

Accurate Jordan Structures of Irreducible Totally Nonnegative Matrices

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Definition

- ▶ *Totally nonnegative* means all minors are nonnegative
- ▶ Examples: Hilbert, Pascal, Vandermonde with increasing nodes

$$\begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$$

- ▶ Nonsymmetric in general
- ▶ Interested in the *irreducible case*

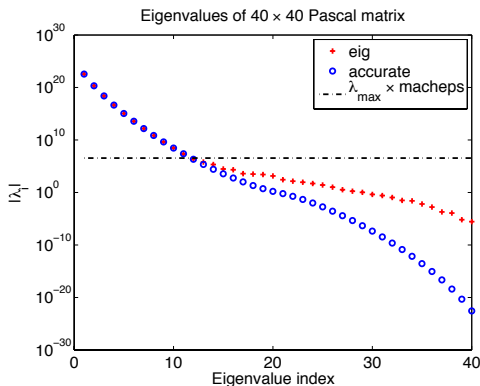
Goal

- ▶ Compute the Jordan structure of $IrTN$ matrix in floating point arithmetic to high relative accuracy:
 - ▶ all eigenavlues will have correct sign and leading digits (including tiniest ones)
 - ▶ the Jordan blocks will be correctly computed

Background

- ▶ Spectral structure well understood (Fallat/Johnson/Gekhtman):
 - ▶ All eigenvalues are nonnegative
 - ▶ Positive eigenvalues are distinct
 - ▶ Zero eigenvalues can have Jordan blocks
- ▶ Computationally hard: TN matrices can be ill conditioned, so accuracy in tiny eigenvalues lost in floating point

Example: 40×40 Pascal matrix, $P = \begin{pmatrix} n \\ k \end{pmatrix}$



$\text{cond}(P) \approx 10^{45}$; eigenvalues $< \lambda_{\max} * 10^{-16}$ lost

Reason accuracy is lost in floating point arithmetic

- ▶ Relative accuracy preserved in \times , $+$, $/$
Proof: $(1 + \delta)$ factors accumulate multiplicatively
- ▶ Subtractions of approximate quantities dangerous:

$$\begin{array}{r} .123456789xxx \\ - .123456789yyy \\ \hline .000000000zzz \end{array}$$

- ▶ Thus, if we avoid subtractions, we get accuracy

Previous results

- ▶ All linear algebra with *nonsingular* TN matrices possible accurately (K., '05)
 - ▶ Eigenvalues
 - ▶ Singular Values
 - ▶ Product
 - ▶ LU
 - ▶ submatrix
 - ▶ R factor of QR (still TN)
 - ▶ Converse, ...
- ▶ Only 3 operations needed and all 3 possible accurately
 - ▶ Subtracting a multiple of one row from next to create a zero
 - ▶ Add a multiple of one row to the previous
 - ▶ Diagonal scaling (trivial)

Bidiagonal decompositions

- ▶ Trick: Work on the *bidiagonal decomposition* (BD), not on the matrix!
- ▶ That representation reveals the TN structure
- ▶ Result of Neville elimination

$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 1 & 1 & \\ & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & \\ & 1 & 3 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & 2 \\ & & 1 \end{bmatrix}$$

- ▶ Then operate on those entries, NOT on the matrix entries!
- ▶ TN-preserving operations require no subtractions \Rightarrow accuracy

Basic Operations

- ▶ Subtracting a multiple of one row from next to create a 0 is equivalent to setting an entry of the BD to 0

$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & \\ & 1 & 3 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & 2 \\ & 1 & \\ & & 1 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 0 & 1 & 7 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & \\ & 1 & 3 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & 2 \\ & 1 & \\ & & 1 \end{bmatrix}$$

- ▶ No subtractions \Rightarrow accuracy
- ▶ New matrix still TN

Basic Operations

- ▶ Adding a multiple of one row/col to next/previous is done by changing the entries of the BD only

$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 0 & 1 & 7 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 1 & 1 & \\ & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & \\ & 1 & 3 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & 2 \\ & & 1 \end{bmatrix}$$

↓

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$$\begin{bmatrix} 1 & 6 & 4 \\ 1 & 12 & 9 \\ 0 & 8 & 7 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 2 & 1 & \\ & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 6 & \\ & & 2 \end{bmatrix} \begin{bmatrix} 1 & 6 & \\ & 1 & \frac{4}{3} \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & 2 \\ & & 1 \end{bmatrix}$$

- ▶ New entries are rational functions with > 0 coefficients
- ▶ Again, no subtractions \Rightarrow accuracy
- ▶ New matrix is still TN (Cauchy–Binet)

Eigenvalues of Nonsingular TN matrices

- ▶ Reduction to tridiagonal form using above similarities
- ▶ To create a 0 in position (3,1) of

$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}$$

- ▶ We use similarity

$$\begin{aligned} & \begin{bmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 0 & 1 & 7 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 4 \\ 1 & 12 & 9 \\ 0 & 8 & 7 \end{bmatrix} \end{aligned}$$

Eigenvalues of Nonsingular TN matrices

- ▶ Reduction to tridiagonal form possible using standard approach (Cryer '76)

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \rightarrow \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \end{bmatrix} \rightarrow \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \end{bmatrix} \rightarrow \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} + & + & + & 0 \\ + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \end{bmatrix} \rightarrow \begin{bmatrix} + & + & 0 & 0 \\ + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \end{bmatrix} \rightarrow \begin{bmatrix} + & + & 0 & 0 \\ + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \end{bmatrix}$$

Irreducible (singular) TN matrices

- ▶ At the end we have an (irreducible TN) tridiagonal in factored form

$$\begin{bmatrix} + & + & 0 & 0 \\ + & + & + & 0 \\ 0 & + & + & + \\ 0 & 0 & + & + \end{bmatrix} = \begin{bmatrix} 1 & & & \\ l_1 & 1 & & \\ & l_2 & 1 & \\ & & l_3 & 1 \end{bmatrix} \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & d_4 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & l_1 & & \\ & & 1 & l_2 \\ & & & 1 & l_3 \\ & & & & 1 \end{bmatrix}$$

- ▶ Eigenvalues readily computable accurately as singular values of bidiagonal factor (Demmel–Kahan, 1990)

Irreducible (singular) TN matrices

- ▶ Bidiagonal decompositions exist, but not unique:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

Irreducible (singular) TN matrices – 2

- ▶ Bidiagonal decompositions exist, but not unique:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

Subtract 2nd row from 3rd

Irreducible (singular) TN matrices – 3

- ▶ Bidiagonal decompositions exist, but not unique:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

Irreducible (singular) TN matrices – 4

- ▶ Bidiagonal decompositions exist, but not unique:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

Then 1st from 2nd

Irreducible (singular) TN matrices – 5

- ▶ Bidiagonal decompositions exist, but not unique:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

Irreducible (singular) TN matrices – 6

- ▶ Bidiagonal decompositions exist, but not unique:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

- ▶ We want to swap rows 2 and 3 using a TN transformation
- ▶ The obvious solution

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

is *not* TN!

Irreducible (singular) TN matrices – 7

- ▶ Bidiagonal decompositions exist, but not unique:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

- ▶ We want to swap rows 2 and 3 using a TN transformation
- ▶ The obvious solution

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

is *not* TN!

- ▶ Dealing with zeros cuts both ways:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{\text{TN}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Irreducible (singular) TN matrices – 8

- ▶ The bidiagonal decompositions may have zeros on the diagonals:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 1 & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 1 & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 0 & \\ & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 1 & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 0 & \\ & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & 3 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{bmatrix}$$

Tridiagonal reduction breaks

- ▶ All TN linear algebra still possible accurately (unaffected by the new zeros)
- ▶ But tridiagonal reduction can no longer be done with EB matrices only:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ \color{red}{1} & 2 & 4 \end{bmatrix}$$

- ▶ To kill the (3,1) entry we need to form

$$\begin{bmatrix} 1 & & \\ & 1 & \\ -1 & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ \color{red}{1} & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

—the similarity is not NOT TN!!!

Preserve the nonzero eigenvalues

- ▶ We can erase zero rows and columns (TN preserving operations)

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}$$

which preserves the nonzero eigenvalues

- ▶ This yields accurate nonzero eigenvalues; how about the zero ones (and those Jordan blocks)?

Jordan blocks corresponding to zero eigenvalues

- ▶ $n - \text{rank}(A) = \#$ Jordan blocks
- ▶ $\text{rank}(A) - \text{rank}(A^2) = \#$ of Jordan blocks of size ≥ 2
- ▶ ...
- ▶ $\text{rank}(A), \text{rank}(A^2), \dots$ readily obtainable from its BD
- ▶ A^2 is TN (as a product of TN) and its BD is a TN-preserving op, thus BD accurate
- ▶ need to form BD of A^2, \dots, A^n , a potential $O(n^4)$ algorithm

Example

A =

```
3 3 2 1
2 2 3 2
1 1 2 3
1 1 2 3
```

```
>> eig(A)
```

ans =

```
7.828427124746188e+00
2.171572875253811e+00
5.247731480861326e-16
-1.110223024625157e-16
```

```
>> TNEigenValues(BD)
```

ans =

```
7.828427124746190e+00
2.171572875253810e+00
0
0
```

```
>> [TNRank(BD), TNRank(TNProduct(BD, BD))]
```

ans=

```
3      2
```