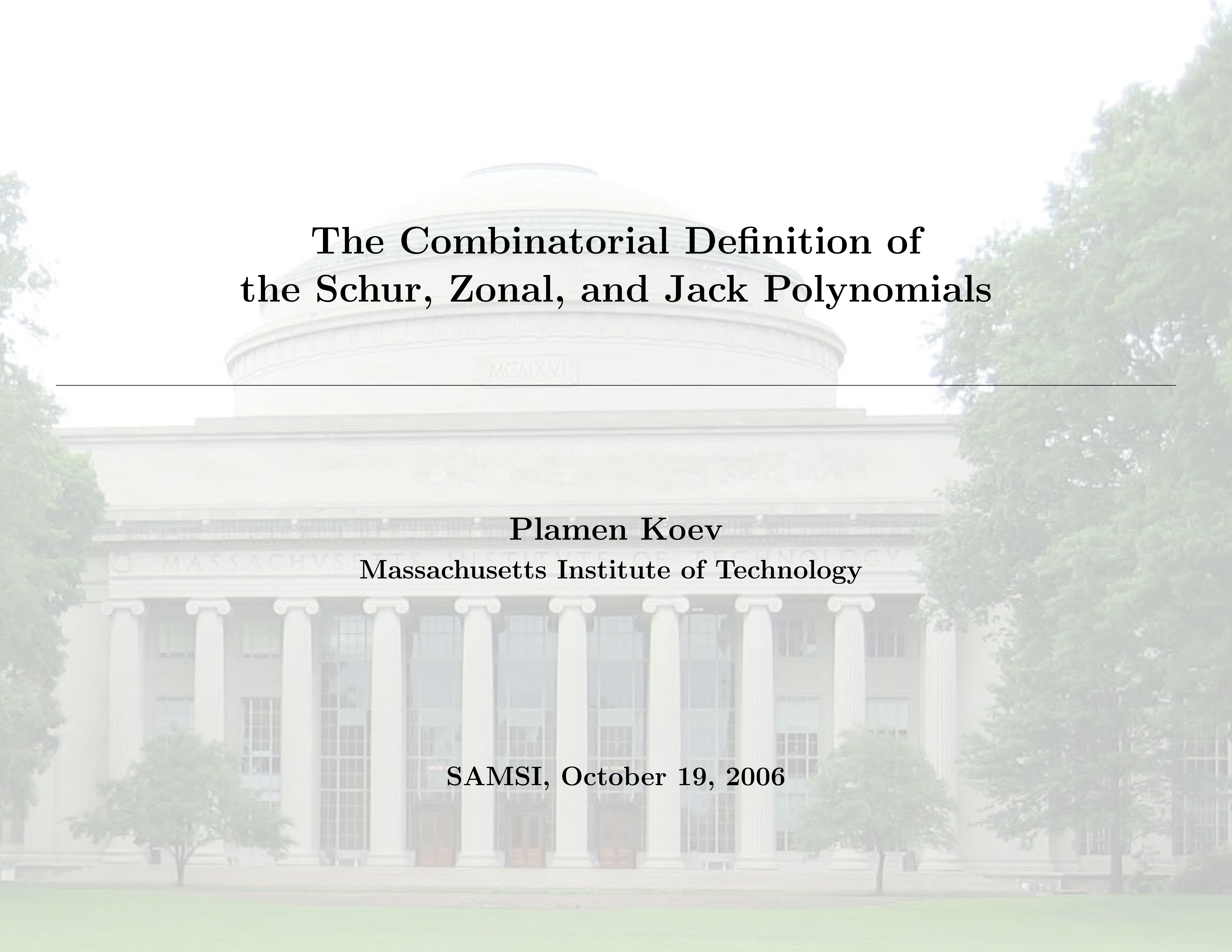
The Combinatorial Definition of the Schur, Zonal, and Jack Polynomials



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Start with the Schur functions. Part One: Partitions

- **Definition: Partition $\boldsymbol{\lambda}$ of the integer n; denoted $\boldsymbol{\lambda} \vdash n$**

  ... is the m-tuple $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ where $n = \lambda_1 + \lambda_2 + \ldots + \lambda_m$ and $\lambda_1 \geq \ldots \geq \lambda_m \geq 0$.

- **Example: Partition**

$$(1) \vdash 1$$

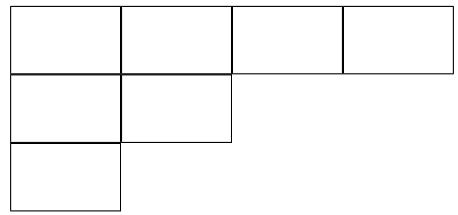
$$(1,1) \vdash 2$$
$$(2) \vdash 2$$

$$(1,1,1) \vdash 3$$
$$(2,1) \vdash 3$$
$$(3) \vdash 3$$

$$\ldots$$
$$(4,2,1) \vdash 7$$

Schur functions. Part Two: Young Diagrams

- **Definition: The Young diagram of the partition $(4, 2, 1) \vdash 7$ is**



- **Coordinates of the boxes**

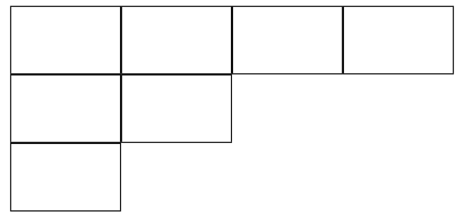
(1,1)	(1,2)	(1,3)	(1,4)
(2,1)	(2,2)		
(3,1)			

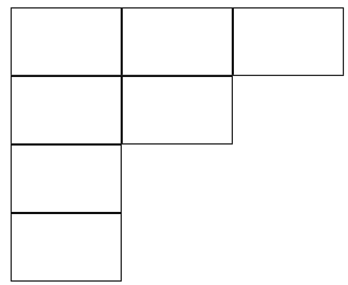
- **The expression "for all $(i, j) \in \lambda$" means:**

  "for all pairs (i, j) such that (i, j) are coordinates of boxes in the Young diagram of λ"

Schur functions. Part Three: Transposed Partitions

- **Definition: If $\boldsymbol{\lambda} = (4, 2, 1)$, $\boldsymbol{\lambda}' = (3, 2, 1, 1)$**





Schur functions. Part Four: Young tableaux

- **Definition: (Semistandard) Young tableaux (SSYT)**

  A Young diagram filled with numbers, strictly increasing by columns, nonstrictly by rows

- **Example: Young tableaux: Two that are and one that is not!**

1	1	2	4
2	2		
3			

1	3	6	7
2	4		
5			

1	2	3	4
6	5		
7			

- **Definition: x^T, where T is a SSYT**

$$x_1^2 x_2^3 x_3 x_4 \qquad x_1 x_2 x_3 x_4 x_5 x_6 x_7$$

Schur functions. Part Five: Definition

$$s_\lambda(x_1, \ldots, x_m) \equiv \sum_T x^T$$

where the summation is over all Young tableaux of shape λ filled with the numbers $1, 2, \ldots, m$

Default case: For $\lambda = (0), s_\lambda \equiv 1$.

- Example 1: $\lambda = (1)$

1	2	3	...	m

$$s_{(1)}(x_1, \ldots, x_m) = x_1 + \cdots + x_m$$

- Example 2: $\lambda = (2)$

i	j

$$s_{(2)}(x_1, \ldots, x_m) = \sum_{i \leq j} x_i x_j$$

Schur functions. Part Five: Definition

- **Example 3:** $\boldsymbol{\lambda = (1,1)}$

$$\begin{array}{|c|}\hline i \\ \hline j \\ \hline \end{array}$$

$$s_{(1,1)}(x_1,\ldots,x_m) = \sum_{i<j} x_i x_j$$

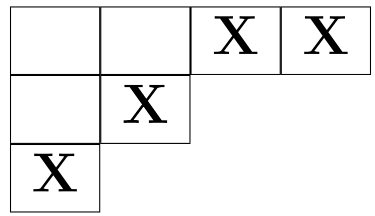
- **Example 4: Schur function,** $\boldsymbol{m = 3, \lambda = (2,1)}$

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1} \end{array}$$

$$s_{(2,1)}(x_1,x_2,x_3) = 2x_1x_2x_3 + x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2$$

Schur functions. Part Six: Recursive formula

- **Key in computing the Schur/Zonal polynomials!**
- **Observe that m can only occur in a "rightmost square."**



The rightmost squares form a "horizontal strip."

- **Therefore**

$$s_{\lambda}(x_1, \ldots, x_m) = \sum_{\mu} s_{\mu}(x_1, \ldots, x_{m-1}) x_m^{|\lambda|-|\mu|}$$

where the summation is over all partitions μ which "fit inside" λ and such that $\lambda - \mu$ is a horizontal strip.

- **Example:**

$$s_{(2)}(x_1, x_2) = s_{(2)}(x_1) + s_{(1)}(x_1)x_2 + s_{(0)}(x_1)x_2^2 = x_1^2 + x_1x_2 + x_2^2$$

Zonal and Jack polynomials

- **That β parameter again!**
- **In combinatorics people use $\alpha = 2/\beta$ instead**
- **Schur $\beta = 2, \alpha = 1$**
- **Zonal $\beta = 1, \alpha = 2$**
- **Jack—general β, α**

Hook lengths

- Definition: For $(i,j) \in \lambda$

$$h_\lambda(i,j) \equiv \lambda'_j - i + \lambda_i - j + 1$$

- Example: For $\lambda = (4,2,1)$, $h_\lambda(1,2) = 4$

	X	X	X
	X		

- In general the hook lengths are

6	4	2	1
3	1		
1			

- α–modifications: Upper and lower hook lengths

$$h^*_\lambda(i,j) \equiv \lambda'_j - i + \alpha(\lambda_i - j + 1)$$
$$h_*^\lambda(i,j) \equiv \lambda'_j - i + 1 + \alpha(\lambda_i - j)$$

Jack function

- Definition: Jack function (for any α; plug in $\alpha = 2$ for the Zonals)

$$J_\kappa(x_1, x_2, \ldots, x_m) = \sum_\mu J_\mu(x_1, x_2, \ldots, x_{m-1}) x_m^{|\kappa/\mu|} \xi_{\kappa\mu},$$

where the summation is over all $\mu \leq \kappa$ such that κ/μ is a horizontal strip, and

$$\xi_{\kappa\mu} \equiv \frac{\prod_{(i,j)\in\kappa} B^\kappa_{\kappa\mu}(i,j)}{\prod_{(i,j)\in\mu} B^\mu_{\kappa\mu}(i,j)}, \quad \text{where} \quad B^\nu_{\kappa\mu}(i,j) \equiv \begin{cases} h^*_\nu(i,j), & \text{if } \kappa'_j = \mu'_j; \\ h^\nu_*(i,j), & \text{otherwise,} \end{cases}$$

(h^*_κ and h^κ_* are the upper and lower hook lengths)

- $\xi_{\kappa\mu}$ = messy-to-describe, yet easy-to-compute rational function of α

- Caveat: Need to be careful about normalization—see our paper with Edelman

References

- **R. Stanley, Some combinatorial properties of Jack symmetric functions, Adv. Math. 77 (1989), no. 1, 76–115.**
- **I. G. Macdonald, Symmetric functions and Hall polynomials, Second ed., Oxford University Press, New York, 1995.**
- **P. Koev, A. Edelman, The Efficient Evaluation of the Hypergeometric Function of a Matrix Argument, Math. Comp. 75 (2006), 833-846.**
- **Ioana Dumitriu's MOPS package**
- **My website: Software for computing (symbolically) Schur, Jack, Zonals, Hypergeometric functions of matrix argument**

`http://math.mit.edu/~plamen`