \end{bmatrix} \begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{bmatrix} \begin{bmatrix} 1 & + & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix}$$

DONE!

Accurate Matrix Computations with TP matrices

- $\mathcal{BD}$(TP $\times$ TP)
  - suffices $\mathcal{BD}$(TP $\times$ BIDIAGONAL) – done
- $\mathcal{BD}(\text{diag}(\pm 1) \cdot (\text{TP})^{-1} \cdot \text{diag}(\pm 1))$
  - OK, still a product of positive bidiagonals
- $\mathcal{BD}$(SchurComplement) – also OK, but barely more complicated
- $\mathcal{BD}(\text{Givens} \cdot TP)$

$$\begin{bmatrix} c & s \\ -s & c \end{bmatrix} = \begin{bmatrix} + & + \\ 0 & + \end{bmatrix} \begin{bmatrix} 1 & 0 \\ - & 1 \end{bmatrix}$$

- SVD is then OK: TP$\rightarrow$BIDIAGONAL$\rightarrow$SVD
  - no loss of accuracy
- Eigenvalues: First reduce to tridiagonal

Reduction of a nonsymmetric matrix to tridiagonal form – 1

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \color{red}{+} & + & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 2

$$\begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & - & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ \color{red}{+} & \color{red}{+} & \color{red}{+} & \color{red}{+} \\ \color{red}{+} & \color{red}{+} & \color{red}{+} & \color{red}{+} \end{bmatrix}$$

$$= \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 3

$$\begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & - & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & + & \mathbf{1} \end{bmatrix}$$

$$= \begin{bmatrix} + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ \mathbf{0} & + & \color{red}{+} & \color{red}{+} \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & + & \mathbf{1} \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 4

$$\begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & - & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & + & \mathbf{1} \end{bmatrix}$$

$$= \begin{bmatrix} + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ \mathbf{0} & + & \color{red}{+} & \color{red}{+} \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & + & \mathbf{1} \end{bmatrix}$$

$$= \begin{bmatrix} + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ \mathbf{0} & + & \color{red}{+} & \color{red}{+} \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 5

$$\begin{bmatrix} + & + & + & \color{red}{+} \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 6

$$\begin{bmatrix} + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ \mathbf{0} & + & \color{red}{+} & \color{red}{+} \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & - \\ & & & \mathbf{1} \end{bmatrix}$$

$$= \begin{bmatrix} + & + & + & \mathbf{0} \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 7

$$\begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & + \\ & & & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ \color{red}{+} & \color{red}{+} & \color{red}{+} & \color{red}{+} \\ \color{red}{\mathbf{0}} & \color{red}{+} & \color{red}{+} & \color{red}{+} \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & - \\ & & & \mathbf{1} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & + \\ & & & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & + & \mathbf{0} \\ + & + & + & + \\ \color{red}{+} & \color{red}{+} & \color{red}{+} & \color{red}{+} \\ \color{red}{\mathbf{0}} & \color{red}{+} & \color{red}{+} & \color{red}{+} \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 8

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ {\color{red}+} & {\color{red}+} & {\color{red}+} & {\color{red}+} \\ {\color{red}0} & {\color{red}+} & {\color{red}+} & {\color{red}+} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & - \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & 0 \\ + & + & + & + \\ {\color{red}+} & {\color{red}+} & {\color{red}+} & {\color{red}+} \\ {\color{red}0} & {\color{red}+} & {\color{red}+} & {\color{red}+} \end{bmatrix}$$

$$= \begin{bmatrix} + & + & + & 0 \\ + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & + \\ & & & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & - \\ & & & \mathbf{1} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & + \\ & & & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & + & \mathbf{0} \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix}$$

$$= \begin{bmatrix} + & + & + & \mathbf{0} \\ + & + & + & + \\ \mathbf{0} & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 10

$$\begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & + \\ & & & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & - \\ & & & \mathbf{1} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & + \\ & & & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & + & \mathbf{0} \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix}$$

$$= \begin{bmatrix} + & + & \color{red}{\mathbf{0}} & \mathbf{0} \\ + & + & + & + \\ \mathbf{0} & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & + \\ & & & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & - \\ & & & \mathbf{1} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & + \\ & & & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & + & \mathbf{0} \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix}$$

$$= \begin{bmatrix} + & + & \mathbf{0} & \mathbf{0} \\ + & + & + & + \\ \mathbf{0} & + & + & + \\ \mathbf{0} & \color{red}{\mathbf{0}} & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 12

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & - \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & 0 \\ + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \end{bmatrix}$$

$$= \begin{bmatrix} + & + & 0 & 0 \\ + & + & + & \color{red}{0} \\ 0 & + & + & + \\ 0 & 0 & + & + \end{bmatrix}$$

Accurate Eigenvalues of Tridiagonals

The eigenvalues of

$$\begin{bmatrix} 1 & & & & \\ b_1 & 1 & & & \\ & \ddots & \ddots & & \\ & & b_{n-2} & 1 & \\ & & & b_{n-1} & 1 \end{bmatrix} \begin{bmatrix} d_1 & & & & \\ & d_2 & & & \\ & & \ddots & & \\ & & & d_{n-1} & \\ & & & & d_n \end{bmatrix} \begin{bmatrix} 1 & c_1 & & & \\ & 1 & c_2 & & \\ & & \ddots & \ddots & \\ & & & 1 & c_{n-1} \\ & & & & 1 \end{bmatrix}$$

are the squares of the singular values of

$$\begin{bmatrix} \sqrt{d_1} & \sqrt{d_1 b_1 c_1} & & & \\ & \sqrt{d_2} & \sqrt{d_2 b_2 c_2} & & \\ & & \ddots & \ddots & \\ & & & \sqrt{d_{n-1}} & \sqrt{d_{n-1} b_{n-1} c_{n-1}} \\ & & & & \sqrt{d_n} \end{bmatrix}$$

- Thus the eigenvalue problem of a TP matrix is reduced to the SVD problem for a bidiagonal matrix
- Forward stable
- Total cost $= 8n^3$.

Computing Accurate $\mathcal{BD}(A)$

- **Bidiagonal decompositions intrinsic for all TP matrices**
- **Often accurate formulas:**
  - **Vandermonde:**

$$\prod_{j=1}^{i-1}(x_i - x_j), \quad \prod_{j=i-k+1}^{i-1} \frac{x_{i+1} - x_{j+1}}{x_i - x_j}, \quad x_{i+n-k}$$

  - **Cauchy:**

$$\prod_{k=1}^{n-1} \frac{(x_n - x_k)(y_n - y_k)}{(x_n + y_k)(y_n + x_k)}, \qquad \frac{x_{n-k} + y_{i-n+k+1}}{x_i + y_{i-n+k+1}} \prod_{l=n-k}^{i-1} \frac{x_{i+1} - x_{l+1}}{x_i - x_l} \cdot \prod_{l=1}^{i-n+k-1} \frac{x_i + y_l}{x_{i+1} + y_l}$$

$$\frac{y_{n-k} + x_{i-n+k+1}}{y_i + x_{i-n+k+1}} \prod_{l=n-k}^{i-1} \frac{y_{i+1} - y_{l+1}}{y_i - y_l} \cdot \prod_{l=1}^{i-n+k-1} \frac{y_i + x_l}{y_{i+1} + x_l},$$

- **Equivalent to Björck-Pereyra Methods**
- **Can be computed iff initial minors can be computed accurately**

$$U^{(k)}_{i,i+1} = \frac{\det(A(1:k, i-k+2:i+1))}{\det(A(1:k-1, i-k+2:i))} \cdot \frac{\det(A(1:k-1, i-k+1:i-1))}{\det(A(1:k, i-k+1:i))}.$$

initial minors = contiguous, include the first row or column

Conclusions

- $O(n^3)$ algorithms for the eigenvalues and the SVD of Totally Positive Matrices to high relative accuracy
- Accurate linear algebra with TP matrices closed under same operations as TP
- Applies to:
  - Oscillatory
  - Totally Nonnegative Matrices
  - Sign Regular (Inverses of TP matrices)
- Open problem: If we perturb an entry in

  $$(\text{PositiveBidiagonal}) \cdot (\text{PositiveBidiagonal})$$

  How much do the signular vectors change?
  Singular values provably change by at most ϵ.
- This talk and Matlab software: `math.mit.edu/~plamen`