Accurate Computations with Totally Positive Matrices

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Supported by NSF

LAPACK Seminar, University of California, Berkeley, September 21, 2005

What is Total Positivity?

$$A \text{ is TP} \iff \text{All minors} > 0$$

Applications Abound

- **Vibrations of mechanical systems (Gantmacher, Krein)**
- **Corner cutting algorithms (Cutting corners from polytopes)**
- **Electrical Impedance Tomography (Y. Chen)**
- **Stochastic analysis**

- **Books by Karlin, Gantmacher–Krein, Gasca–Micchelli**
- **Motivated by work of Whitney, Björck, Pereyra, Higham, Gasca, Peña, Kahan, C. Johnson, Fallat,...**

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 4 & 4^2 & 4^3 \end{bmatrix}$$

Vandermonde

$$\begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/3 & 1/4 & 1/5 \\ 1/3 & 1/4 & 1/5 & 1/6 \\ 1/4 & 1/5 & 1/6 & 1/7 \end{bmatrix}$$

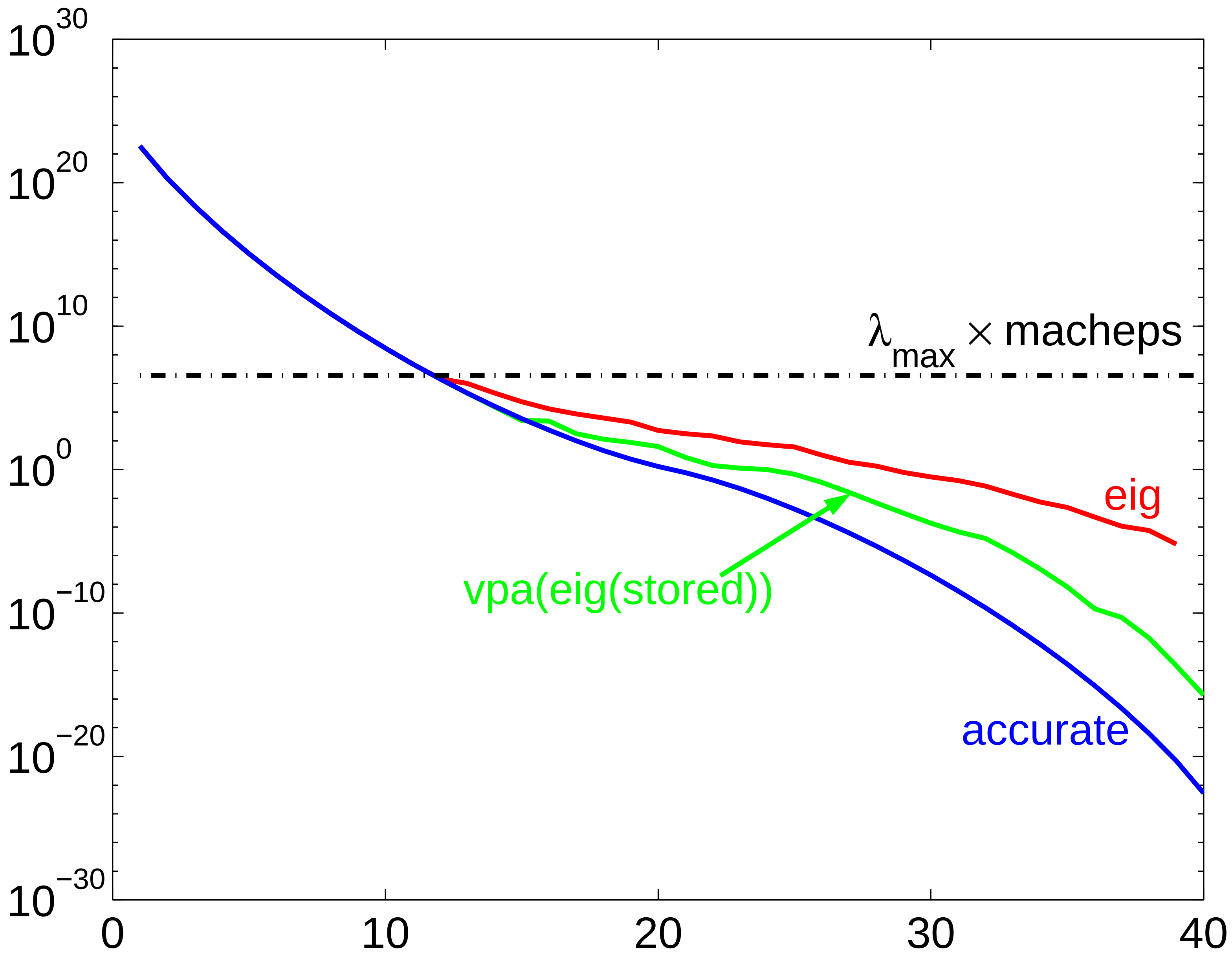
Hilbert (Cauchy)

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$$

Pascal

Notoriously illconditioned $\Rightarrow$ conventional algorithms fail

Computed Eigenvalues of Pascal(40)



Properties of TP matrices

- **Eigenvalues are real, distinct, positive**
- **TP preserved by**
  - **multiplication**
  - **Schur complementation**
  - **Taking and inverse and resigning ($|A^{-1}|$)**
  - **Taking a *converse***
  - **Taking a submatrix**
  - R **in** $A = QR$ **is TP**
  - **One step of QR iteration, if** A **is symmetric**

GOALS

- **Accurate computations within the class of TP matrices**
- **In particular:**
  - λ_i, σ_i, entries of A^{-1}, LDU, to high relative accuracy

$$|\hat{\lambda}_i - \lambda_i| \le O(\epsilon)|\lambda_i| \qquad |\hat{\sigma}_i - \sigma_i| \le O(\epsilon)\sigma_i$$

  - Contrast: Conventional abs error bounds:

$$|\hat{\lambda}_i - \lambda_i| \le O(\epsilon)\frac{||A||}{|y^* x|} \qquad |\hat{\sigma}_i - \sigma_i| \le O(\epsilon)\sigma_1$$

- **More complicated computations, e.g.,**

$$\lambda_i(\text{SchurComplement}(\text{Hilbert} \times \text{Pascal}))$$

Main Result

Virtually all TP linear algebra possible accurately

First Issue: Proper Representation

- Matrix entries are a bad choice of parameters:
  - ϵ perturbation in $(2,2)$ entry of the TP matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 1+\varepsilon \end{bmatrix} = \begin{bmatrix} 1 & \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & \\ & \varepsilon \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1+2\varepsilon \end{bmatrix} = \begin{bmatrix} 1 & \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & \\ & 2\varepsilon \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}$$

  - Leads to 100% relative perturbation in $\lambda_{\min}$
- How do we know if a matrix is TP?
  Clue: Factorizations: If Cholesky is real $\Rightarrow$ s.p.d.
- If Neville elimination succeeds with positive pivots $\Rightarrow$ TP

Neville Elimination – eliminate using adjacent rows

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix}$$

Neville Elimination

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 0 & 1 & 7 & 37 \end{bmatrix}$$

Neville Elimination

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 0 & 1 & 5 & 19 \\ 0 & 1 & 7 & 37 \end{bmatrix}$$

Neville Elimination

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & \color{red}{1} & 1 \end{bmatrix} \begin{bmatrix} \color{red}{1} & \color{red}{1} & \color{red}{1} & \color{red}{1} \\ \color{red}{1} & \color{red}{2} & \color{red}{4} & \color{red}{8} \\ 0 & 1 & 5 & 19 \\ 0 & 0 & 2 & 18 \end{bmatrix}$$

Neville Elimination – We can choose the pattern

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ \color{red}{1} & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & \color{red}{1} & \color{red}{3} & \color{red}{7} \\ 0 & \color{red}{1} & \color{red}{5} & \color{red}{19} \\ 0 & 0 & 2 & 18 \end{bmatrix}$$

Neville Elimination

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & 1 & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 2 & 12 \\ 0 & 0 & 2 & 18 \end{bmatrix}$$

Neville Elimination

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & 1 & 1 & \\ & & {\color{red}1} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 2 & 12 \\ 0 & 0 & 0 & {\color{red}6} \end{bmatrix}$$

Neville Elimination

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 2 & \\ & & & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Neville Elimination

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 2 & \\ & & & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}$$

Neville Elimination

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 2 & \\ & & & 6 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 1 & & \\ & 1 & 2 & \\ & & 1 & 3 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & 1 & \\ & & 1 & 2 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}}_{U}$$

Thm: A is TP $\Leftrightarrow$ Red entries > 0

Pascal matrices are all of a sudden just ones

$$\mathbf{PASCAL(4)} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix}}_{\boldsymbol{L}} \underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}}_{\boldsymbol{D}} \underbrace{\begin{bmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & 1 & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}}_{\boldsymbol{U}}$$

Any TP = Product of Nonnegative Bidiagonals

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & l_{41} & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & l_{31} & 1 & \\ & & l_{42} & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ l_{21} & 1 & & \\ & l_{32} & 1 & \\ & & l_{43} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & d_4 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & u_{12} & & \\ & 1 & u_{23} & \\ & & 1 & u_{34} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & u_{13} & \\ & & 1 & u_{24} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & u_{14} \\ & & & 1 \end{bmatrix}}_{U}$$

- Theorem: A is TP $\iff l_{ij}, d_i, u_{ij} > 0$
- This representation reveals the TP structure
- Thm (K.): $\mathcal{BD}(A)$ determines $\lambda_i, \sigma_i, A^{-1}_{ij}, \ldots$ accurately
- Red entries $= \frac{\det A(1{:}k, i-k+2{:}i+1)}{\det A(1{:}k-1, i-k+2{:}i)} \cdot \frac{\det A(1{:}k-1, i-k+1{:}i-1)}{\det A(1{:}k, i-k+1{:}i)}$

  $\Rightarrow$ Formulas

Accurate $\mathcal{BD}$ of Vandermonde and Cauchy

- $V = \left[x_i^{j-1}\right]_{i,j=1}^n$ (TP if $0 < x_1 < x_2 < \cdots < x_n$)

$$D_{ii} = \prod_{j=1}^{i-1}(x_i - x_j), \qquad L_{i+1,i}^{(k)} = \prod_{j=n-k}^{i-1} \frac{x_{i+1} - x_{j+1}}{x_i - x_j}, \qquad U_{i,i+1}^{(k)} = x_{i+n-k}$$

- $C = \left[\frac{1}{x_i + y_j}\right]_{i,j=1}^n$ (TP if $0 < x_1 < \cdots < x_n$, $0 < y_1 < \cdots < y_n$)

$$\begin{aligned} D_{ii} &= \prod_{k=1}^{i-1} \frac{(x_i - x_k)(y_i - y_k)}{(x_i + y_k)(y_i + x_k)} \\ L_{i,i+1}^{(k)} &= \frac{x_{n-k} + y_{i-n+k+1}}{x_i + y_{i-n+k+1}} \prod_{l=n-k}^{i-1} \frac{x_{i+1} - x_{l+1}}{x_i - x_l} \cdot \prod_{l=1}^{i-n+k-1} \frac{x_i + y_l}{x_{i+1} + y_l} \\ U_{i+1,i}^{(k)} &= \frac{y_{n-k} + x_{i-n+k+1}}{y_i + x_{i-n+k+1}} \prod_{l=n-k}^{i-1} \frac{y_{i+1} - y_{l+1}}{y_i - y_l} \cdot \prod_{l=1}^{i-n+k-1} \frac{y_i + x_l}{y_{i+1} + x_l} \end{aligned}$$

- Similar formulas for Cauchy–Vandermonde, confluent (José Javier Martínez)
- No subtractive cancellation $\Rightarrow$ accurate

Representation of TP Matrices

Every TP matrix A represented by the entries of $\mathcal{BD}(A)$

Idea in Accurate Computations

- TP matrices represented by $\mathcal{BD}$, not entries
- Work on $\mathcal{BD}(A)$, not on A
- Use only *Elementary Elimination Transformations (EETs)*:
  1. Positive diagonal scaling
  2. Subtract a row from next to make a zero;
  3. Add a positive multiple of row to next/previous;
- In matrix form

$$\text{EET} = \begin{bmatrix} + & & & \\ & + & & \\ & x & + & \\ & & & + \end{bmatrix}$$

- EETs preserve TN and every operation that preserves total non-negativity = sequence of EETs.
- Applying an EET to $\mathcal{BD}(A)$ accurate, since it involves no subtractions!

Why Can Subtractions be Bad?

- if $\hat{a} \approx a > 0$ and $\hat{b} \approx b > 0$ to (say) 9 digits, then

$$\left.\begin{array}{rcl} \hat{a} \cdot \hat{b} & \approx & a \cdot b \\ \hat{a} + \hat{b} & \approx & a + b \\ \hat{a} / \hat{b} & \approx & a / b \end{array}\right\} \quad \text{to about 9 digits}$$

BUT

$$\hat{a} - \hat{b}$$

may have no correct digits if $a \approx b$, e.g.,

$$\begin{array}{rl} & .123456789xxx \\ - & .123456789yyy \\ \hline & .000000000zzz \end{array}$$

- Some subtractions are OK, e.g.,

$$x_i - x_j$$

where x_i are initial data.

- **Subtracting a row from next to make a zero**

$$\begin{bmatrix} 1 & 2 & 6 \\ 3 & 10 & 50 \\ 21 & 102 & 615 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & 7 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 3 & 1 & \\ & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 4 & \\ & & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 & \\ & 1 & 5 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & 3 \\ & & 1 \end{bmatrix}$$

Applying EETs on $\mathcal{BD}(A)$ – II

- **Subtracting a row from next to make a zero**

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & -7 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 6 \\ 3 & 10 & 50 \\ 21 & 102 & 615 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & -7 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & 7 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 3 & 1 & \\ & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 4 & \\ & & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 & \\ & 1 & 5 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & 3 \\ & & 1 \end{bmatrix}$$

Applying EETs on $\mathcal{BD}(A)$ – III

- **Subtracting a row from next to make a zero**

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & -7 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 6 \\ 3 & 10 & 50 \\ 21 & 102 & 615 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & -7 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & 7 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 3 & 1 & \\ & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 4 & \\ & & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 & \\ & 1 & 5 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & 3 \\ & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 3 & 1 & \\ & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 4 & \\ & & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 & \\ & 1 & 5 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & 3 \\ & & 1 \end{bmatrix}$$

- **Is equivalent to setting an entry of $\mathcal{BD}(A)$ to zero and performing *no arithmetic*.**

Adding a multiple of a column to previous – I

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & a & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ b & 1 & \\ & c & 1 \end{bmatrix} \begin{bmatrix} d & & \\ & e & \\ & & f \end{bmatrix} \begin{bmatrix} 1 & g & \\ & 1 & h \\ & & 1 \end{bmatrix} \underline{\begin{bmatrix} 1 & & \\ & 1 & k \\ & & 1 \end{bmatrix} \color{red}{\begin{bmatrix} 1 & & \\ & y & \\ & x & z \end{bmatrix}}}$$

Adding a multiple of a column to previous – II

$$\begin{bmatrix}1&&\\&1&\\&a&1\end{bmatrix}\begin{bmatrix}1&&\\b&1&\\&c&1\end{bmatrix}\begin{bmatrix}d&&\\&e&\\&&f\end{bmatrix}\begin{bmatrix}1&g&\\&1&h\\&&1\end{bmatrix}\underline{\begin{bmatrix}1&&\\&1&k\\&&1\end{bmatrix}\color{red}{\begin{bmatrix}1&&\\&y&\\&x&z\end{bmatrix}}}$$

$$=\begin{bmatrix}1&&\\&1&\\&a&1\end{bmatrix}\begin{bmatrix}1&&\\b&1&\\&c&1\end{bmatrix}\begin{bmatrix}d&&\\&e&\\&&f\end{bmatrix}\begin{bmatrix}1&g&\\&1&h\\&&1\end{bmatrix}\underline{\color{red}{\begin{bmatrix}1&&\\&y'&\\&x'&z'\end{bmatrix}}\begin{bmatrix}1&&\\&1&\color{blue}{k'}\\&&1\end{bmatrix}}$$

$$\begin{aligned}x' &= x\\ y' &= y+kx\\ z' &= 1/y'\\ k' &= kz/y_1\end{aligned}$$

... it's all qd recurrences

Adding a multiple of a column to previous – III

$$\begin{bmatrix}1&&\\&1&\\&a&1\end{bmatrix}\begin{bmatrix}1&&\\b&1&\\&c&1\end{bmatrix}\begin{bmatrix}d&&\\&e&\\&&f\end{bmatrix}\begin{bmatrix}1&g&\\&1&h\\&&1\end{bmatrix}\begin{bmatrix}1&&\\&1&k\\&&1\end{bmatrix}\color{red}{\begin{bmatrix}1&&\\&y&\\&x&z\end{bmatrix}}$$

$$=\begin{bmatrix}1&&\\&1&\\&a&1\end{bmatrix}\begin{bmatrix}1&&\\b&1&\\&c&1\end{bmatrix}\begin{bmatrix}d&&\\&e&\\&&f\end{bmatrix}\underline{\begin{bmatrix}1&g&\\&1&h\\&&1\end{bmatrix}\color{red}{\begin{bmatrix}1&&\\&y'&\\&x'&z'\end{bmatrix}}}\begin{bmatrix}1&&\\&1&\color{blue}{k'}\\&&1\end{bmatrix}$$

$$=\begin{bmatrix}1&&\\&1&\\&a&1\end{bmatrix}\begin{bmatrix}1&&\\b&1&\\&c&1\end{bmatrix}\begin{bmatrix}d&&\\&e&\\&&f\end{bmatrix}\underline{\color{red}{\begin{bmatrix}1&&\\&y''&\\&x''&z''\end{bmatrix}}\begin{bmatrix}1&\color{blue}{g'}&\\&1&\color{blue}{h'}\\&&1\end{bmatrix}}\begin{bmatrix}1&&\\&1&\color{blue}{k'}\\&&1\end{bmatrix}$$

Adding a multiple of a column to previous – IV

$$\begin{bmatrix}1&&\\&1&\\&a&1\end{bmatrix}\begin{bmatrix}1&&\\b&1&\\&c&1\end{bmatrix}\begin{bmatrix}d&&\\&e&\\&&f\end{bmatrix}\begin{bmatrix}1&g&\\&1&h\\&&1\end{bmatrix}\begin{bmatrix}1&&\\&1&k\\&&1\end{bmatrix}\color{red}{\begin{bmatrix}1&&\\&y&\\&x&z\end{bmatrix}}$$

$$=\begin{bmatrix}1&&\\&1&\\&a&1\end{bmatrix}\begin{bmatrix}1&&\\b&1&\\&c&1\end{bmatrix}\begin{bmatrix}d&&\\&e&\\&&f\end{bmatrix}\begin{bmatrix}1&g&\\&1&h\\&&1\end{bmatrix}\color{red}{\begin{bmatrix}1&&\\&y'&\\&x'&z'\end{bmatrix}}\begin{bmatrix}1&&\\&1&\color{blue}{k'}\\&&1\end{bmatrix}$$

$$=\begin{bmatrix}1&&\\&1&\\&a&1\end{bmatrix}\begin{bmatrix}1&&\\b&1&\\&c&1\end{bmatrix}\underline{\begin{bmatrix}d&&\\&e&\\&&f\end{bmatrix}\color{red}{\begin{bmatrix}1&&\\&y''&\\&x''&z''\end{bmatrix}}}\begin{bmatrix}1&\color{blue}{g'}&\\&1&\color{blue}{h'}\\&&1\end{bmatrix}\begin{bmatrix}1&&\\&1&\color{blue}{k'}\\&&1\end{bmatrix}$$

$$=\begin{bmatrix}1&&\\&1&\\&a&1\end{bmatrix}\begin{bmatrix}1&&\\b&1&\\&c&1\end{bmatrix}\underline{\color{red}{\begin{bmatrix}1&&\\&1&\\&x'''&1\end{bmatrix}}\begin{bmatrix}d&&\\&\color{blue}{e'}&\\&&\color{blue}{f'}\end{bmatrix}}\begin{bmatrix}1&\color{blue}{g'}&\\&1&\color{blue}{h'}\\&&1\end{bmatrix}\begin{bmatrix}1&&\\&1&\color{blue}{k'}\\&&1\end{bmatrix}$$

Adding a multiple of a column to previous – V

$$\begin{bmatrix}1&&\\&1&\\&a&1\end{bmatrix}\begin{bmatrix}1&&\\b&1&\\&c&1\end{bmatrix}\begin{bmatrix}d&&\\&e&\\&&f\end{bmatrix}\begin{bmatrix}1&g&\\&1&h\\&&1\end{bmatrix}\begin{bmatrix}1&&\\&1&k\\&&1\end{bmatrix}\begin{bmatrix}1&&\\&y&\\&x&z\end{bmatrix}$$

$$=\begin{bmatrix}1&&\\&1&\\&a&1\end{bmatrix}\begin{bmatrix}1&&\\b&1&\\&c&1\end{bmatrix}\begin{bmatrix}d&&\\&e&\\&&f\end{bmatrix}\begin{bmatrix}1&g&\\&1&h\\&&1\end{bmatrix}\begin{bmatrix}1&&\\&y'&\\&x'&z'\end{bmatrix}\begin{bmatrix}1&&\\&1&k'\\&&1\end{bmatrix}$$

$$=\begin{bmatrix}1&&\\&1&\\&a&1\end{bmatrix}\begin{bmatrix}1&&\\b&1&\\&c&1\end{bmatrix}\begin{bmatrix}d&&\\&e&\\&&f\end{bmatrix}\begin{bmatrix}1&&\\&y''&\\&x''&z''\end{bmatrix}\begin{bmatrix}1&g'&\\&1&h'\\&&1\end{bmatrix}\begin{bmatrix}1&&\\&1&k'\\&&1\end{bmatrix}$$

$$=\begin{bmatrix}1&&\\&1&\\&a&1\end{bmatrix}\underline{\begin{bmatrix}1&&\\b&1&\\&c&1\end{bmatrix}\begin{bmatrix}1&&\\&1&\\&x'''&1\end{bmatrix}}\begin{bmatrix}d&&\\&e'&\\&&f'\end{bmatrix}\begin{bmatrix}1&g'&\\&1&h'\\&&1\end{bmatrix}\begin{bmatrix}1&&\\&1&k'\\&&1\end{bmatrix}$$

$$=\begin{bmatrix}1&&\\&1&\\&a&1\end{bmatrix}\underline{\begin{bmatrix}1&&\\b&1&\\&c+x'''&1\end{bmatrix}\begin{bmatrix}d&&\\&e'&\\&&f'\end{bmatrix}}\begin{bmatrix}1&g'&\\&1&h'\\&&1\end{bmatrix}\begin{bmatrix}1&&\\&1&k'\\&&1\end{bmatrix}$$

Done.

The Story so Far

- **TP matrices represented as products of bidiagonals**
- **We can perform EETs without subtractions $\Rightarrow$ accuracy**
- **Remains to see that all TP linear algebra is a sequence of EETs**

Idea in Accurate Eigenvalue Computations

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & l_{41} & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & l_{31} & 1 & \\ & & l_{42} & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ l_{21} & 1 & & \\ & l_{32} & 1 & \\ & & l_{43} & 1 \end{bmatrix}\begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & d_4 \end{bmatrix}\begin{bmatrix} 1 & u_{12} & & \\ & 1 & u_{23} & \\ & & 1 & u_{34} \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & u_{13} & \\ & & 1 & u_{24} \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & u_{14} \\ & & & 1 \end{bmatrix}$$

$\downarrow$ **(Well known EET similarity)**

$$\begin{bmatrix} 1 & & & \\ l_1 & 1 & & \\ & l_2 & 1 & \\ & & l_3 & 1 \end{bmatrix}\begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & d_4 \end{bmatrix}\begin{bmatrix} 1 & l_1 & & \\ & 1 & l_2 & \\ & & 1 & l_3 \\ & & & 1 \end{bmatrix}$$

$\downarrow$

$$\mathbf{SVD}\left(\begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & d_4 \end{bmatrix}^{\frac{1}{2}}\begin{bmatrix} 1 & l_1 & & \\ & 1 & l_2 & \\ & & 1 & l_3 \\ & & & 1 \end{bmatrix}\right)$$

Recall: Reduction of a nonsymmetric matrix to tridiagonal form

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \color{red}{+} & + & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 2

$$\begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & - & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ \color{red}{+} & \color{red}{+} & \color{red}{+} & \color{red}{+} \\ \color{red}{+} & \color{red}{+} & \color{red}{+} & \color{red}{+} \end{bmatrix}$$

$$= \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 3

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & - & 1 \end{bmatrix} \begin{bmatrix} + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}$$

$$= \begin{bmatrix} + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ \mathbf{0} & + & \color{red}{+} & \color{red}{+} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 4

$$\begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & - & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & + & \mathbf{1} \end{bmatrix}$$

$$= \begin{bmatrix} + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ \mathbf{0} & + & \color{red}{+} & \color{red}{+} \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & + & \mathbf{1} \end{bmatrix}$$

$$= \begin{bmatrix} + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ \mathbf{0} & + & \color{red}{+} & \color{red}{+} \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 5

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ {\color{red}+} & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 6

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & - & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \end{bmatrix}$$

$$= \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 7

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & - & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ \color{red}{+} & \color{red}{+} & \color{red}{+} & \color{red}{+} \\ \color{red}{+} & \color{red}{+} & \color{red}{+} & \color{red}{+} \\ 0 & + & + & + \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} + & \color{red}{+} & \color{red}{+} & + \\ + & \color{red}{+} & \color{red}{+} & + \\ 0 & \color{red}{+} & \color{red}{+} & + \\ 0 & \color{red}{+} & \color{red}{+} & + \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & & 1 \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 8

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & - & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 9

$$\begin{bmatrix} + & + & + & \color{red}{+} \\ + & + & + & + \\ \mathbf{0} & + & + & + \\ \mathbf{0} & \mathbf{0} & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 10

$$\begin{bmatrix} + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ \mathbf{0} & + & \color{red}{+} & \color{red}{+} \\ \mathbf{0} & \mathbf{0} & \color{red}{+} & \color{red}{+} \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & - \\ & & & \mathbf{1} \end{bmatrix}$$

$$= \begin{bmatrix} + & + & + & \color{red}{\mathbf{0}} \\ + & + & + & + \\ \mathbf{0} & + & + & + \\ \mathbf{0} & \mathbf{0} & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 11

$$\begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & + \\ & & & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ \mathbf{0} & + & \color{red}{+} & \color{red}{+} \\ \mathbf{0} & \mathbf{0} & \color{red}{+} & \color{red}{+} \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & - \\ & & & \mathbf{1} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & + \\ & & & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & + & \mathbf{0} \\ + & + & + & + \\ \color{red}{\mathbf{0}} & \color{red}{+} & \color{red}{+} & \color{red}{+} \\ \color{red}{\mathbf{0}} & \color{red}{\mathbf{0}} & \color{red}{+} & \color{red}{+} \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 12

$$\begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & + \\ & & & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & {\color{red}+} & {\color{red}+} \\ + & + & {\color{red}+} & {\color{red}+} \\ \mathbf{0} & + & {\color{red}+} & {\color{red}+} \\ \mathbf{0} & \mathbf{0} & {\color{red}+} & {\color{red}+} \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & - \\ & & & \mathbf{1} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & + \\ & & & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & + & \mathbf{0} \\ + & + & + & + \\ {\color{red}\mathbf{0}} & {\color{red}+} & {\color{red}+} & {\color{red}+} \\ {\color{red}\mathbf{0}} & {\color{red}\mathbf{0}} & {\color{red}+} & {\color{red}+} \end{bmatrix}$$

$$= \begin{bmatrix} + & + & + & \mathbf{0} \\ + & + & + & + \\ \mathbf{0} & + & + & + \\ \mathbf{0} & \mathbf{0} & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 13

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \color{red}{+} & + & + & + \end{bmatrix} \Rightarrow \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ \color{red}{+} & + & + & + \\ \color{blue}{\mathbf{0}} & + & + & + \end{bmatrix} \Rightarrow \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ \color{blue}{\mathbf{0}} & + & + & + \\ \color{blue}{\mathbf{0}} & \color{red}{+} & + & + \end{bmatrix} \Rightarrow \begin{bmatrix} + & + & + & \color{red}{+} \\ + & + & + & + \\ \color{blue}{\mathbf{0}} & + & + & + \\ \color{blue}{\mathbf{0}} & \color{blue}{\mathbf{0}} & + & + \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} + & + & \color{red}{+} & \color{blue}{\mathbf{0}} \\ + & + & + & + \\ \color{blue}{\mathbf{0}} & + & + & + \\ \color{blue}{\mathbf{0}} & \color{blue}{\mathbf{0}} & + & + \end{bmatrix} \Rightarrow \begin{bmatrix} + & + & \color{blue}{\mathbf{0}} & \color{blue}{\mathbf{0}} \\ + & + & + & \color{red}{+} \\ \color{blue}{\mathbf{0}} & + & + & + \\ \color{blue}{\mathbf{0}} & \color{blue}{\mathbf{0}} & + & + \end{bmatrix} \Rightarrow \begin{bmatrix} + & + & \color{blue}{\mathbf{0}} & \color{blue}{\mathbf{0}} \\ + & + & + & \color{blue}{\mathbf{0}} \\ \color{blue}{\mathbf{0}} & + & + & + \\ \color{blue}{\mathbf{0}} & \color{blue}{\mathbf{0}} & + & + \end{bmatrix}$$

- **EETs performed implicitly $\Rightarrow$ high relative accurcy**
- **The first example of a *nonsymmetric* eigenvalue algorithm delivering high relative accuracy**
- **Cost: $7n^3$**

The SVD, QR Decompositions

- **Givens rotation = product of 3 EETs**

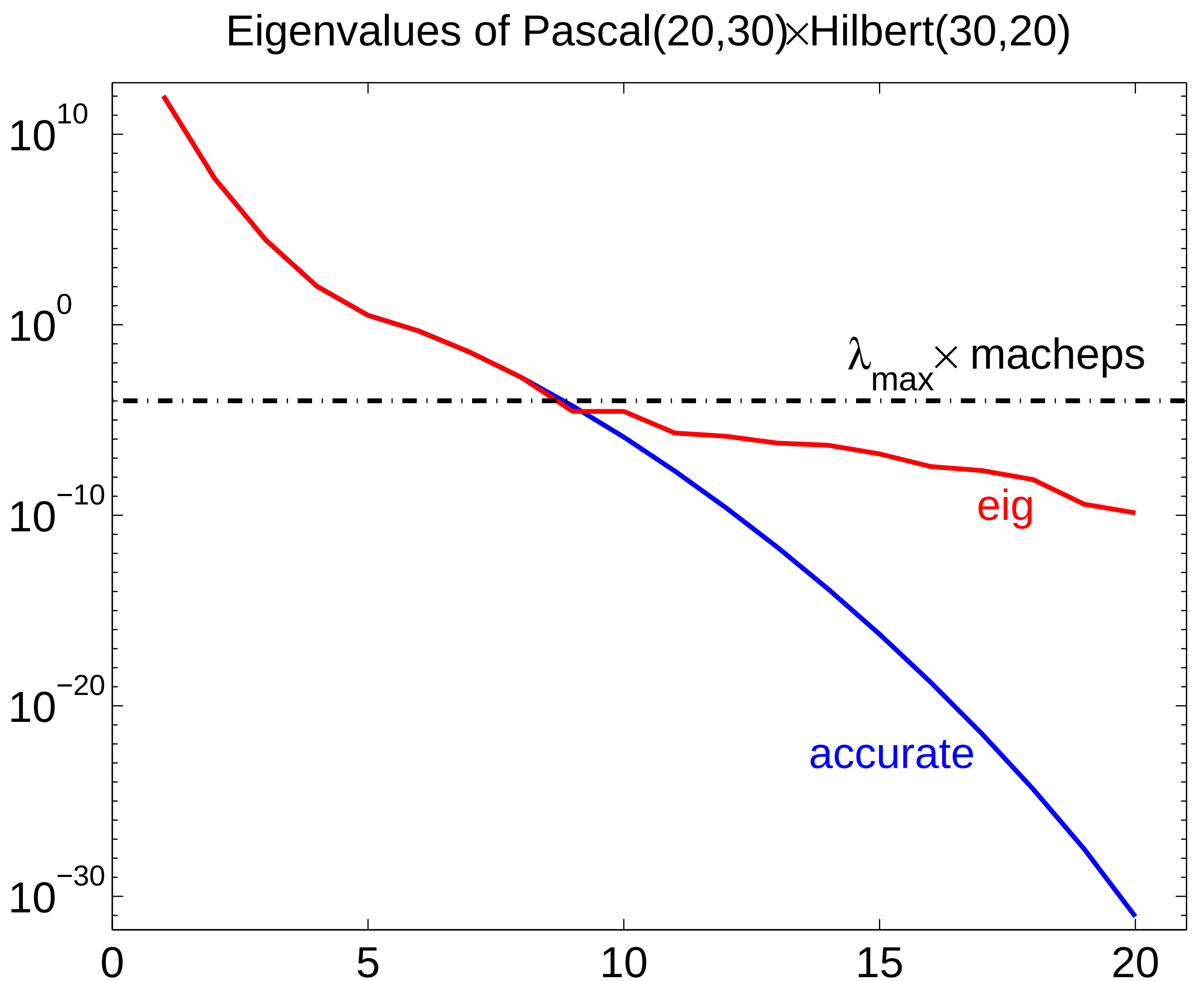
$$\begin{bmatrix} c & s \\ -s & c \end{bmatrix} = \begin{bmatrix} 1 & s/c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/c & \\ & c \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -s/c & 1 \end{bmatrix}$$

- **Implies Hessenberg reduction, SVD, QR accurate**

$\mathcal{BD}$(Product)

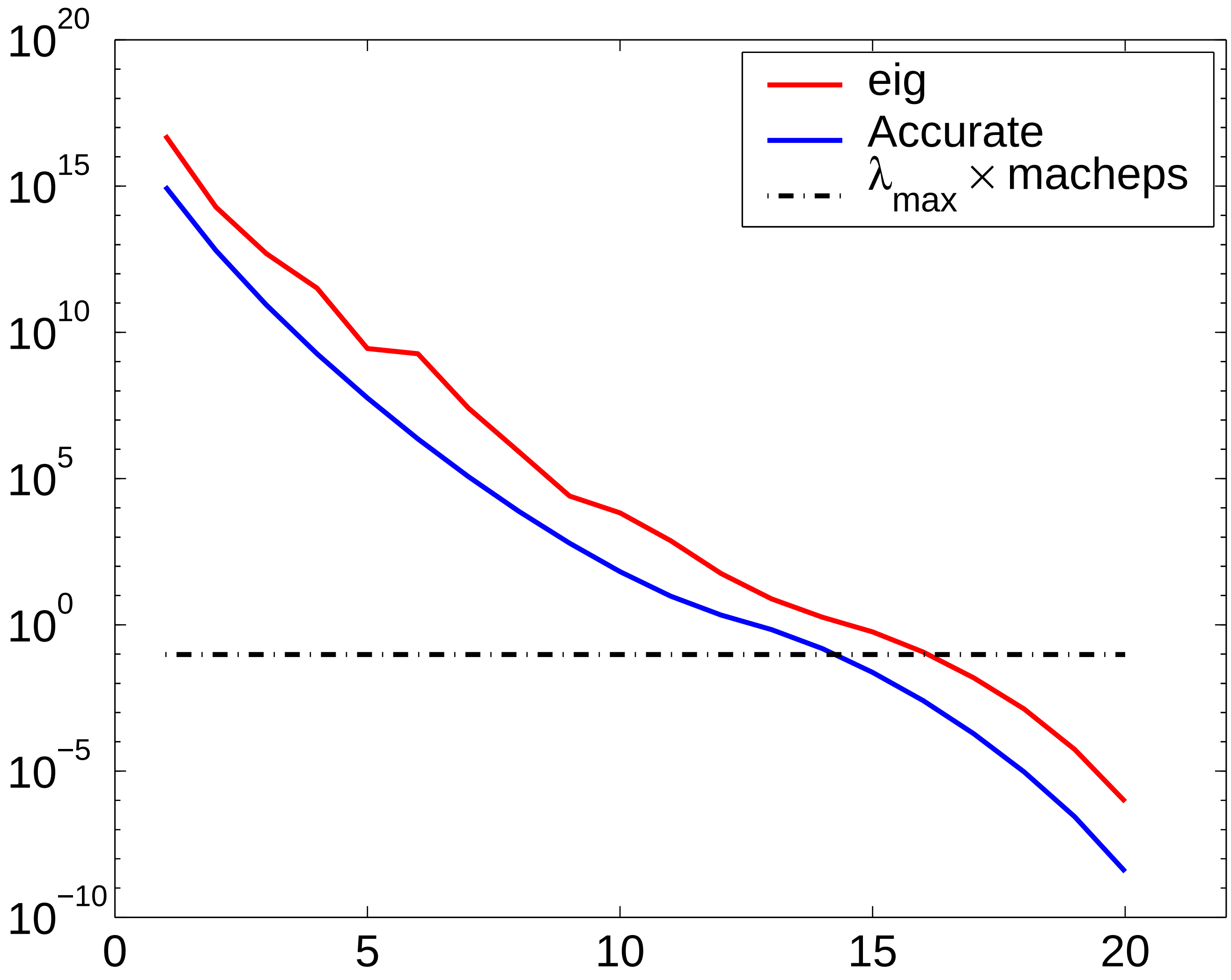
- **Any TP matrix = product of bidiagonals = product of EETs**
- **Thus $\mathcal{BD}(AB)$ equivalent to applying a number of EETs to $\mathcal{BD}(A)$**
- **All TP linear algebra = sequence of EETs $\Rightarrow$ accurate**

Example



Eigenvalues of Pascal(20,30)×Hilbert(30,20)
10^10
10^0
10^−10
10^−20
10^−30
0
5
10
15
20
λmax× macheps
eig
accurate

Extreme Example: eig(20th Schur Complement of Vand(1:40))



Bidiagonal have repercussions beyond accurate computations

- **Variation-diminishing property:**

$$\text{EET} \times \text{vector} = \begin{bmatrix} 1 & & \\ x & 1 & \\ & & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ b + ax \\ c \end{bmatrix}, x > 0$$

$$\begin{array}{ccc} \begin{array}{ccc} + & & + \\ + & \longrightarrow & + \\ + & & + \end{array} & \begin{array}{ccc} + & & + \\ * & \longrightarrow & * \\ - & & - \end{array} & \begin{array}{ccc} + & & + \\ - & \longrightarrow & ? \\ + & & + \end{array} \\ \\ \begin{array}{ccc} - & & - \\ - & \longrightarrow & - \\ - & & - \end{array} & \begin{array}{ccc} - & & - \\ * & \longrightarrow & * \\ + & & + \end{array} & \begin{array}{ccc} - & & - \\ + & \longrightarrow & ? \\ - & & - \end{array} \\ \hline \\ 0 \longrightarrow 0 & 1 \longrightarrow 1 & 2 \longrightarrow 0, 2 \end{array}$$

Therefore: **#Sign Changes(Az) $\leq$ #Sign Changes(z)**

Oscillating Vectors and TP Matrices

Let A be TP, $A = V \cdot \Lambda \cdot V^{-1}$, then

$$V = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ + & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{bmatrix} \begin{bmatrix} 1 & - & & \\ & 1 & - & \\ & & 1 & - \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & - & \\ & & 1 & - \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & - \\ & & & 1 \end{bmatrix}$$

- Lowerly Totally Positive (LTP) matrix
- Variation Diminishing Property $\Rightarrow S(v_j) \leq j - 1$
- V^{-T} eigenvecs of A^T $\Rightarrow V^{-T}$ LTP $\Rightarrow S(v_j) \geq j - 1$

- Call these LTP^2 matrices (V *and* V^{-T} are LTP)
- Thm (Dopico, K.): LTP^2 $\Leftrightarrow S(v_j) = j - 1$
- Cor: Q in $A = QR$ is LTP; $Q = Q^{-T}$ $\Rightarrow S(q_j) = j - 1$
- Also works numerically

Conclusions

- **Virtually all linear algebra with TP (also nonsingular TN) matrices done to high relative accuracy in $O(n^3)$ time**
- **New: Understanding of oscillating properties of TP matrices**
- **Key: Bidiagonal Decompositions**
- **Slides, papers, software: Google('Plamen Koev')**