ACCURATE EIGENVALUES AND SVDs OF TOTALLY POSITIVE MATRICES

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Goals

- Compute Eigenvalues and Singular Values in $O(n^3)$ time to High Relative Accuracy:

$$|\hat{\lambda}_i - \lambda_i| \le O(\epsilon)|\lambda_i|, \qquad |\hat{\sigma}_i - \sigma_i| \le O(\epsilon)\sigma_i,$$

  meaning compute correct sign and leading digits

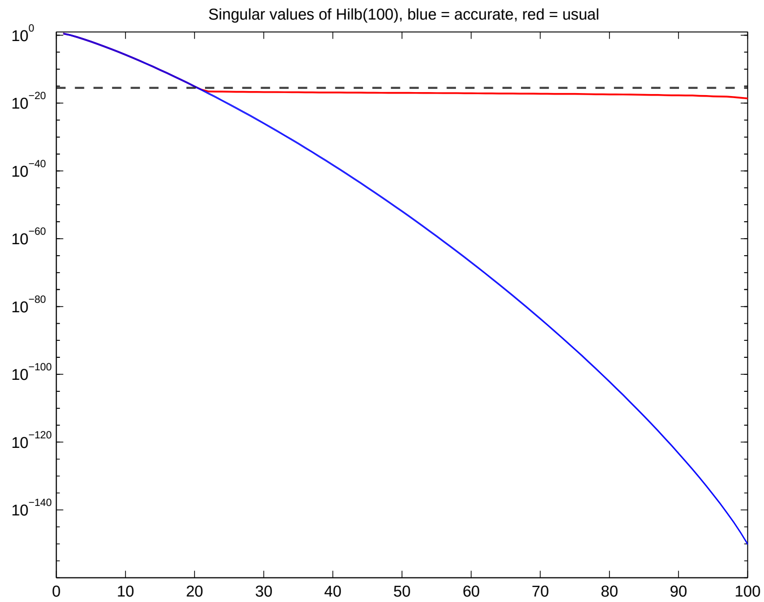
- Traditional algorithms (Symmetric Case)

$$|\hat{\lambda}_i - \lambda_i| \le O(\epsilon)|\lambda_1|, \qquad |\hat{\sigma}_i - \sigma_i| \le O(\epsilon)\sigma_1$$

- Traditional algorithms (Non-Symmetric Case)

$$|\hat{\lambda}_i - \lambda_i| \le O(\epsilon)\frac{||A||}{|y^*x|}, \qquad |\hat{\sigma}_i - \sigma_i| \le O(\epsilon)\sigma_1$$

Example: 100 × 100 Hilbert Matrix $H = 1/(i+j)$



Bidiagonal Matrices

- Demmel–Kahan '91

$$B = \begin{bmatrix} a_1 & b_1 & & & \\ & a_2 & b_2 & & \\ & & \ddots & \ddots & \\ & & & a_{n-1} & b_{n-1} \\ & & & & a_n \end{bmatrix}$$

- a_i, b_i determine σ_i accurately
- dqds preserves the relative accuracy
- Eigenvalues of SPD tridiagonal matrices: $\lambda_i(B^T B) = \sigma_i^2(B)$
  – computable accurately given the Cholesky factors

Outline

- Reduction of a nonsymmetric matrix to tridiagonal form
- Parametrization of Totally Positive Matrices
- Computing with bidiagonals
- Accurate reduction to tridiagonal form
- The SVD

Reduction of a nonsymmetric matrix to tridiagonal form – 1

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ {\color{red}+} & + & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 2

$$\begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & \mathbf{-} & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ {\color{red}+} & {\color{red}+} & {\color{red}+} & {\color{red}+} \\ {\color{red}+} & {\color{red}+} & {\color{red}+} & {\color{red}+} \end{bmatrix}$$

$$= \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 3

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & - & 1 \end{bmatrix} \begin{bmatrix} + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}$$

$$= \begin{bmatrix} + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ 0 & + & \color{red}{+} & \color{red}{+} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 4

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & - & 1 \end{bmatrix} \begin{bmatrix} + & + & {\color{red}+} & {\color{red}+} \\ + & + & {\color{red}+} & {\color{red}+} \\ + & + & {\color{red}+} & {\color{red}+} \\ + & + & {\color{red}+} & {\color{red}+} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}$$

$$= \begin{bmatrix} + & + & {\color{red}+} & {\color{red}+} \\ + & + & {\color{red}+} & {\color{red}+} \\ + & + & {\color{red}+} & {\color{red}+} \\ \mathbf{0} & + & {\color{red}+} & {\color{red}+} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}$$

$$= \begin{bmatrix} + & + & {\color{red}+} & {\color{red}+} \\ + & + & {\color{red}+} & {\color{red}+} \\ + & + & {\color{red}+} & {\color{red}+} \\ \mathbf{0} & + & {\color{red}+} & {\color{red}+} \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 5

$$\begin{bmatrix} + & + & + & \color{red}{+} \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 6

$$\begin{bmatrix} + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ \mathbf{0} & + & \color{red}{+} & \color{red}{+} \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & - \\ & & & \mathbf{1} \end{bmatrix}$$

$$= \begin{bmatrix} + & + & + & \mathbf{0} \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 7

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ {\color{red}+} & {\color{red}+} & {\color{red}+} & {\color{red}+} \\ {\color{red}0} & {\color{red}+} & {\color{red}+} & {\color{red}+} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & - \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & 0 \\ + & + & + & + \\ {\color{red}+} & {\color{red}+} & {\color{red}+} & {\color{red}+} \\ {\color{red}0} & {\color{red}+} & {\color{red}+} & {\color{red}+} \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 8

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ \color{red}{+} & \color{red}{+} & \color{red}{+} & \color{red}{+} \\ \color{red}{\mathbf{0}} & \color{red}{+} & \color{red}{+} & \color{red}{+} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & - \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & \mathbf{0} \\ + & + & + & + \\ \color{red}{+} & \color{red}{+} & \color{red}{+} & \color{red}{+} \\ \color{red}{\mathbf{0}} & \color{red}{+} & \color{red}{+} & \color{red}{+} \end{bmatrix}$$

$$= \begin{bmatrix} + & + & + & \mathbf{0} \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 9

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & - \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & 0 \\ + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \end{bmatrix}$$

$$= \begin{bmatrix} + & + & + & 0 \\ + & + & + & + \\ \color{red}{0} & + & + & + \\ 0 & + & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 10

$$\begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & + \\ & & & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & - \\ & & & \mathbf{1} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & + \\ & & & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & + & \mathbf{0} \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix}$$

$$= \begin{bmatrix} + & + & \color{red}{\mathbf{0}} & \mathbf{0} \\ + & + & + & + \\ \mathbf{0} & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 11

$$\begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & + \\ & & & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & - \\ & & & \mathbf{1} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & + \\ & & & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & + & \mathbf{0} \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix}$$

$$= \begin{bmatrix} + & + & \mathbf{0} & \mathbf{0} \\ + & + & + & + \\ \mathbf{0} & + & + & + \\ \mathbf{0} & \color{red}{\mathbf{0}} & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 12

$$\begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & + \\ & & & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & - \\ & & & \mathbf{1} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & + \\ & & & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & + & \mathbf{0} \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix}$$

$$= \begin{bmatrix} + & + & \mathbf{0} & \mathbf{0} \\ + & + & + & \mathbf{0} \\ \mathbf{0} & + & + & + \\ \mathbf{0} & \mathbf{0} & + & + \end{bmatrix}$$

Totally Positive Matrices

- **Def: Matrices with all minors positive**
- **Appear in many areas of mathematics, engineering, etc.**
- **Are non-symmetric in general**
- **Very often ill conditioned in practice**
- **Real distinct positive eigenvalues**
- **Examples:**
  - **Hilbert $H = 1/(i+j)$**
  - **Vandermonde**

$$V(x) = \begin{bmatrix} 1 & x_1 & \dots & x_1^{n-1} \\ 1 & x_2 & \dots & x_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \dots & x_n^{n-1} \end{bmatrix} \quad \text{for} \quad 0 < x_1 < x_2 < \dots < x_n$$

- **Motivation:**
  - **Eigenvalues and SVD well determined**
  - **Right answer costs $O(n^3)$, same as the wrong**

Total Positivity and Neville Elimination – 1

- **Gaussian Elimination**

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix}$$

- **Gaussian Elimination**

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{4} & \mathbf{8} \\ \mathbf{1} & \mathbf{3} & \mathbf{9} & \mathbf{27} \\ \mathbf{0} & \mathbf{3} & \mathbf{15} & \mathbf{63} \end{bmatrix}$$

- **Gaussian Elimination**

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{4} & \mathbf{8} \\ \mathbf{0} & \mathbf{2} & \mathbf{8} & \mathbf{26} \\ \mathbf{0} & \mathbf{3} & \mathbf{15} & \mathbf{63} \end{bmatrix}$$

- **Gaussian Elimination**

$$\begin{bmatrix} \color{red}{1} & \color{red}{1} & \color{red}{1} & \color{red}{1} \\ 0 & 1 & 3 & 7 \\ 0 & 2 & 8 & 26 \\ 0 & 3 & 15 & 63 \end{bmatrix}$$

- **In Matrix form:**

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 0 & 1 & \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 2 & 8 & 26 \\ 0 & 3 & 15 & 63 \end{bmatrix}$$

- **Neville Elimination**

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{4} & \mathbf{8} \\ \mathbf{1} & \mathbf{3} & \mathbf{9} & \mathbf{27} \\ \mathbf{1} & \mathbf{4} & \mathbf{16} & \mathbf{64} \end{bmatrix}$$

- **Neville Elimination**

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{4} & \mathbf{8} \\ \mathbf{1} & \mathbf{3} & \mathbf{9} & \mathbf{27} \\ \mathbf{0} & \mathbf{1} & \mathbf{7} & \mathbf{37} \end{bmatrix}$$

- **Neville Elimination**

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 0 & 1 & 5 & 19 \\ 0 & 1 & 7 & 37 \end{bmatrix}$$

- **Neville Elimination**

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{3} & \mathbf{7} \\ \mathbf{0} & \mathbf{1} & \mathbf{5} & \mathbf{19} \\ \mathbf{0} & \mathbf{1} & \mathbf{7} & \mathbf{37} \end{bmatrix}$$

Neville Elimination – We can choose the pattern

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 0 & 1 & 7 & 37 \end{bmatrix}$$

Neville Elimination – We can choose the pattern

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & \color{red}{1} & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 0 & \color{red}{1} & \color{red}{5} & \color{red}{19} \\ 0 & \color{red}{1} & \color{red}{7} & \color{red}{37} \end{bmatrix}$$

Neville Elimination – We can choose the pattern

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & {\color{red}1} & 1 \end{bmatrix} \begin{bmatrix} {\color{red}1} & {\color{red}1} & {\color{red}1} & {\color{red}1} \\ {\color{red}1} & {\color{red}2} & {\color{red}4} & {\color{red}8} \\ 0 & 1 & 5 & 19 \\ 0 & 0 & 2 & 18 \end{bmatrix}$$

Neville Elimination – We can choose the pattern

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ \color{red}{1} & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & \color{red}{1} & \color{red}{3} & \color{red}{7} \\ 0 & \color{red}{1} & \color{red}{5} & \color{red}{19} \\ 0 & 0 & 2 & 18 \end{bmatrix}$$

Neville Elimination – We can choose the pattern

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & 1 & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 2 & 12 \\ 0 & 0 & 2 & 18 \end{bmatrix}$$

Neville Elimination – We can choose the pattern

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & 1 & 1 & \\ & & \color{red}{1} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 2 & 12 \\ 0 & 0 & 0 & \color{red}{6} \end{bmatrix}$$

Neville Elimination – We can choose the pattern

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 2 & \\ & & & 6 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Neville Elimination – We can choose the pattern

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 2 & \\ & & & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}$$

Neville Elimination – We can choose the pattern

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 2 & \\ & & & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 & & \\ & 1 & 2 & \\ & & 1 & 3 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & 1 & \\ & & 1 & 2 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}$$

Notice: positive bidiagonal decomposition

$$U_{i,i+1}^{(k)} = \frac{\det(A(1:k, i-k+2:i+1))}{\det(A(1:k-1, i-k+2:i))} \cdot \frac{\det(A(1:k-1, i-k+1:i-1))}{\det(A(1:k, i-k+1:i))}.$$

Total Non-Negativity is preserved in Schur complementation

- **Neville Elimination – Any Totally Positive Matrix**

$$
\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} =
$$

$$
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix}
\begin{bmatrix} 1 & & & \\ + & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix}
\begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{bmatrix}
\begin{bmatrix} 1 & + & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix}
\begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix}
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix}
$$

- **Any minor determined accurately – Cauchy-Binet**
- **Then k-th compound matrix determined accurately**
- **Perron root of k-th compound matrix $\lambda_1 \ldots \lambda_k$ thus λ_i determined accurately**
- **The bidiagonal decomposition will be the starting point of our eigenvalue computation**

How do we lose accuracy in floating point?

- $\mathrm{fl}(a \otimes b) = (a \otimes b)(1 + \delta)$, no under/overflow
- How can we lose accuracy in computing in floating point?
  - OK to multiply, divide, add positive numbers
    Proof: $1 + \delta$ factors can be factored out
  - $x_i \pm x_j$, where x_i and x_j are initial data (so exact)
  - Cancellation when subtracting approximate results dangerous:

```
   .12345xxx
-  .12345yyy
------------
   .00000zzz
```

- We will compute everything using only allowable expressions

“Commuting” of positive bidiagonals

- Assume $x, y, z > 0$, $b_i \geq 0$. Then

$$\begin{bmatrix} 1 & b_1 & & & \\ & 1 & b_2 & & \\ & & \ddots & \ddots & \\ & & & 1 & b_{n-1} \\ & & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & y & & & \\ & & x & z & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & y' & & & \\ & & x' & z' & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} \begin{bmatrix} 1 & b'_1 & & & \\ & 1 & b'_2 & & \\ & & \ddots & \ddots & \\ & & & 1 & b'_{n-1} \\ & & & & 1 \end{bmatrix}$$

where

$$\begin{aligned} x' &= x \\ y' &= y + b_i x \\ z' &= 1/y' \\ b'_{i-1} &= b_{i-1} y \\ b'_i &= b_i z / y_1 \end{aligned}$$

- No loss of relative accuracy
- “Commute accurately”.

“Commuting” of positive bidiagonals

- Assume $x > 0$, $b_i \geq 0$. Then

$$\begin{bmatrix} 1 & & & & \\ b_1 & 1 & & & \\ & \ddots & \ddots & & \\ & & b_{n-2} & 1 & \\ & & & b_{n-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & x & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & x' & 1 & \\ & & & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & & \\ b'_1 & 1 & & & \\ & \ddots & \ddots & & \\ & & b'_{n-2} & 1 & \\ & & & b'_{n-1} & 1 \end{bmatrix}$$

where

$$\begin{aligned} x' &= x b_{i+1}/(b_i + x) \\ b'_i &= b_i + x \\ b'_{i+1} &= x b_i / b'_i \end{aligned}$$

- No loss of relative accuracy

Applying similarity to bidiagonal form – 1

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & - & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & - & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ + & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{bmatrix} \begin{bmatrix} 1 & + & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}$$

Applying similarity to bidiagonal form – 2

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & - & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ + & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{bmatrix} \begin{bmatrix} 1 & + & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}$$

Applying similarity to bidiagonal form – 3

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & - & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ + & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{bmatrix} \begin{bmatrix} 1 & + & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & y & \\ & & + & z \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix}$$

Applying similarity to bidiagonal form – 4

$$\begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & - & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & + & \mathbf{1} \end{bmatrix} =$$

$$\begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & + & \mathbf{1} & \\ & & + & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ + & \mathbf{1} & & \\ & + & \mathbf{1} & \\ & & + & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{bmatrix} \begin{bmatrix} \mathbf{1} & + & & \\ & \mathbf{1} & + & \\ & & \mathbf{1} & + \\ & & & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \boldsymbol{y'} & \\ & & + & \boldsymbol{z'} \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & + & \\ & & \mathbf{1} & + \\ & & & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & + \\ & & & \mathbf{1} \end{bmatrix}$$

Applying similarity to bidiagonal form – 5

$$\begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & - & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & + & \mathbf{1} \end{bmatrix} =$$

$$\begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & + & \mathbf{1} & \\ & & + & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ + & \mathbf{1} & & \\ & + & \mathbf{1} & \\ & & + & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \boldsymbol{y''} & \\ & & + & \boldsymbol{z''} \end{bmatrix} \begin{bmatrix} \mathbf{1} & + & & \\ & \mathbf{1} & + & \\ & & \mathbf{1} & + \\ & & & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & + & \\ & & \mathbf{1} & + \\ & & & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & + \\ & & & \mathbf{1} \end{bmatrix}$$

Applying similarity to bidiagonal form – 6

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & - & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ + & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{bmatrix} \begin{bmatrix} 1 & + & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix}$$

Applying similarity to bidiagonal form – 7

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & - & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ + & 1 & & \\ & + & 1 & \\ & & {\color{red}+} & 1 \end{bmatrix} \begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{bmatrix} \begin{bmatrix} 1 & + & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix}$$

Applying similarity to bidiagonal form – 8

$$\begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & + \\ & & & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & - \\ & & & \mathbf{1} \end{bmatrix} =$$

Applying similarity to bidiagonal form – 9

$$\begin{bmatrix} + & + & 0 & 0 \\ + & + & + & 0 \\ 0 & + & + & + \\ 0 & 0 & + & + \end{bmatrix} \qquad =$$

$$\begin{bmatrix} 1 & & & \\ + & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{bmatrix} \begin{bmatrix} 1 & + & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix}$$

The eigenvalues of

$$\begin{bmatrix} 1 & & & & \\ b_1 & 1 & & & \\ & \ddots & \ddots & & \\ & & b_{n-2} & 1 & \\ & & & b_{n-1} & 1 \end{bmatrix} \begin{bmatrix} d_1 & & & & \\ & d_2 & & & \\ & & \ddots & & \\ & & & d_{n-1} & \\ & & & & d_n \end{bmatrix} \begin{bmatrix} 1 & c_1 & & & \\ & 1 & c_2 & & \\ & & \ddots & \ddots & \\ & & & 1 & c_{n-1} \\ & & & & 1 \end{bmatrix}$$

are the squares of the singular values of

$$\begin{bmatrix} \sqrt{d_1} & \sqrt{d_1 b_1 c_1} & & & \\ & \sqrt{d_2} & \sqrt{d_2 b_2 c_2} & & \\ & & \ddots & \ddots & \\ & & & \sqrt{d_{n-1}} & \sqrt{d_{n-1} b_{n-1} c_{n-1}} \\ & & & & \sqrt{d_n} \end{bmatrix}$$

- Thus the eigenvalue problem of a TP matrix is reduced to the SVD problem for a bidiagonal matrix
- Forward stable
- Total cost $= 8n^3$.

The SVD

- **First a matrix is reduced to a bidiagonal by Givens rotations**
- **Suffices to show how to apply one Givens rotation accurately**

$$\begin{bmatrix} c & s \\ -s & c \end{bmatrix} = \begin{bmatrix} + & + \\ 0 & + \end{bmatrix} \begin{bmatrix} 1 & 0 \\ - & 1 \end{bmatrix}$$

Applying a Givens Rotation

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & c & s \\ & & -s & c \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & + & + \\ & & & + \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & - & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & + & + \\ & & & + \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & - & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ + & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{bmatrix} \begin{bmatrix} 1 & + & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & + & + \\ & & & + \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ + & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{bmatrix} \begin{bmatrix} 1 & + & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix}$$

Computing Accurate Bidiagonal Decompositions

- **Bidiagonal decompositions intrinsic for all TP matrices**
- **Often accurate formulas**
- **Equivalent to Björck-Pereyra Methods**
- **Can be computed iff initial minors can be computed accurately**

$$U_{i,i+1}^{(k)} = \frac{\det(A(1:k, i-k+2:i+1))}{\det(A(1:k-1, i-k+2:i))} \cdot \frac{\det(A(1:k-1, i-k+1:i-1))}{\det(A(1:k, i-k+1:i))}.$$

initial minors = contiguous, include the first row or column

- **Accurate for Vandermonde, Cauchy, Cauchy-Vandermonde and Generalized Vandermonde Matrices**

Conclusions

- $O(n^3)$ algorithms for the eigenvalues and the SVD of Totally Positive Matrices
- Computes all eigenvalues and singular values to high relative accuracy
- Applies to:
  - Oscillatory
  - Totally Non-Negative Matrices
  - Sign Regular (Inverses of TP matrices)
- This talk and Matlab software: `math.mit.edu/~plamen`