The Complexity of Accurate Computations with Structured Matrices

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Goals

- **Accurate and Efficient Linear Algebra**
  - **$\det(A)$**
  - **Any minor**
  - **A^{-1}**
  - **Decompositions: LDU, QR, Bidiagonal**
  - **Solution to $Ax = b$**
  - **SVD**
  - **Eigenvalues**
- **With Structured Matrices**
  - **Vandermonde**
    - **Polynomial / Generalized / Totally Positive**
  - **Cauchy**
  - **Totally Positive (All minors > 0)**
  - **M-matrices**
  - **Green Matrices ...**

Goals

- Efficient means $O(n^3)$
  - Usual arithmetic model $fl(a \odot b) = (a \odot b)(1 + \delta), \; |\delta| \le \epsilon$
- Accurate means correct sign and leading digits
  - $\det(A)$
  - minors
  - componentwise A^{-1}, LDU, solution to $Ax = b$
  - Eigenvalues, SVD:

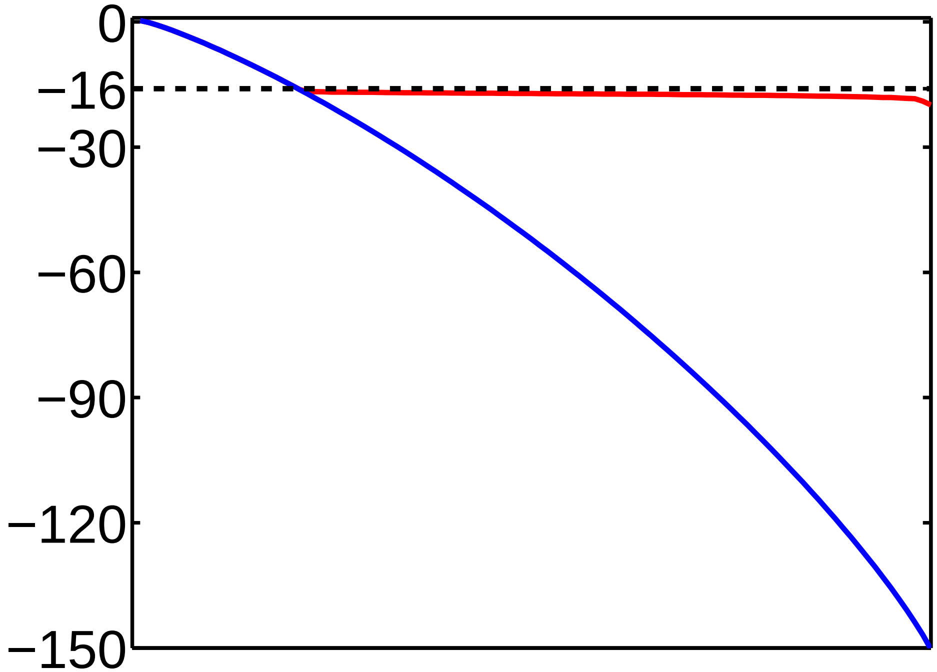
$$|\hat{\lambda}_i - \lambda_i| \le O(\epsilon)|\lambda_i|, \qquad |\hat{\sigma}_i - \sigma_i| \le O(\epsilon)\sigma_i,$$

- Contrast: Traditional algorithms

$$|\hat{\lambda}_i - \lambda_i| \le O(\epsilon)\frac{||A||}{|y^* x|}, \qquad |\hat{\sigma}_i - \sigma_i| \le O(\epsilon)\sigma_1$$

Example: 100×100 Hilbert Matrix $H = 1/(i+j-1)$

- Eigenvalues range from 1 to 10^{-150}
- Old Algorithm, New algorithm, both in 16 digits



- Exploits Cauchy Structure

Cost of Accuracy in TM (1)

Matrix Type	$\det(A)$	A^{-1}	Minor	GENP	GEPP	GECP	SVD	NENP
Cauchy								
TP Cauchy								
Vandermonde								
TP Vandermonde								
Confluent Vandermonde								
TP Confluent Vandermonde								
Vandermonde 3 Term Orth. Poly.								
Generalized Vandermonde								
TP Generalized Vandermonde								

GENP/PP/CP = Gaussian Elimination with No/Partial/Complete Pivoting
SVD = Singular Value Decomposition
NENP = Neville Elimination (bidiagonal factorization) with No Pivoting

Cost of Accuracy in TM (2)

TP = Totally Positive (all minors nonnegative)

Matrix Type	
Cauchy	$C_{ij} = 1/(x_i + y_j)$
TP Cauchy	$x_i \nearrow$, $y_j \nearrow$, $x_1 + y_1 > 0$
Vandermonde	$V_{ij} = x_i^{j-1}$, x_i distinct
TP Vandermonde	$0 < x_i \nearrow$
Confluent Vandermonde	if some x_i coincide, differentiate rows of V
TP Confluent Vandermonde	$0 < x_i \nearrow$
Vandermonde 3 Term Orth. Poly.	$V_{ij} = P_j(x_i)$, P_j orthogonal polynomial from 3-term recurrence
Generalized Vandermonde	$G_{ij} = x_i^{\lambda_j + j - 1}$, λ_j nonnegative increasing integer sequence
TP Generalized Vandermonde	$0 < x_i \nearrow$

How do we lose relative accuracy in floating point?

- **MODEL:** $\mathrm{fl}(a \otimes b) = (a \otimes b)(1+\delta)$, no under/overflow
- **ACCURATE:**
  - Products, Quotients, Sums of positive numbers
    Proof: $1+\delta$ factors can be factored out
  - $x_i \pm x_j$, where x_i and x_j are initial data (so exact)

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- **POSSIBLE LOSS OF ACCURACY:**
  - Subtractive Cancellation when subtracting approximate results:

```
    .12345xxx
 -  .12345yyy
 -------------
    .00000zzz
```

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```
   .12345xxx
-  .12345yyy
------------
   .00000zzz
```

$$.12345(1+\delta_1) - .12345(1+\delta_2) = .12345(\delta_1 - \delta_2) \text{ – BAD}$$
$$.12345(1+\delta_1) + .12345(1+\delta_2) = .23490(1+(\delta_1+\delta_2)/2) \text{ – OK}$$

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- **TRICK:** Only compute:
  - $1/(1+xy)$; $1/x$; xy; $1/(x-y)$; $x, y > 0$
  - never subtract approximate quantities

Central Role of Minors

- Being able to compute $\det(A)$ accurately and efficiently is necessary for
  - $A = LU$ with pivoting
  - $A = QR$
  - Eigenvalues λ_i of A ...
    - Proof: $\det(A) = \pm \prod_i U_{ii} = \pm \prod_i R_{ii} = \prod_i \lambda_i = \cdots$
- Being able to compute all minors of A is sufficient for
  - A^{-1}
    - Proof: Cramer's rule, only need $n^2 + 1$ minors
  - $A = LU$ or $A = LDU$ with pivoting
    - Proof: Each entry of L, D, U a quotient of minors; $O(n^3)$ needed
  - Singular values of A (SVD)
    - Proof: $A = LDU$ with complete pivoting, then SVD of LDU
  - Eigenvalues of Totally Positive matrices (yesterday's talk)
- Other methods for getting accurate λ_i, σ_i developed

Accurate det(A), minors

- det(A):
  - Vandermonde $V = x_i^{j-1}$

$$\det(V) = \prod_{i>j}(x_i - x_j)$$

  - Cauchy $C = 1/(x_i + y_j)$

$$\det(C) = \frac{\prod_{i<j}(x_i - x_j)(y_i - y_j)}{\prod_{i,j}(x_i + y_j)}$$

  - TP Generalized Vandermonde $G = x_i^{a_j}$, $x_i > 0$, increasing

$$\det(G) = \det(V) \cdot s_\lambda(x_1, x_2, \ldots, x_n)$$

    where $s_\lambda(x_1, x_2, \ldots, x_n) = \sum x^\mu$ – sum of positives
- Minors: Cauchy – OK, Vandermonde – OK, but not efficient
- LDU: Each entry of L, D, U – quotient of minors of A

Accurate Solution to $Ax = b$

- Classical Example: Björck-Pereyra Methods for Vandermonde systems

$$\begin{bmatrix} 1 & x_1 & \ldots & x_1^{n-1} \\ 1 & x_2 & \ldots & x_2^{n-1} \\ 1 & x_3 & \ldots & x_3^{n-1} \\ & & \ddots & \\ 1 & x_n & \ldots & x_n^{n-1} \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} + \\ - \\ + \\ \vdots \\ - \end{bmatrix}$$

and $0 < x_1 < \ldots < x_n$.

  - In $O(n^2)$ time
  - With small forward error: $|y_i - \hat{y}_i| \le O(\epsilon)|y_i|$
  - With small backward error:
    If $\hat{V}\hat{y} = b$ then $|V_{ij} - \hat{V}_{ij}| \le O(\epsilon)|V_{ij}|$.

Accuracy of the Björck-Pereyra Method for Vandermondes

$$\begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -x_1 & & \\ & 1 & -x_1 & \\ & & 1 & -x_1 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & -x_2 & \\ & & 1 & -x_2 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & -x_3 \\ & & & 1 \end{bmatrix} \times$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & \frac{-1}{x_4-x_1} & \frac{1}{x_4-x_1} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & \frac{-1}{x_3-x_1} & \frac{1}{x_3-x_1} & \\ & & \frac{-1}{x_4-x_2} & \frac{1}{x_4-x_2} \end{bmatrix} \begin{bmatrix} 1 & & & \\ \frac{-1}{x_2-x_1} & \frac{1}{x_2-x_1} & & \\ & \frac{-1}{x_3-x_2} & \frac{1}{x_3-x_2} & \\ & & \frac{-1}{x_4-x_3} & \frac{1}{x_4-x_3} \end{bmatrix}$$

The solution to $y = V^{-1}b$ is then computed without subtractions

$$= \begin{bmatrix} 1 & - & & \\ & 1 & - & \\ & & 1 & - \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & - & \\ & & 1 & - \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & - \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & - & + \end{bmatrix} \begin{bmatrix} + & & & \\ & + & & \\ & - & + & \\ & & - & + \end{bmatrix} \begin{bmatrix} + & & & \\ - & + & & \\ & - & + & \\ & & - & + \end{bmatrix} \begin{bmatrix} + \\ - \\ + \\ - \end{bmatrix}$$

Accuracy ... OK

Other TP matrices? ... Yes. Cauchy

The Björck-Pereyra Method for Cauchy Matrices

- *Cauchy* matrices

$$C_{ij} = \frac{1}{x_i - y_j}$$

- TP, if $x_1 > \ldots > x_n > y_1 > \ldots > y_n$

$$\begin{bmatrix} \frac{1}{x_1-y_1} & \frac{1}{x_1-y_2} & \frac{1}{x_1-y_3} \\ \frac{1}{x_2-y_1} & \frac{1}{x_2-y_2} & \frac{1}{x_2-y_3} \\ \frac{1}{x_3-y_1} & \frac{1}{x_3-y_2} & \frac{1}{x_3-y_3} \end{bmatrix}^{-1} = \begin{bmatrix} 1 & \frac{-(x_1-y_1)}{y_1-y_2} & 0 \\ 0 & \frac{x_1-y_2}{y_1-y_2} & \frac{-(x_1-y_2)}{y_2-y_3} \\ 0 & 0 & \frac{x_1-y_3}{y_2-y_3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{-(x_2-y_1)}{y_1-y_3} \\ 0 & 0 & \frac{x_2-y_3}{y_1-y_3} \end{bmatrix} \times$$

$$\begin{bmatrix} x_1 - y_1 & & \\ & x_2 - y_2 & \\ & & x_3 - y_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{-(x_1-y_2)}{x_3-x_1} & \frac{x_3-y_2}{x_3-x_1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \frac{-(x_1-y_1)}{x_2-x_1} & \frac{x_2-y_1}{x_2-x_1} & 0 \\ 0 & \frac{-(x_2-y_1)}{x_3-x_2} & \frac{x_3-y_1}{x_3-x_2} \end{bmatrix}$$

(Vadim Olshevsky, 1995)

- Unifying Characteristic?

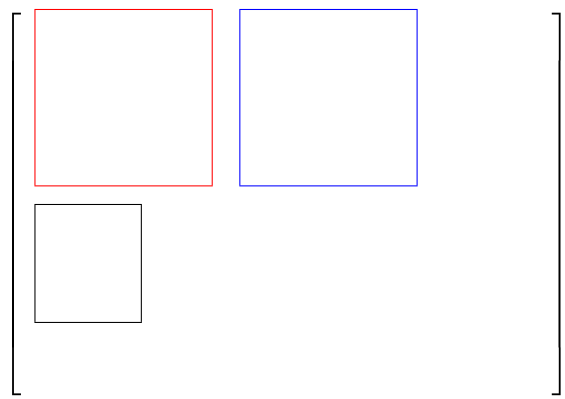
The Connection with Minors

- Which TP matrices permit *accurate* bidiagonal decomposition?
- Each entry is ***product of quotients of initial minors***

$$L^{(k)}_{i+1,i} = -\frac{\det(A(i-k+2:i+1,1:k))}{\det(A(i-k+2:i,1:k-1))} \cdot \frac{\det(A(i-k+1:i-1,1:k-1))}{\det(A(i-k+1:i,1:k))}$$

INITIAL MINORS
– Contiguous
– Include first row or column

$A =$



- Accurate Initial Minors ⇒ Björck-Pereyra Methods
- Initial Minors easier than all minors
- New results: TP Generalized Vandermonde, $x_i > 0$ increasing

$$G = \begin{bmatrix} 1 & x_1 & x_1^6 & x_1^8 \\ 1 & x_2 & x_2^6 & x_2^8 \\ 1 & x_3 & x_3^6 & x_3^8 \\ 1 & x_3 & x_4^6 & x_4^8 \end{bmatrix}$$

- **Demmel, Kahan 1991: Accurate SVD of bidiagonals**

$$B = \begin{bmatrix} a_1 & b_1 & & & \\ & a_2 & b_2 & & \\ & & \ddots & \ddots & \\ & & & a_{n-1} & b_{n-1} \\ & & & & a_n \end{bmatrix}$$

- **Demmel et al. 1997:**
  Given an accurate LDU from GECP

$$A = LDU$$

⇒ Accurate SVD

  - **LDU – computed accurately through minors**
  - **Works for Cauchy, totally unimodular, Green matrices, unit displacement rank, total signed compound**
  - **New result: Diagonally dominant M-matrices**

The SVD – 2

- **New results:**
  - Given A, first decompose as $A = B \cdot (\text{WELL-CONDITIONED})$
  - Polynomial Vandermonde involving orthonormal polynomials

$$V_P = \begin{bmatrix} P_0(x_1) & P_1(x_1) & \dots & P_{n-1}(x_1) \\ P_0(x_2) & P_1(x_2) & \dots & P_{n-1}(x_2) \\ & & \ddots & \\ P_0(x_n) & P_1(x_n) & \dots & P_{n-1}(x_n) \end{bmatrix}$$
$$= (CAUCHY) \cdot (ORTHOGONAL)$$

- Interesting case – unsymmetric
- TP Tridiagonals: If $b_i, d_i, c_i > 0$ then the eigenvalues of

$$T = \begin{bmatrix} 1 & & & & \\ b_1 & 1 & & & \\ & \ddots & \ddots & & \\ & & b_{n-2} & 1 & \\ & & & b_{n-1} & 1 \end{bmatrix} \begin{bmatrix} d_1 & & & & \\ & d_2 & & & \\ & & \ddots & & \\ & & & d_{n-1} & \\ & & & & d_n \end{bmatrix} \begin{bmatrix} 1 & c_1 & & & \\ & 1 & c_2 & & \\ & & \ddots & \ddots & \\ & & & 1 & c_{n-1} \\ & & & & 1 \end{bmatrix}$$

are the squares of the singular values of

$$\begin{bmatrix} \sqrt{d_1} & \sqrt{d_1 b_1 c_1} & & & \\ & \sqrt{d_2} & \sqrt{d_2 b_2 c_2} & & \\ & & \ddots & \ddots & \\ & & & \sqrt{d_{n-1}} & \sqrt{d_{n-1} b_{n-1} c_{n-1}} \\ & & & & \sqrt{d_n} \end{bmatrix}$$

- Computed to high relative accuracy

Accurate Eigenvalues

- **Thm: If A is TP**

$$A = (TP) \cdot T \cdot (TP)^{-1}$$

- **If we start with a bidiagonal decomposition (think Björck-Pereyra decomposition)**

$$A = \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ + & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{bmatrix} \begin{bmatrix} 1 & + & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix}$$

- **New Result: Accurate $A \longrightarrow T \longrightarrow \lambda_i$**

SVD – 3

- **New approach to the accurate SVD of TP matrices:**
  - Applying accurate Givens rotations
- **Start with the bidiagonal decomposition of a TP Matrix A**

$$A = \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ + & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{bmatrix} \begin{bmatrix} 1 & + & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix}$$

- **Givens Rotation =**
  - Subtract a row from next to make a 0
  - Add a multiple of row with 0 to previous
  - Scale both rows

Cost of Accuracy in TM (1)

Matrix Type	det(A)	A^{-1}	Minor	GENP	GEPP	GECP	SVD	NENP
Cauchy								
TP Cauchy								
Vandermonde								
TP Vandermonde								
Confluent Vandermonde								
TP Confluent Vandermonde								
Vandermonde 3 Term Orth. Poly.								
Generalized Vandermonde								
TP Generalized Vandermonde								

GENP/PP/CP = Gaussian Elimination with No/Partial/Complete Pivoting
SVD = Singular Value Decomposition
NENP = Neville Elimination (bidiagonal factorization) with No Pivoting

Cost of Accuracy in TM (3)
Known results

Matrix Type	det(A)	A^{-1}	Minor	GENP	GEPP	GECP	SVD	NENP
Cauchy	n^2	n^2	n^2	n^3	n^3	n^3		n^2
TP Cauchy	n^2	n^2	n^2	n^3	n^3	n^3		n^2
Vandermonde								
TP Vandermonde								
Confluent Vandermonde								
TP Confluent Vandermonde								
Vandermonde 3 Term Orth. Poly.								
Generalized Vandermonde								
TP Generalized Vandermonde								

Proof: Exploit $\det(C) = \Pi_{i<j}(x_j - x_i)(y_j - y_i) / \Pi_{ij}(x_i + y_j)$

Cost of Accuracy in TM (4)
Known results + New Results

Matrix Type	$\det(A)$	A^{-1}	Minor	GENP	GEPP	GECP	SVD	NENP
Cauchy	n^2	n^2	n^2	n^2	n^2	n^3	n^3	n^2
TP Cauchy	n^2	n^2	n^2	n^2	n^2	n^3	n^3	n^2
Vandermonde								
TP Vandermonde								
Confluent Vandermonde								
TP Confluent Vandermonde								
Vandermonde 3 Term Orth. Poly.								
Generalized Vandermonde								
TP Generalized Vandermonde								

Proof: Do GECP, apply new SVD algorithm

Cost of Accuracy in TM (5)
Known results

Matrix Type	det(A)	A^{-1}	Minor	GENP	GEPP	GECP	SVD	NENP
Cauchy	n^2	n^2	n^2	n^2	n^2	n^3	n^3	n^2
TP Cauchy	n^2	n^2	n^2	n^2	n^2	n^3	n^3	n^2
Vandermonde	n^2							n^2
TP Vandermonde	n^2	n^3						n^2
Confluent Vandermonde								
TP Confluent Vandermonde								
Vandermonde 3 Term Orth. Poly.								
Generalized Vandermonde								
TP Generalized Vandermonde								

Proof: Björck-Pereyra

Cost of Accuracy in TM (6)
Known results + New Results

Matrix Type	$\det(A)$	A^{-1}	Minor	GENP	GEPP	GECP	SVD	NENP
Cauchy	n^2	n^2	n^2	n^2	n^2	n^3	n^3	n^2
TP Cauchy	n^2	n^2	n^2	n^3	n^3	n^3	n^3	n^2
Vandermonde	n^2						n^3	n^2
TP Vandermonde	n^2	n^3					n^3	n^2
Confluent Vandermonde								
TP Confluent Vandermonde								
Vandermonde 3 Term Orth. Poly.								
Generalized Vandermonde								
TP Generalized Vandermonde								

Proof: Vandermonde = Cauchy $\times$ DFT

Cost of Accuracy in TM (7)

Known results + New Results

Matrix Type	$\det(A)$	A^{-1}	Minor	GENP	GEPP	GECP	SVD	NENP
Cauchy	n^2	n^2	n^2	n^2	n^2	n^3	n^3	n^2
TP Cauchy	n^2	n^2	n^2	n^2	n^2	n^3	n^3	n^2
Vandermonde	n^2						n^3	n^2
TP Vandermonde	n^2	n^3	exp	n^2	n^2	exp	n^3	n^2
Confluent Vandermonde								
TP Confluent Vandermonde								
Vandermonde 3 Term Orth. Poly.								
Generalized Vandermonde								
TP Generalized Vandermonde								

Proof: Special case of TP Generalized Vandermonde

Cost of Accuracy in TM (8)

Known results + New Results

Matrix Type	det(A)	A^{-1}	Minor	GENP	GEPP	GECP	SVD	NENP
Cauchy	n^2	n^2	n^2	n^2	n^2	n^3	n^3	n^2
TP Cauchy	n^2	n^2	n^2	n^2	n^2	n^3	n^3	n^2
Vandermonde	n^2	No	No	No	No	No	n^3	n^2
TP Vandermonde	n^2	n^3	exp	n^2	n^2	exp	n^3	n^2
Confluent Vandermonde								
TP Confluent Vandermonde								
Vandermonde 3 Term Orth. Poly.								
Generalized Vandermonde								
TP Generalized Vandermonde								

Proof: Can't add $x + y + z$ accurately

Cost of Accuracy in TM (9)
Known results

Matrix Type	det(A)	A^{-1}	Minor	GENP	GEPP	GECP	SVD	NENP
Cauchy	n^2	n^2	n^2	n^2	n^2	n^3	n^3	n^2
TP Cauchy	n^2	n^2	n^2	n^2	n^2	n^3	n^3	n^2
Vandermonde	n^2	No	No	No	No	No	n^3	n^2
TP Vandermonde	n^2	n^3	exp	n^2	n^2	exp	n^3	n^2
Confluent Vandermonde	n^2							n^2
TP Confluent Vandermonde	n^2	n^3		n^3				n^2
Vandermonde 3 Term Orth. Poly.								
Generalized Vandermonde								
TP Generalized Vandermonde								

Proof: Higham

Cost of Accuracy in TM (10)

Known results + New Results

Matrix Type	$\det(A)$	A^{-1}	Minor	GENP	GEPP	GECP	SVD	NENP
Cauchy	n^2	n^2	n^2	n^2	n^2	n^3	n^3	n^2
TP Cauchy	n^2	n^2	n^2	n^2	n^2	n^3	n^3	n^2
Vandermonde	n^2	No	No	No	No	No	n^3	n^2
TP Vandermonde	n^2	n^3	exp	n^2	n^2	exp	n^3	n^2
Confluent Vandermonde	n^2	No	No	No	No	No		n^2
TP Confluent Vandermonde	n^2	n^3		n^3				n^2
Vandermonde 3 Term Orth. Poly.								
Generalized Vandermonde								
TP Generalized Vandermonde								

Proof: Can't add $x + y + z$ accurately

Cost of Accuracy in TM (11)
Known results

Matrix Type	$\det(A)$	A^{-1}	Minor	GENP	GEPP	GECP	SVD	NENP
Cauchy	n^2	n^2	n^2	n^2	n^2	n^3	n^3	n^2
TP Cauchy	n^2	n^2	n^2	n^2	n^2	n^3	n^3	n^2
Vandermonde	n^2	No	No	No	No	No	n^3	n^2
TP Vandermonde	n^2	n^3	exp	n^2	n^2	exp	n^3	n^2
Confluent Vandermonde	n^2	No	No	No	No	No		n^2
TP Confluent Vandermonde	n^2	n^3		n^3				n^2
Vandermonde 3 Term Orth. Poly.	n^2							
Generalized Vandermonde								
TP Generalized Vandermonde								

Proof: Higham

Cost of Accuracy in TM (12)
Known results + New Results

Matrix Type	$\det(A)$	A^{-1}	Minor	GENP	GEPP	GECP	SVD	NENP
Cauchy	n^2	n^2	n^2	n^2	n^2	n^3	n^3	n^2
TP Cauchy	n^2	n^2	n^2	n^2	n^2	n^3	n^3	n^2
Vandermonde	n^2	No	No	No	No	No	n^3	n^2
TP Vandermonde	n^2	n^3	exp	n^2	n^2	exp	n^3	n^2
Confluent Vandermonde	n^2	No	No	No	No	No		n^2
TP Confluent Vandermonde	n^2	n^3		n^3				n^2
Vandermonde 3 Term Orth. Poly.	n^2						n^3	
Generalized Vandermonde								
TP Generalized Vandermonde								

Proof: Poly_Vand(x) = Cauchy(x,y) $\times$ Poly_Vand(y)

Choose y as roots of Orth Poly $\Rightarrow$ Poly_Vand(y) = diagonal $\times$ orthogonal

Cost of Accuracy in TM (13)

New Results

Matrix Type	det(A)	A^{-1}	Minor	GENP	GEPP	GECP	SVD	NENP
Cauchy	n^2	n^2	n^2	n^2	n^2	n^3	n^3	n^2
TP Cauchy	n^2	n^2	n^2	n^2	n^2	n^3	n^3	n^2
Vandermonde	n^2	No	No	No	No	No	n^3	n^2
TP Vandermonde	n^2	n^3	exp	n^2	n^2	exp	n^3	n^2
Confluent Vandermonde	n^2	No	No	No	No	No		n^2
TP Confluent Vandermonde	n^2	n^3		n^3				n^2
Vandermonde 3 Term Orth. Poly.	n^2						n^3	
Generalized Vandermonde	No	No	No	No	No	No	No	No
TP Generalized Vandermonde								

Proof: Can't add $x + y + z$ accurately

Cost of Accuracy in TM (14)

New Results

Matrix Type	det(A)	A^{-1}	Minor	GENP	GEPP	GECP	SVD	NENP
Cauchy	n^2	n^2	n^2	n^2	n^2	n^3	n^3	n^2
TP Cauchy	n^2	n^2	n^2	n^2	n^2	n^3	n^3	n^2
Vandermonde	n^2	No	No	No	No	No	n^3	n^2
TP Vandermonde	n^2	n^3	exp	n^2	n^2	exp	n^3	n^2
Confluent Vandermonde	n^2	No	No	No	No	No		n^2
TP Confluent Vandermonde	n^2	n^3		n^3				n^2
Vandermonde 3 Term Orth. Poly.	n^2						n^3	
Generalized Vandermonde	No	No	No	No	No	No	No	No
TP Generalized Vandermonde	Λn^2	Λn^3	exp	Λn^2	Λn^2	exp	exp	Λn^2

- $G_{ij} = x_i^{\lambda_j+j-1}$, $0 \le \lambda_i \nearrow$
- $\Lambda \le 2(\lambda_1+1)\cdot(\lambda_2+1)^2 \cdots (\lambda_n+1)^2 \cdot (\#\lambda_i > 1)$
- Exponential speedup over previous best algorithm: $n^{\lambda_1+\cdots+\lambda_n}$
- Proof: Divide-and-conquer to evaluate Schur polynomials

Cost of Accuracy in TM (15)

Known results + New Results

Matrix Type	det(A)	A^{-1}	Minor	GENP	GEPP	GECP	SVD	NENP	EVD
Cauchy	n^2	n^2	n^2	n^2	n^2	n^3	n^3	n^2	
TP Cauchy	n^2	n^2	n^2	n^2	n^2	n^3	n^3	n^2	n^3
Vandermonde	n^2	No	No	No	No	No	n^3	n^2	
TP Vandermonde	n^2	n^3	exp	n^2	n^2	exp	n^3	n^2	n^3
Confluent Vandermonde	n^2	No	No	No	No	No		n^2	
TP Confluent Vandermonde	n^2	n^3		n^3			n^3	n^2	n^3
Vandermonde 3 Term Orth. Poly.	n^2						n^3		
Generalized Vandermonde	No	No	No	No	No	No	No	No	
TP Generalized Vandermonde	Λn^2	Λn^3	exp	Λn^2	Λn^2	exp	Λn^3	Λn^2	Λn^3
Any TP	n	n^3	exp	n^3	exp	exp	n^3	0	n^3

- Any TP: assuming NENP already done

Conclusions

- **We have identified many classes of structured matrices that permit accurate and efficient matrix computations**
- **New results:**
  - **$O(n^3)$ algorithms for the eigenvalues and the SVD of (*unsymmetric*) TP Matrices to high relative accuracy**
  - **Björck-Pereyra Methods for Generalized Vandermondes**
  - **Accurate SVDs of M-matrices**
  - **Accurate SVDs of Polynomial Vandermonde matrices**
- **Open problems remain**
  - **TP eigenvectors, singular vectors**
  - **Non-TP eigenproblem**
- **This talk and Matlab software:** `math.mit.edu/~plamen`