Accurate and Efficient Matrix Computations with Totally Positive Matrices

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SIAM Conference on Applied Linear Algebra, Williamsburg, VA, July, 2003

Goals

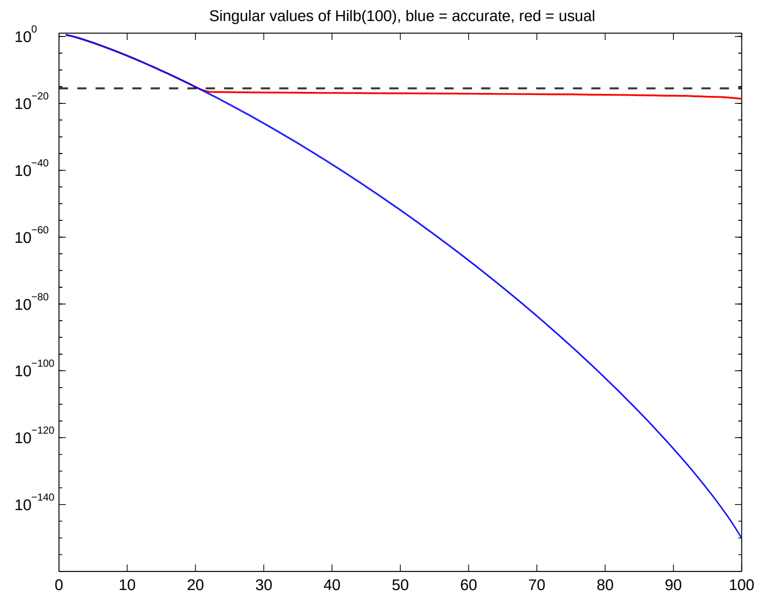
- Accurate and Efficient Linear Algebra with Totally Positive Matrices
  - Linear Equation Solving, Inversion
  - Eigenvalues, Eigenvectors
  - Singular Values, Singular Vectors
  - Schur Complementation, Multiplication
- Efficient means $O(n^3)$
  - Usual arithmetic model $\mathrm{fl}(a \odot b) = (a \odot b)(1 + \delta)$, $|\delta| \le \epsilon$
- Accurate means correct sign and leading digits

$$|\hat{\lambda}_i - \lambda_i| \le O(\epsilon)|\lambda_i|, \qquad |\hat{\sigma}_i - \sigma_i| \le O(\epsilon)\sigma_i,$$

  - Contrast: Traditional algorithms

$$|\hat{\lambda}_i - \lambda_i| \le O(\epsilon)\frac{||A||}{|y^* x|}, \qquad |\hat{\sigma}_i - \sigma_i| \le O(\epsilon)\sigma_1$$

Example: 100 × 100 Hilbert Matrix $H = 1/(i + j)$



How do we lose relative accuracy in floating point?

- **MODEL:** $\mathrm{fl}(a \otimes b) = (a \otimes b)(1+\delta)$, no under/overflow
- **ACCURATE:**
  - Products, Quotients, Sums of positive numbers
    Proof: $1+\delta$ factors can be factored out
  - $x_i \pm x_j$, where x_i and x_j are initial data (so exact)

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  - Subtractive Cancellation when subtracting approximate results:

```
   .12345xxx
 - .12345yyy
 -----------
   .00000zzz
```

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$$.12345(1 + \delta_1) - .12345(1 + \delta_2) = .12345(\delta_1 - \delta_2) \text{ – BAD}$$
$$.12345(1 + \delta_1) + .12345(1 + \delta_2) = .23490(1 + (\delta_1 + \delta_2)/2) \text{ – OK}$$

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- **TRICK:** Only compute:
  - $1/(1+xy)$; $1/x$; xy; $1/(x-y)$; $x, y > 0$
  - never subtract approximate quantities

Bidiagonal Matrices

- **Demmel–Kahan '91**

$$B = \begin{bmatrix} a_1 & b_1 & & & \\ & a_2 & b_2 & & \\ & & \ddots & \ddots & \\ & & & a_{n-1} & b_{n-1} \\ & & & & a_n \end{bmatrix}$$

- a_i, b_i **determine** σ_i **accurately**
- **dqds preserves the relative accuracy**
- **Eigenvalues of SPD tridiagonal matrices:** $\lambda_i(B^T B) = \sigma_i^2(B)$
  – computable accurately given the Cholesky factors

Total Positivity (TP)

- **Def: A is TP if all minors > 0:**

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix}$$

  - Hilbert $H = 1/(i+j)$
  - Cauchy $C = 1/(x_i + y_j)$, if $0 < x_i, y_j$-increasing
  - Generalized Vandermonde $V = x_i^{a_j}$, $0 < x_i, a_i$ - increasing
  - TP·TP
  - Schur-complement(TP)
  - $\text{diag}(\pm 1) \cdot (\text{TP})^{-1} \cdot \text{diag}(\pm 1)$
  - See books by Karlin, Gantmacher, Krein
- **Theory extends to Totally Nonnegative, Oscillatory, Sign-regular**
- **Motivation:**
  - Eigenvalues and SVD well determined
  - Right answer costs $O(n^3)$, same as the wrong

Structure of Totally Positive Matrices

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix}$$

- **A lot simpler than it seems**
- **Matrix entries – bad choice of parameters (4^n conditions)**
- **Best represented as products of positive bidiagonals**

$$\left(\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix}\right) =$$

$$\underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ + & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix}}_{\boldsymbol{L}}\underbrace{\begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{bmatrix}}_{\boldsymbol{D}}\underbrace{\begin{bmatrix} 1 & + & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix}}_{\boldsymbol{U}}$$

- **n^2 independent positive parameters**

- **Neville Elimination**

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{4} & \mathbf{8} \\ \color{red}{\mathbf{1}} & \color{red}{\mathbf{3}} & \color{red}{\mathbf{9}} & \color{red}{\mathbf{27}} \\ \color{red}{\mathbf{1}} & \color{red}{\mathbf{4}} & \color{red}{\mathbf{16}} & \color{red}{\mathbf{64}} \end{bmatrix}$$

Total Positivity and Neville Elimination – 2

- **Neville Elimination**

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 0 & 1 & 7 & 37 \end{bmatrix}$$

- **Neville Elimination**

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{4} & \mathbf{8} \\ \mathbf{0} & \mathbf{1} & \mathbf{5} & \mathbf{19} \\ \mathbf{0} & \mathbf{1} & \mathbf{7} & \mathbf{37} \end{bmatrix}$$

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Neville Elimination – We can choose the pattern

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ \color{red}{1} & \color{red}{3} & \color{red}{9} & \color{red}{27} \\ \color{red}{1} & \color{red}{4} & \color{red}{16} & \color{red}{64} \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & \color{red}{1} & 1 \end{bmatrix} \begin{bmatrix} \color{red}{1} & \color{red}{1} & \color{red}{1} & \color{red}{1} \\ \color{red}{1} & \color{red}{2} & \color{red}{4} & \color{red}{8} \\ 0 & 1 & 5 & 19 \\ 0 & 0 & 2 & 18 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

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$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

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Notice: positive bidiagonal decomposition

$$U_{i,i+1}^{(k)} = \frac{\det(A(1:k, i-k+2:i+1))}{\det(A(1:k-1, i-k+2:i))} \cdot \frac{\det(A(1:k-1, i-k+1:i-1))}{\det(A(1:k, i-k+1:i))}.$$

- **Bidiagonal Decomposition of Any TP Matrix**

$$\mathcal{BD}\left(\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix}\right) =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ + & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{bmatrix} \begin{bmatrix} 1 & + & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix}$$

- **n^2 entries of factors intrinsically parametrize all TP matrices**
- **For numerical computation:**
  - **Discard A**
    Instead: Use $\mathcal{BD}(A)$
  - **Let $A \longrightarrow A'$ be a linear transformation**
    - **Subtract a row from another, apply Givens rot, etc.**
  - **We will compute $\mathcal{BD}(A')$ from $\mathcal{BD}(A)$**
    without losing relative accuracy

Accurate Linear Algebra with $\mathcal{BD}(A)$

- **With $\mathcal{BD}(A)$ we can do accurately**
  - **Positive diagonal scaling – trivial**
  - **Subtract a row/column from next to create a zero**
  - **Multiply by a positive bidiagonal matrix**
- **All linear algebra for eigenvalues, SVD, etc. can be "assembled" from above**

Subtract a row/column from next to create a zero – 1

- **Equivalent to setting an entry in $\mathcal{BD}(A)$ to 0 and doing no arithmetic**

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ + & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{bmatrix} \begin{bmatrix} 1 & + & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix}}_{\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \end{bmatrix}}$$

Subtract a row/column from next to create a zero – 2

- **Equivalent to setting an entry in $\mathcal{BD}(A)$ to 0 and doing no arithmetic**

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ + & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{bmatrix} \begin{bmatrix} 1 & + & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix}}_{\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \end{bmatrix}}$$

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ + & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{bmatrix} \begin{bmatrix} 1 & + & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix}$$

Accurate TP×BIDIAGONAL

$$\mathcal{BD}\left(\mathcal{BD}(A) \times \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & & 1 \end{bmatrix}\right) = ?$$

"Commuting" of positive bidiagonals

- Assume $x, y, z > 0$, $b_i \geq 0$. Then

$$\begin{bmatrix} 1 & b_1 & & & \\ & 1 & b_2 & & \\ & & \ddots & \ddots & \\ & & & 1 & b_{n-1} \\ & & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & y & & & \\ & & x & z & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & y' & & & \\ & & x' & z' & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} \begin{bmatrix} 1 & b'_1 & & & \\ & 1 & b'_2 & & \\ & & \ddots & \ddots & \\ & & & 1 & b'_{n-1} \\ & & & & 1 \end{bmatrix}$$

where

$$\begin{aligned} x' &= x \\ y' &= y + b_i x \\ z' &= 1/y' \\ b'_{i-1} &= b_{i-1} y \\ b'_i &= b_i z / y_1 \end{aligned}$$

- No loss of relative accuracy
- "Commute accurately".

“Commuting” of positive bidiagonals - 2

- Assume $x > 0$, $b_i \geq 0$. Then

$$
\begin{bmatrix}
1 & & & & \\
b_1 & 1 & & & \\
 & \ddots & \ddots & & \\
 & & b_{n-2} & 1 & \\
 & & & b_{n-1} & 1
\end{bmatrix}
\begin{bmatrix}
1 & & & & & \\
 & 1 & & & & \\
 & & 1 & & & \\
 & & x & 1 & & \\
 & & & & 1 & \\
 & & & & & 1
\end{bmatrix}
=
\begin{bmatrix}
1 & & & & & \\
 & 1 & & & & \\
 & & 1 & & & \\
 & & & 1 & & \\
 & & & x' & 1 & \\
 & & & & & 1
\end{bmatrix}
\begin{bmatrix}
1 & & & & \\
b'_1 & 1 & & & \\
 & \ddots & \ddots & & \\
 & & b'_{n-2} & 1 & \\
 & & & b'_{n-1} & 1
\end{bmatrix}
$$

where

$$
\begin{aligned}
x' &= x b_{i+1}/(b_i + x) \\
b'_i &= b_i + x \\
b'_{i+1} &= x b_i / b'_i
\end{aligned}
$$

- No loss of relative accuracy

Applying similarity to bidiagonal form – 1

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ + & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix}\begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{bmatrix}\begin{bmatrix} 1 & + & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}$$

Applying similarity to bidiagonal form – 2

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ + & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{bmatrix} \begin{bmatrix} 1 & + & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix} {\color{blue}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & y & \\ & & + & z \end{bmatrix}} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & {\color{red}+} \\ & & & 1 \end{bmatrix}$$

Applying similarity to bidiagonal form – 3

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ + & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix}\begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{bmatrix}\begin{bmatrix} 1 & + & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & y' & \\ & & + & z' \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix}$$

Applying similarity to bidiagonal form – 4

$$
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix}
\begin{bmatrix} 1 & & & \\ + & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix}
\begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{bmatrix}
{\color{blue}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & y'' & \\ & & + & z'' \end{bmatrix}}
\begin{bmatrix} 1 & + & & \\ & 1 & + & \\ & & 1 & {\color{red}+} \\ & & & 1 \end{bmatrix}
\begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix}
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix}
$$

Applying similarity to bidiagonal form – 5

$$
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix}
\begin{bmatrix} 1 & & & \\ + & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix}
\color{blue}{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}}
\begin{bmatrix} + & & & \\ & + & & \\ & & \color{red}{+} & \\ & & & \color{red}{+} \end{bmatrix}
\begin{bmatrix} 1 & + & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix}
\begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix}
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix}
$$

Applying similarity to bidiagonal form – 6

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ + & 1 & & \\ & + & 1 & \\ & & {\color{red}+} & 1 \end{bmatrix} \begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{bmatrix} \begin{bmatrix} 1 & + & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix}$$

DONE!

- $\mathcal{BD}$(TP × TP)
  - suffices $\mathcal{BD}$(TP × BIDIAGONAL) – done
- $\mathcal{BD}(\text{diag}(\pm 1) \cdot (\text{TP})^{-1} \cdot \text{diag}(\pm 1))$
  - OK, still a product of positive bidiagonals
- $\mathcal{BD}$(SchurComplement) – also OK, but barely more complicated
- $\mathcal{BD}(\text{Givens} \cdot TP)$

$$\begin{bmatrix} c & s \\ -s & c \end{bmatrix} = \begin{bmatrix} + & + \\ 0 & + \end{bmatrix} \begin{bmatrix} 1 & 0 \\ - & 1 \end{bmatrix}$$

- SVD is then OK: TP→BIDIAGONAL→SVD
  - no loss of accuracy
- Eigenvalues: First reduce to tridiagonal

Reduction of a nonsymmetric matrix to tridiagonal form – 1

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ {\color{red}+} & + & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 2

$$\begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & - & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ \color{red}{+} & \color{red}{+} & \color{red}{+} & \color{red}{+} \\ \color{red}{+} & \color{red}{+} & \color{red}{+} & \color{red}{+} \end{bmatrix}$$

$$= \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 3

$$\begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & - & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & + & \mathbf{1} \end{bmatrix}$$

$$= \begin{bmatrix} + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ \mathbf{0} & + & \color{red}{+} & \color{red}{+} \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & + & \mathbf{1} \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 4

$$\begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & - & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & + & \mathbf{1} \end{bmatrix}$$

$$= \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & + & \mathbf{1} \end{bmatrix}$$

$$= \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 5

$$\begin{bmatrix} + & + & + & \color{red}{+} \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 6

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & - \\ & & & \mathbf{1} \end{bmatrix}$$

$$= \begin{bmatrix} + & + & + & \mathbf{0} \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 7

$$\begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & + \\ & & & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ \color{red}{+} & \color{red}{+} & \color{red}{+} & \color{red}{+} \\ \color{red}{\mathbf{0}} & \color{red}{+} & \color{red}{+} & \color{red}{+} \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & - \\ & & & \mathbf{1} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & + \\ & & & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & + & \mathbf{0} \\ + & + & + & + \\ \color{red}{+} & \color{red}{+} & \color{red}{+} & \color{red}{+} \\ \color{red}{\mathbf{0}} & \color{red}{+} & \color{red}{+} & \color{red}{+} \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 8

$$\begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & + \\ & & & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ \color{red}{+} & \color{red}{+} & \color{red}{+} & \color{red}{+} \\ \color{red}{\mathbf{0}} & \color{red}{+} & \color{red}{+} & \color{red}{+} \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & - \\ & & & \mathbf{1} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & + \\ & & & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & + & \mathbf{0} \\ + & + & + & + \\ \color{red}{+} & \color{red}{+} & \color{red}{+} & \color{red}{+} \\ \color{red}{\mathbf{0}} & \color{red}{+} & \color{red}{+} & \color{red}{+} \end{bmatrix}$$

$$= \begin{bmatrix} + & + & + & \mathbf{0} \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 9

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ {\color{red}+} & + & + & + \end{bmatrix} \Rightarrow \begin{bmatrix} + & + & + & {\color{red}+} \\ + & + & + & + \\ + & + & + & + \\ {\color{blue}\mathbf{0}} & + & + & + \end{bmatrix} \Rightarrow \begin{bmatrix} + & + & + & {\color{blue}\mathbf{0}} \\ + & + & + & + \\ {\color{red}+} & + & + & + \\ {\color{blue}\mathbf{0}} & + & + & + \end{bmatrix} \Rightarrow \begin{bmatrix} + & + & {\color{red}+} & {\color{blue}\mathbf{0}} \\ + & + & + & + \\ {\color{blue}\mathbf{0}} & + & + & + \\ {\color{blue}\mathbf{0}} & + & + & + \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} + & + & {\color{blue}\mathbf{0}} & {\color{blue}\mathbf{0}} \\ + & + & + & + \\ {\color{blue}\mathbf{0}} & + & + & + \\ {\color{blue}\mathbf{0}} & {\color{red}+} & + & + \end{bmatrix} \Rightarrow \begin{bmatrix} + & + & {\color{blue}\mathbf{0}} & {\color{blue}\mathbf{0}} \\ + & + & + & {\color{red}+} \\ {\color{blue}\mathbf{0}} & + & + & + \\ {\color{blue}\mathbf{0}} & {\color{blue}\mathbf{0}} & + & + \end{bmatrix} \Rightarrow \begin{bmatrix} + & + & {\color{blue}\mathbf{0}} & {\color{blue}\mathbf{0}} \\ + & + & + & {\color{blue}\mathbf{0}} \\ {\color{blue}\mathbf{0}} & + & + & + \\ {\color{blue}\mathbf{0}} & {\color{blue}\mathbf{0}} & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 10

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \Rightarrow \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix} \Rightarrow \begin{bmatrix} + & + & + & \mathbf{0} \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix} \Rightarrow \begin{bmatrix} + & + & + & \mathbf{0} \\ + & + & + & + \\ \mathbf{0} & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} + & + & \mathbf{0} & \mathbf{0} \\ + & + & + & + \\ \mathbf{0} & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix} \Rightarrow \begin{bmatrix} + & + & \mathbf{0} & \mathbf{0} \\ + & + & + & + \\ \mathbf{0} & + & + & + \\ \mathbf{0} & \mathbf{0} & + & + \end{bmatrix} \Rightarrow \begin{bmatrix} + & + & \mathbf{0} & \mathbf{0} \\ + & + & + & \mathbf{0} \\ \mathbf{0} & + & + & + \\ \mathbf{0} & \mathbf{0} & + & + \end{bmatrix}$$

... Obtaining $\mathcal{BD}$(TP Tridiagonal)=

$$\begin{bmatrix} + & + & \mathbf{0} & \mathbf{0} \\ + & + & + & \mathbf{0} \\ \mathbf{0} & + & + & + \\ \mathbf{0} & \mathbf{0} & + & + \end{bmatrix} = \begin{bmatrix} \mathbf{1} & & & \\ + & \mathbf{1} & & \\ & + & \mathbf{1} & \\ & & + & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{bmatrix} \begin{bmatrix} \mathbf{1} & + & & \\ & \mathbf{1} & + & \\ & & \mathbf{1} & + \\ & & & \mathbf{1} \end{bmatrix}$$

Accurate Eigenvalues of Tridiagonals

The eigenvalues of

$$\begin{bmatrix} 1 & & & & \\ b_1 & 1 & & & \\ & \ddots & \ddots & & \\ & & b_{n-2} & 1 & \\ & & & b_{n-1} & 1 \end{bmatrix} \begin{bmatrix} d_1 & & & & \\ & d_2 & & & \\ & & \ddots & & \\ & & & d_{n-1} & \\ & & & & d_n \end{bmatrix} \begin{bmatrix} 1 & c_1 & & & \\ & 1 & c_2 & & \\ & & \ddots & \ddots & \\ & & & 1 & c_{n-1} \\ & & & & 1 \end{bmatrix}$$

are the squares of the singular values of

$$\begin{bmatrix} \sqrt{d_1} & \sqrt{d_1 b_1 c_1} & & & \\ & \sqrt{d_2} & \sqrt{d_2 b_2 c_2} & & \\ & & \ddots & \ddots & \\ & & & \sqrt{d_{n-1}} & \sqrt{d_{n-1} b_{n-1} c_{n-1}} \\ & & & & \sqrt{d_n} \end{bmatrix}$$

- Thus the eigenvalue problem of a TP matrix is reduced to the SVD problem for a bidiagonal matrix
- Forward stable
- Total cost $= 8n^3$.

Computing Accurate $\mathcal{BD}(A)$

- **Bidiagonal decompositions intrinsic for all TP matrices**
- **Often accurate formulas:**
  - **Vandermonde** $V = x_i^{j-1}$:

$$\prod_{j=1}^{i-1}(x_i - x_j), \quad \prod_{j=i-k+1}^{i-1} \frac{x_{i+1} - x_{j+1}}{x_i - x_j}, \quad x_{i+n-k}$$

  - **Cauchy** $C = 1/(x_i + y_j)$:

$$\prod_{k=1}^{n-1} \frac{(x_n - x_k)(y_n - y_k)}{(x_n + y_k)(y_n + x_k)}, \qquad \frac{x_{n-k} + y_{i-n+k+1}}{x_i + y_{i-n+k+1}} \prod_{l=n-k}^{i-1} \frac{x_{i+1} - x_{l+1}}{x_i - x_l} \cdot \prod_{l=1}^{i-n+k-1} \frac{x_i + y_l}{x_{i+1} + y_l}$$

$$\frac{y_{n-k} + x_{i-n+k+1}}{y_i + x_{i-n+k+1}} \prod_{l=n-k}^{i-1} \frac{y_{i+1} - y_{l+1}}{y_i - y_l} \cdot \prod_{l=1}^{i-n+k-1} \frac{y_i + x_l}{y_{i+1} + x_l},$$

- **Equivalent to Björck-Pereyra Methods**
- **Can be computed iff initial minors can be computed accurately**

$$U_{i,i+1}^{(k)} = \frac{\det(A(1:k, i-k+2:i+1))}{\det(A(1:k-1, i-k+2:i))} \cdot \frac{\det(A(1:k-1, i-k+1:i-1))}{\det(A(1:k, i-k+1:i))}.$$

initial minors = contiguous, include the first row or column

Conclusions

- **$O(n^3)$ algorithms for the eigenvalues and the SVD of (*unsymmetric*) TP Matrices to high relative accuracy**
- **Accurate linear algebra with TP matrices closed under same operations as TP**
- **Applies to:**
  - **Oscillatory**
  - **Totally Nonnegative**
  - **Sign Regular (Inverses of TP)**
- **Open problem: If we perturb an entry in**

  $$\textbf{(PositiveBidiagonal)} \cdot \textbf{(PositiveBidiagonal)}$$

  How much do the signular vectors change?
  Singular values provably change by at most ϵ.
- **This talk and Matlab software:** `math.mit.edu/~plamen`