# Accurate Eigenvalues of Sign Regular Matrices

**Plamen Koev**
**M.I.T.**

**joint work with Froilán Dopico**

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# Recall: Totally Positive Means All Minors $> 0$

- **Examples:**

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 4 & 4^2 & 4^3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/3 & 1/4 & 1/5 \\ 1/3 & 1/4 & 1/5 & 1/6 \\ 1/4 & 1/5 & 1/6 & 1/7 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$$

| Vandermonde | Hilbert | Pascal |
|---|---|---|

- **Eigenvalues**

| | | |
|---|---|---|
| 71.5987 | 1.5002 | 26.3047 |
| 3.6199 | 0.1691 | 2.2034 |
| 0.7168 | 0.0067 | 0.4538 |
| 0.0646 | 0.0001 | 0.0380 |

- **Note: Despite possible nonsymmetry: $\lambda_i > 0$, real!**
- **Virtually all linear algebra possible accurately (2 papers in SIMAX)**

# This Talk: Sign Regular Matrices

- Consider

$$\text{(Totally positive)} \times \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

—called Sign Regular, meaning

**All minors of order $k$ have the same sign $\rho_k \in \{-1, +1\}$**

- In our case $(\rho_1, \rho_2, \ldots) = (+1, -1, -1, +1, \ldots)$
- Generally unsymmetric; NOT similar to TP

# Eigenproblem for Sign Regular (SR) Matrices

- **Examples of SR matrices**

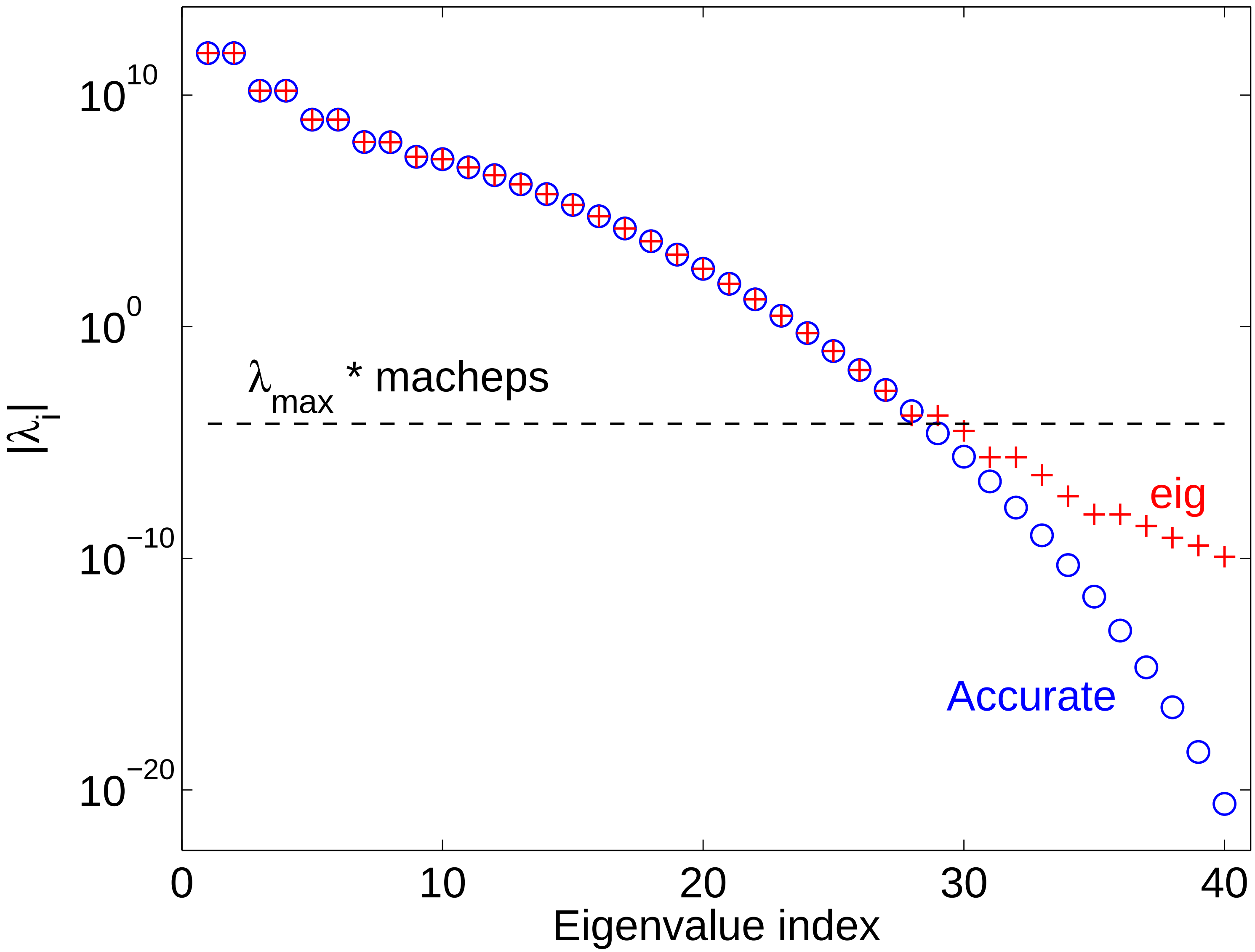
$$\begin{bmatrix} 1 & 4 & 4^2 & 4^3 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1/4 & 1/5 & 1/6 & 1/7 \\ 1/3 & 1/4 & 1/5 & 1/6 \\ 1/2 & 1/3 & 1/4 & 1/5 \\ 1 & 1/2 & 1/3 & 1/4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 4 & 10 & 20 \\ 1 & 3 & 6 & 10 \\ 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

- **Eigenvalues:**

$$\begin{array}{r} 15.4897 \\ -7.7063 \\ 1.2942 \\ -0.0777 \end{array} \qquad \begin{array}{r} 1.1194 \\ -\,0.1261 \\ 0.0068 \\ -\,0.0002 \end{array} \qquad \begin{array}{r} 11.2006 \\ -3.7655 \\ 0.6041 \\ -0.0392 \end{array}$$

- **Note: Despite nonsymmetry, $\lambda_k$ are real; $\text{sign}(\lambda_k) = (-1)^{k-1}$**
- **Notoriously ill conditioned, just like TP**

# Eigenvalues of a $40 \times 40$ Sign Regular Vandermonde




$$V = \left[ x_i^{j-1} \right]_{i,j=1}^{40}, \quad x = (4.0, 3.9, 3.8, \ldots, 0.1)$$

# New Result: Accurate Eigenvalue Algorithm for SR Matrices

- **Takes $O(n^3)$ time**
- **Uses only working precision**
- **Accuracy unaffected by angle between left and right eigenvectors**
- **All eigenvalues computed to high relative accuracy:**

$$|\boldsymbol{\lambda}_i - \hat{\boldsymbol{\lambda}}_i| = \mathcal{O}(\varepsilon)|\boldsymbol{\lambda}_i|$$

**as opposed to**

$$|\boldsymbol{\lambda}_i - \hat{\boldsymbol{\lambda}}_i| = \mathcal{O}(\varepsilon)|\boldsymbol{\lambda}_{\max}|\frac{\mathbf{1}}{\boldsymbol{y}_i^T \boldsymbol{x}_i}$$

- **First example of negative eigenvalues of a nonsymmetric matrix computed accurately**

# Outline

- (Almost) any matrix is similar to a symmetric anti-bidiagonal (eig of latter then easy)

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \longrightarrow \begin{bmatrix} & & + & + \\ & + & + & \\ + & + & & \\ + & & & \end{bmatrix}$$

- Trick: no subtractions in above reduction
- We need a structure revealing representation—product of bidiagonals

# Outline

- **(Almost) any matrix is similar to a symmetric anti-bidiagonal (eig of latter then easy)**

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \longrightarrow \begin{bmatrix} & & + & + \\ & + & + & \\ + & + & & \\ + & & & \end{bmatrix}$$

- **Trick: no subtractions in above reduction**
- **We need a structure revealing representation—product of bidiagonals**

# Similarity Reduction to an Anti-Bidiagonal

- **Goal:**

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \longrightarrow \begin{bmatrix} & & + & + \\ & + & + & \\ + & + & & \\ + & & & \end{bmatrix}$$

**Sign regular** $\longrightarrow$ **anti-bidiagonal**

- **Equivalently:**

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \cdot \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \longrightarrow \begin{bmatrix} + & + & & \\ & + & + & \\ & & + & + \\ & & & + \end{bmatrix} \cdot \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

**Totally Positive** $\cdot$ $J$ $\longrightarrow$ **Bidiagonal** $\cdot$ $J$

**(Prefer TP $\cdot J$ since we know how to do TP linear algebra accurately)**

# Similarity Reduction to an Anti-Bidiagonal

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \cdot \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \longrightarrow \begin{bmatrix} + & + & & \\ & + & + & \\ & & + & + \\ & & & + \end{bmatrix} \cdot \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

**Totally Positive** $\cdot$ $J$ $\longrightarrow$ **Bidiagonal** $\cdot$ $J$

- **Nevermind the possible instability; it won't be an issue**
- **Only 2 operations required; both performed on the TP factor:**
  - **Subtract a multiple of row from next to make a zero**
  - **Add a positive multiple of a column to next/previous**
- **Both operations preserve the TP structure of the left factor**
  $\Rightarrow$ **SR structure preserved at every step!**
- **We know how to perform both operations accurately (later)**

# Similarity Reduction to an Anti-Bidiagonal

- **First reduce to (Upper Triangular) $\times J$**

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ {\color{red}+} & + & + & + \end{bmatrix} \begin{bmatrix} & & & \mathbf{1} \\ & & \mathbf{1} & \\ & \mathbf{1} & & \\ \mathbf{1} & & & \end{bmatrix}$$

# Similarity Reduction to an Anti-Bidiagonal 2

- **First reduce to (Upper Triangular) $\times J$**

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -1 & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

# Similarity Reduction to an Anti-Bidiagonal 3

- **First reduce to (Upper Triangular)** $\times J$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & - & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}$$

# Similarity Reduction to an Anti-Bidiagonal 4

- **First reduce to (Upper Triangular)** $\times J$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & - & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}$$

# Similarity Reduction to an Anti-Bidiagonal 5

- First reduce to (Upper Triangular) $\times J$

$$
\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \color{red}{\mathbf{0}} & + & + & + \end{bmatrix}
\begin{bmatrix} & & & \mathbf{1} \\ & & \mathbf{1} & \\ & \mathbf{1} & & \\ \mathbf{1} & & & \end{bmatrix}
\begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & + & \mathbf{1} \end{bmatrix}
$$

# Similarity Reduction to an Anti-Bidiagonal 6

- **First reduce to (Upper Triangular) $\times J$**

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \color{red}{0} & + & + & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix} \color{blue}{\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}} \color{blue}{\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}}$$

# Similarity Reduction to an Anti-Bidiagonal 7

- **First reduce to (Upper Triangular) $\times J$**

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \color{red}{\mathbf{0}} & + & + & + \end{bmatrix} \color{blue}{\begin{bmatrix} & & & \mathbf{1} \\ & & \mathbf{1} & \\ & \mathbf{1} & & \\ \mathbf{1} & & & \end{bmatrix}} \color{blue}{\begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & + & \mathbf{1} \end{bmatrix}} \color{blue}{\begin{bmatrix} & & & \mathbf{1} \\ & & \mathbf{1} & \\ & \mathbf{1} & & \\ \mathbf{1} & & & \end{bmatrix}} \begin{bmatrix} & & & \mathbf{1} \\ & & \mathbf{1} & \\ & \mathbf{1} & & \\ \mathbf{1} & & & \end{bmatrix}$$

# Similarity Reduction to an Anti-Bidiagonal 8

- **First reduce to (Upper Triangular)** $\times J$

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \color{red}{0} & + & + & + \end{bmatrix} \qquad \color{blue}{\begin{bmatrix} 1 & + & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}} \qquad \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

# Similarity Reduction to an Anti-Bidiagonal 9

- First reduce to (Upper Triangular) $\times J$

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix} \qquad \begin{bmatrix} \mathbf{1} & + & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & & \mathbf{1} \end{bmatrix} \qquad \begin{bmatrix} & & & \mathbf{1} \\ & & \mathbf{1} & \\ & \mathbf{1} & & \\ \mathbf{1} & & & \end{bmatrix}$$

# Similarity Reduction to an Anti-Bidiagonal 10

- **First reduce to (Upper Triangular) $\times J$**

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix} \qquad \begin{bmatrix} \mathbf{1} & + & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & & \mathbf{1} \end{bmatrix} \qquad \begin{bmatrix} & & & \mathbf{1} \\ & & \mathbf{1} & \\ & \mathbf{1} & & \\ \mathbf{1} & & & \end{bmatrix}$$

# Similarity Reduction to an Anti-Bidiagonal 11

- **First reduce to (Upper Triangular)** $\times J$

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix} \begin{bmatrix} & & & \mathbf{1} \\ & & \mathbf{1} & \\ & \mathbf{1} & & \\ \mathbf{1} & & & \end{bmatrix}$$

# Similarity Reduction to an Anti-Bidiagonal 12

- **Making more zeros analogous**

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ \color{red}{+} & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix} \begin{bmatrix} & & & \mathbf{1} \\ & & \mathbf{1} & \\ & \mathbf{1} & & \\ \mathbf{1} & & & \end{bmatrix}$$

# Similarity Reduction to an Anti-Bidiagonal 13

- **Making more zeros analogous**

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & - & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ {\color{red}+} & + & + & + \\ 0 & + & + & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

# Similarity Reduction to an Anti-Bidiagonal 14

- **Making more zeros analogous**

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & - & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ {\color{red}+} & + & + & + \\ 0 & + & + & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

$$\begin{bmatrix} + & {\color{blue}+} & {\color{blue}+} & + \\ + & {\color{blue}+} & {\color{blue}+} & + \\ {\color{red}0} & {\color{blue}+} & {\color{blue}+} & + \\ 0 & {\color{blue}+} & {\color{blue}+} & + \end{bmatrix} \qquad \begin{bmatrix} 1 & & & \\ & 1 & {\color{blue}+} & \\ & & 1 & \\ & & & 1 \end{bmatrix} \qquad \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

# Similarity Reduction to an Anti-Bidiagonal 15

- **Making more zeros analogous**

$$
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & - & 1 & \\ & & & 1 \end{bmatrix}
\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ {\color{red}+} & + & + & + \\ 0 & + & + & + \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & & 1 \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
$$

$$
\begin{bmatrix} + & {\color{blue}+} & {\color{blue}+} & + \\ + & {\color{blue}+} & {\color{blue}+} & + \\ {\color{red}0} & {\color{blue}+} & {\color{blue}+} & + \\ 0 & {\color{blue}+} & {\color{blue}+} & + \end{bmatrix}
\qquad
\begin{bmatrix} 1 & & & \\ & 1 & {\color{blue}+} & \\ & & 1 & \\ & & & 1 \end{bmatrix}
\qquad
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
$$

$$
\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
$$

# Similarity Reduction to an Anti-Bidiagonal 16

- **Making more zeros analogous**

$$
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & - & 1 & \\ & & & 1 \end{bmatrix}
\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & & 1 \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
$$

$$
\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \end{bmatrix}
\qquad
\begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & \\ & & & 1 \end{bmatrix}
\qquad
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
$$

$$
\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
$$

# Similarity Reduction to an Anti-Bidiagonal 17

- **Making more zeros analogous**

$$
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & - & 1 & \\ & & & 1 \end{bmatrix}
\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & & 1 \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
$$

$$
\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \end{bmatrix}
\qquad
\begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & \\ & & & 1 \end{bmatrix}
\qquad
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
$$

$$
\begin{bmatrix} + & + & + & + \\ \color{red}{0} & + & + & + \\ 0 & + & + & + \\ 0 & \color{red}{+} & + & + \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
$$

# Similarity Reduction to an Anti-Bidiagonal 18

- **Making more zeros analogous**

$$
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & - & 1 & \\ & & & 1 \end{bmatrix}
\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & & 1 \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
$$

$$
\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \end{bmatrix}
\qquad
\begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & \\ & & & 1 \end{bmatrix}
\qquad
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
$$

$$
\begin{bmatrix} + & + & + & + \\ 0 & + & + & + \\ 0 & \color{red}{+} & + & + \\ 0 & \color{red}{0} & + & + \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
$$

# Similarity Reduction to an Anti-Bidiagonal 19

- **Making more zeros analogous**

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & - & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \end{bmatrix} \qquad \begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & \\ & & & 1 \end{bmatrix} \qquad \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

$$\begin{bmatrix} + & + & + & + \\ 0 & + & + & + \\ 0 & \color{red}{0} & + & + \\ 0 & 0 & \color{red}{+} & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

# Similarity Reduction to an Anti-Bidiagonal 20

- **Making more zeros analogous**

$$
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & - & 1 & \\ & & & 1 \end{bmatrix}
\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & & 1 \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
$$

$$
\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \end{bmatrix}
\qquad
\begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & \\ & & & 1 \end{bmatrix}
\qquad
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
$$

$$
\begin{bmatrix} + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
$$

# Similarity Reduction to an Anti-Bidiagonal 21

- **(Upper Triangular)** $\times J \longrightarrow$ **(Upper bidiagonal)** $\times J$

$$\begin{bmatrix} + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

# Similarity Reduction to an Anti-Bidiagonal 22

- **(Upper Triangular)$\times J \longrightarrow$ (Upper bidiagonal) $\times J$**

$$\begin{bmatrix} 1 & - & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & \color{red}{+} \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} 1 & + & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

# Similarity Reduction to an Anti-Bidiagonal 

- **(Upper Triangular)**$\times J \longrightarrow$ **(Upper bidiagonal)** $\times J$

$$
\begin{bmatrix} 1 & - & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}
\begin{bmatrix} + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
\begin{bmatrix} 1 & + & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
$$

$$
\begin{bmatrix} + & + & + & 0 \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix}
\qquad
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}
\qquad
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
$$

# Similarity Reduction to an Anti-Bidiagonal 

- (Upper Triangular)$\times J \longrightarrow$ (Upper bidiagonal) $\times\ J$

$$\begin{bmatrix} 1 & - & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & \color{red}{+} \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} 1 & + & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

$$\begin{bmatrix} + & + & \color{red}{+} & \color{red}{0} \\ 0 & + & \color{red}{+} & \color{red}{+} \\ 0 & 0 & \color{red}{+} & \color{red}{+} \\ 0 & 0 & \color{red}{0} & \color{red}{+} \end{bmatrix} \qquad \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix} \qquad \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

$$\begin{bmatrix} + & + & + & 0 \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & \color{red}{+} & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

# Similarity Reduction to an Anti-Bidiagonal 25

- **(Upper Triangular)**$\times J \longrightarrow$ **(Upper bidiagonal)** $\times J$

$$
\begin{bmatrix} 1 & - & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}
\begin{bmatrix} + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
\begin{bmatrix} 1 & + & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
$$

$$
\begin{bmatrix} + & + & + & 0 \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix}
\qquad
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}
\qquad
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
$$

$$
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & - & 1 \end{bmatrix}
\begin{bmatrix} + & + & + & 0 \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & \textcolor{red}{+} & + \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
$$

# Similarity Reduction to an Anti-Bidiagonal 26

- (Upper Triangular)$\times J \longrightarrow$ (Upper bidiagonal) $\times J$

$$\begin{bmatrix} 1 & - & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} 1 & + & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

$$\begin{bmatrix} + & + & + & 0 \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix} \qquad \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix} \qquad \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

$$\begin{bmatrix} + & + & + & 0 \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix} \qquad \begin{bmatrix} 1 & + & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \qquad \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

# Similarity Reduction to an Anti-Bidiagonal 27

- **(Upper Triangular)**$\times J \longrightarrow$ **(Upper bidiagonal)** $\times\ J$

$$\begin{bmatrix} 1 & - & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} 1 & + & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

$$\begin{bmatrix} + & + & + & 0 \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix} \qquad \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix} \qquad \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

$$\begin{bmatrix} + & + & + & 0 \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & {\color{red}0} & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

# Similarity Reduction to an Anti-Bidiagonal 28

- **(Upper Triangular)**$\times J \longrightarrow$ **(Upper bidiagonal)** $\times J$

$$\begin{bmatrix} 1 & - & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} 1 & + & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

$$\begin{bmatrix} + & + & + & 0 \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix} \qquad \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix} \qquad \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

$$\begin{bmatrix} + & + & + & 0 \\ 0 & + & + & \color{red}{+} \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

# Similarity Reduction to an Anti-Bidiagonal 29

- **(Upper Triangular)**$\times J \longrightarrow$ **(Upper bidiagonal)** $\times\ J$

$$
\begin{bmatrix} 1 & - & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}
\begin{bmatrix} + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
\begin{bmatrix} 1 & + & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
$$

$$
\begin{bmatrix} + & + & + & 0 \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix}
\qquad
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}
\qquad
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
$$

$$
\begin{bmatrix} + & + & \color{red}{+} & 0 \\ 0 & + & + & \color{red}{0} \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
$$

# Similarity Reduction to an Anti-Bidiagonal 30

- **(Upper Triangular)**$\times J \longrightarrow$ **(Upper bidiagonal)** $\times J$

$$\begin{bmatrix} 1 & - & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} 1 & + & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

$$\begin{bmatrix} + & + & + & 0 \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix} \qquad \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix} \qquad \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

$$\begin{bmatrix} + & + & 0 & 0 \\ 0 & + & + & 0 \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

# Similarity Reduction to an Anti-Bidiagonal 

- **(Upper Triangular)**$\times J \longrightarrow$ **(Upper bidiagonal)** $\times\, J$

$$
\begin{bmatrix} 1 & - & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}
\begin{bmatrix} + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
\begin{bmatrix} 1 & + & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
$$

$$
\begin{bmatrix} + & + & + & 0 \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix}
\qquad
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}
\qquad
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
$$

$$
\begin{bmatrix} + & + & 0 & 0 \\ 0 & + & + & 0 \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
=
\begin{bmatrix} & & + & + \\ & + & + & \\ + & + & & \\ + & & & \end{bmatrix}
$$

# Similarity Reduction to an Anti-Bidiagonal 32

- **Finally, symmetrizing**

$$\begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & d_4 \end{bmatrix} \begin{bmatrix} & & b_1 & a_1 \\ & b_2 & a_2 & \\ b_3 & a_3 & & \\ a_4 & & & \end{bmatrix} \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & d_4 \end{bmatrix}^{-1}$$

# Similarity Reduction to an Anti-Bidiagonal 32

- **Finally, symmetrizing**

$$\begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & d_4 \end{bmatrix} \begin{bmatrix} & & b_1 & a_1 \\ & b_2 & a_2 & \\ b_3 & a_3 & & \\ a_4 & & & \end{bmatrix} \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & d_4 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} & & \sqrt{b_1 b_3} & \sqrt{a_1 a_4} \\ & b_2 & \sqrt{a_2 a_3} & \\ \sqrt{b_1 b_3} & \sqrt{a_2 a_3} & & \\ \sqrt{a_1 a_4} & & & \end{bmatrix}$$

# Eigenvalues of a Sign Regular Matrix

$$\text{eig(Sign Regular)} = \text{eig}\left(\begin{bmatrix} & & & b_1 & a_1 \\ & & b_2 & a_2 & \\ & b_2 & a_3 & & \\ b_1 & a_2 & & & \\ a_1 & & & & \end{bmatrix}\right)$$

- The latter is symmetric, thus $|\lambda_i| = \sigma_i$
- sign $\lambda_i = (-1)^{i-1}$ (from theory)
- Thus suffices to compute

$$\text{svd}\left(\begin{bmatrix} & & & b_1 & a_1 \\ & & b_2 & a_2 & \\ & b_2 & a_3 & & \\ b_1 & a_2 & & & \\ a_1 & & & & \end{bmatrix}\right) = \text{svd}\left(\begin{bmatrix} a_1 & b_1 & & & \\ & a_1 & b_2 & & \\ & & a_3 & b_2 & \\ & & & a_2 & b_1 \\ & & & & a_1 \end{bmatrix}\right)$$

—the latter solved by Demmel and Kahan 1990

# Outline

- **(Almost) any matrix is similar to a symmetric anti-bidiagonal (eig of latter then easy)**

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \longrightarrow \begin{bmatrix} & & + & + \\ & + & + & \\ + & + & & \\ + & & & \end{bmatrix}$$

- **Trick: no subtractions in above reduction**
- **We need a structure revealing representation—product of bidiagonals**

# No Subtractive Cancellation

- Relative accuracy preserved in $\times, +, /$
  Proof: $(1+\delta)$ factors accumulate multiplicatively
- Subtractions of approximate quantities dangerous:

$$\begin{array}{rl} & .123456789xxx \\ - & .123456789yyy \\ \hline & .000000000zzz \end{array}$$

- Initial data is OK to subtract: $(x_i - y_j)$
- Reduction to anti-bidiagonal involves no subtractions

# Outline

- (Almost) any matrix is similar to a symmetric anti-bidiagonal (eig of latter then easy)

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \longrightarrow \begin{bmatrix} & & + & + \\ & + & + & \\ + & + & & \\ + & & & \end{bmatrix}$$

- Trick: no subtractions in above reduction
- We need a structure revealing representation—product of bidiagonals

# The Need for a Structure Revealing Representation

- Matrix entries are poor choice of parameters
- An $\varepsilon$ perturbation in $(1,2)$ entry of the SR matrix

$$\begin{bmatrix} 1 & 1+\varepsilon \\ 1 & 1 \end{bmatrix} \qquad \lambda_1 \approx 1, \quad \lambda_2 \approx -\varepsilon$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 1+2\varepsilon \\ 1 & 1 \end{bmatrix} \qquad \lambda_1 \approx 1, \quad \lambda_2 \approx -2\varepsilon$$

results in a 100% perturbation in $\lambda_2$

- Better idea: product of bidiagonals

# Bidiagonal Decomposition of Totally Positive Matrices

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 2 & \\ & & & 6 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 1 & & \\ & 1 & 2 & \\ & & 1 & 3 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & 1 & \\ & & 1 & 2 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}}_{U}$$

- **Any TP matrix**

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{43} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} =$$

$$\underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & b_{41} & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & b_{31} & 1 & \\ & & b_{42} & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ b_{21} & 1 & & \\ & b_{32} & 1 & \\ & & b_{43} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} b_{11} & & & \\ & b_{22} & & \\ & & b_{33} & \\ & & & b_{44} \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & b_{12} & & \\ & 1 & b_{23} & \\ & & 1 & b_{34} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & b_{13} & \\ & & 1 & b_{24} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & b_{14} \\ & & & 1 \end{bmatrix}}_{U}$$

- $b_{ij}$ = **multiplier used to make zero in position** $(i, j)$

# Representation of Sign Regular Matrices

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{43} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{bmatrix} =$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{43} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & b_{41} & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & b_{31} & 1 & \\ & & b_{42} & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ b_{21} & 1 & & \\ & b_{32} & 1 & \\ & & b_{43} & 1 \end{bmatrix} \begin{bmatrix} b_{11} & & & \\ & b_{22} & & \\ & & b_{33} & \\ & & & b_{44} \end{bmatrix} \begin{bmatrix} 1 & b_{12} & & \\ & 1 & b_{23} & \\ & & 1 & b_{34} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & b_{13} & \\ & & 1 & b_{24} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & b_{14} \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

- Any $b_{ij} > 0$ yield a SR matrix $\Rightarrow$ SR $=$ octant in $n^2$ space
- For most famous SR matrices, $b_{ij}$ computable accurately
- We choose $b_{ij}$ as inputs:
  - $b_{ij}$ reveal SR structure
  - $b_{ij}$ determine $\lambda_i$ accurately

# Representation of Sign Regular Matrices

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{43} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{bmatrix} =$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{43} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & b_{41} & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & b_{31} & 1 & \\ & & b_{42} & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ b_{21} & 1 & & \\ & b_{32} & 1 & \\ & & b_{43} & 1 \end{bmatrix} \begin{bmatrix} b_{11} & & & \\ & b_{22} & & \\ & & b_{33} & \\ & & & b_{44} \end{bmatrix} \begin{bmatrix} 1 & b_{12} & & \\ & 1 & b_{23} & \\ & & 1 & b_{34} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & b_{13} & \\ & & 1 & b_{24} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & b_{14} \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

- Any $b_{ij} > 0$ yield a SR matrix $\Rightarrow$ SR $=$ octant in $n^2$ space
- For most famous SR matrices, $b_{ij}$ computable accurately
- We choose $b_{ij}$ as inputs:
  - $b_{ij}$ reveal SR structure
  - $b_{ij}$ determine $\lambda_i$ accurately

**Rationale: $\hat{\lambda}_i$, as rational functions of $b_{ij}$, likely contain no subtractions**

# How we avoid subtractions

- Recall: We only need to apply 2 operations to TP:
  - Subtract a multiple of row from next to make a zero
  - Add a positive multiple of a row to next/previous
- Both preserve TP structure
- When applied implicitly, no subtractions are required

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & b_{31} & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ b_{21} & 1 & \\ & b_{32} & 1 \end{bmatrix} \begin{bmatrix} b_{11} & & \\ & b_{22} & \\ & & b_{33} \end{bmatrix} \begin{bmatrix} 1 & b_{12} & \\ & 1 & b_{23} \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & b_{13} \\ & & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & b'_{31} & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ b'_{21} & 1 & \\ & b'_{32} & 1 \end{bmatrix} \begin{bmatrix} b'_{11} & & \\ & b'_{22} & \\ & & b'_{33} \end{bmatrix} \begin{bmatrix} 1 & b'_{12} & \\ & 1 & b'_{23} \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & b'_{13} \\ & & 1 \end{bmatrix}$$

# Conclusions

- New $O(n^3)$ algorithm
- Computes all eigenvalues of a SR matrix to high relative accuracy
- First example of negative eigenvalues of a nonsymmetric matrix computed to high relative accuracy

# Open problems

- Sign regular matrices with other signatures—no parameterization known, let alone algorithms
- Eigenvectors: Products of accurate factors, but what does accuracy mean?

# Other work

- Eliptic PDEs with tetrahedral finite elements (with Demmel and Vavasis)

- This talk, papers, software: `http://math.mit.edu/~plamen`

# Perturbation Theory

$$C = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & b_{41} & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & b_{31} & 1 & \\ & & b_{42} & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ b_{21} & 1 & & \\ & b_{32} & 1 & \\ & & b_{43} & 1 \end{bmatrix} \begin{bmatrix} b_{11} & & & \\ & b_{22} & & \\ & & b_{33} & \\ & & & b_{44} \end{bmatrix} \begin{bmatrix} 1 & b_{12} & & \\ & 1 & b_{23} & \\ & & 1 & b_{34} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & b_{13} & \\ & & 1 & b_{24} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & b_{14} \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

- Fact: Any minor of $C$ is a linear function in any $b_{ij}$ with coefficients of same sign

  $\Rightarrow$ All minors are accurately determined by $b_{ij}$

- In particular, $c_{ij}$ are accurately determined, and in turn, the Perron root, $\lambda_1$, is accurately determined
- Next, consider the second compound matrix, $\mathcal{C}_2$, —the $\binom{n}{2} \times \binom{n}{2}$ matrix with entries all $2 \times 2$ minors of $C$
- The entries of $\mathcal{C}_2$ are negative and accurately determined by $b_{ij}$
- The Perron root of $\mathcal{C}_2$, $\lambda_1\lambda_2$, is also accurately determined
- Thus, $\lambda_2$ is accurately determined

# Accurate $\mathcal{BD}$ of Vandermonde and Cauchy

- $V = \left[x_i^{j-1}\right]_{i,j=1}^{n}$ (TP if $0 < x_1 < x_2 < \cdots < x_n$)

$$D_{ii} = \prod_{j=1}^{i-1}(x_i - x_j), \quad L_{i+1,i}^{(k)} = \prod_{j=n-k}^{i-1} \frac{x_{i+1} - x_{j+1}}{x_i - x_j}, \quad U_{i,i+1}^{(k)} = x_{i+n-k}$$

- $C = \left[\frac{1}{x_i + y_j}\right]_{i,j=1}^{n}$ (TP if $0 < x_1 < \cdots < x_n,\ 0 < y_1 < \cdots < y_n$)

$$\begin{aligned} D_{ii} &= \prod_{k=1}^{i-1} \frac{(x_i - x_k)(y_i - y_k)}{(x_i + y_k)(y_i + x_k)} \\ L_{i,i+1}^{(k)} &= \frac{x_{n-k} + y_{i-n+k+1}}{x_i + y_{i-n+k+1}} \prod_{l=n-k}^{i-1} \frac{x_{i+1} - x_{l+1}}{x_i - x_l} \cdot \prod_{l=1}^{i-n+k-1} \frac{x_i + y_l}{x_{i+1} + y_l} \\ U_{i+1,i}^{(k)} &= \frac{y_{n-k} + x_{i-n+k+1}}{y_i + x_{i-n+k+1}} \prod_{l=n-k}^{i-1} \frac{y_{i+1} - y_{l+1}}{y_i - y_l} \cdot \prod_{l=1}^{i-n+k-1} \frac{y_i + x_l}{y_{i+1} + x_l} \end{aligned}$$

- Similar formulas for Cauchy–Vandermonde, confluent (José Javier Martínez)
- No subtractive cancellation $\Rightarrow$ accurate

# Applying EETs on $\mathcal{BD}(A)$ – I

- **Subtracting a row from next to make a zero**

$$\begin{bmatrix} 1 & 2 & 6 \\ 3 & 10 & 50 \\ 21 & 102 & 615 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & 7 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 3 & 1 & \\ & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 4 & \\ & & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 & \\ & 1 & 5 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & 3 \\ & & 1 \end{bmatrix}$$

# Applying EETs on $\mathcal{BD}(A)$ – II

- **Subtracting a row from next to make a zero**

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & -7 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 6 \\ 3 & 10 & 50 \\ 21 & 102 & 615 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & -7 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & 7 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 3 & 1 & \\ & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 4 & \\ & & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 & \\ & 1 & 5 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & 3 \\ & & 1 \end{bmatrix}$$

# Applying EETs on $\mathcal{BD}(A)$ – III

- **Subtracting a row from next to make a zero**

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & -7 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 6 \\ 3 & 10 & 50 \\ 21 & 102 & 615 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & -7 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & 7 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 3 & 1 & \\ & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 4 & \\ & & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 & \\ & 1 & 5 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & 3 \\ & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 3 & 1 & \\ & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 4 & \\ & & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 & \\ & 1 & 5 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & 3 \\ & & 1 \end{bmatrix}$$

- **Is equivalent to setting an entry of $\mathcal{BD}(A)$ to zero and performing no arithmetic.**

# Adding a multiple of a column to previous – I

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & a & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ b & 1 & \\ & c & 1 \end{bmatrix} \begin{bmatrix} d & & \\ & e & \\ & & f \end{bmatrix} \begin{bmatrix} 1 & g & \\ & 1 & h \\ & & 1 \end{bmatrix} \underline{\begin{bmatrix} 1 & & \\ & 1 & k \\ & & 1 \end{bmatrix} \color{red}{\begin{bmatrix} 1 & & \\ & y & \\ & x & z \end{bmatrix}}}$$

# Adding a multiple of a column to previous – II

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & a & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ b & 1 & \\ & c & 1 \end{bmatrix} \begin{bmatrix} d & & \\ & e & \\ & & f \end{bmatrix} \begin{bmatrix} 1 & g & \\ & 1 & h \\ & & 1 \end{bmatrix} \underline{\begin{bmatrix} 1 & & \\ & 1 & k \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & y & \\ & x & z \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & a & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ b & 1 & \\ & c & 1 \end{bmatrix} \begin{bmatrix} d & & \\ & e & \\ & & f \end{bmatrix} \begin{bmatrix} 1 & g & \\ & 1 & h \\ & & 1 \end{bmatrix} \underline{\begin{bmatrix} 1 & & \\ & y' & \\ & x' & z' \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & k' \\ & & 1 \end{bmatrix}}$$

$$\begin{aligned} x' &= x \\ y' &= y + kx \\ z' &= 1/y' \\ k' &= kz/y_1 \end{aligned}$$

**... it's all qd recurrences**

# Adding a multiple of a column to previous – III

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & a & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ b & 1 & \\ & c & 1 \end{bmatrix} \begin{bmatrix} d & & \\ & e & \\ & & f \end{bmatrix} \begin{bmatrix} 1 & g & \\ & 1 & h \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & k \\ & & 1 \end{bmatrix} \color{red}{\begin{bmatrix} 1 & & \\ & y & \\ & x & z \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & a & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ b & 1 & \\ & c & 1 \end{bmatrix} \begin{bmatrix} d & & \\ & e & \\ & & f \end{bmatrix} \underline{\begin{bmatrix} 1 & g & \\ & 1 & h \\ & & 1 \end{bmatrix} \color{red}{\begin{bmatrix} 1 & & \\ & y' & \\ & x' & z' \end{bmatrix}}} \begin{bmatrix} 1 & & \\ & 1 & \color{blue}{k'} \\ & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & a & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ b & 1 & \\ & c & 1 \end{bmatrix} \begin{bmatrix} d & & \\ & e & \\ & & f \end{bmatrix} \underline{\color{red}{\begin{bmatrix} 1 & & \\ & y'' & \\ & x'' & z'' \end{bmatrix}} \begin{bmatrix} 1 & \color{blue}{g'} & \\ & 1 & \color{blue}{h'} \\ & & 1 \end{bmatrix}} \begin{bmatrix} 1 & & \\ & 1 & \color{blue}{k'} \\ & & 1 \end{bmatrix}$$

# Adding a multiple of a column to previous – IV

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & a & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ b & 1 & \\ & c & 1 \end{bmatrix} \begin{bmatrix} d & & \\ & e & \\ & & f \end{bmatrix} \begin{bmatrix} 1 & g & \\ & 1 & h \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & k \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & y & \\ & x & z \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & a & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ b & 1 & \\ & c & 1 \end{bmatrix} \begin{bmatrix} d & & \\ & e & \\ & & f \end{bmatrix} \begin{bmatrix} 1 & g & \\ & 1 & h \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & y' & \\ & x' & z' \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & k' \\ & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & a & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ b & 1 & \\ & c & 1 \end{bmatrix} \underline{\begin{bmatrix} d & & \\ & e & \\ & & f \end{bmatrix} \begin{bmatrix} 1 & & \\ & y'' & \\ & x'' & z'' \end{bmatrix}} \begin{bmatrix} 1 & g' & \\ & 1 & h' \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & k' \\ & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & a & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ b & 1 & \\ & c & 1 \end{bmatrix} \underline{\begin{bmatrix} 1 & & \\ & 1 & \\ & x''' & 1 \end{bmatrix} \begin{bmatrix} d & & \\ & e' & \\ & & f' \end{bmatrix}} \begin{bmatrix} 1 & g' & \\ & 1 & h' \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & k' \\ & & 1 \end{bmatrix}$$

# Adding a multiple of a column to previous – V

$$\begin{bmatrix}1&&\\&1&\\&a&1\end{bmatrix}\begin{bmatrix}1&&\\b&1&\\&c&1\end{bmatrix}\begin{bmatrix}d&&\\&e&\\&&f\end{bmatrix}\begin{bmatrix}1&g&\\&1&h\\&&1\end{bmatrix}\begin{bmatrix}1&&\\&1&k\\&&1\end{bmatrix}\begin{bmatrix}1&&\\&y&\\&x&z\end{bmatrix}$$

$$=\begin{bmatrix}1&&\\&1&\\&a&1\end{bmatrix}\begin{bmatrix}1&&\\b&1&\\&c&1\end{bmatrix}\begin{bmatrix}d&&\\&e&\\&&f\end{bmatrix}\begin{bmatrix}1&g&\\&1&h\\&&1\end{bmatrix}\begin{bmatrix}1&&\\&y'&\\&x'&z'\end{bmatrix}\begin{bmatrix}1&&\\&1&k'\\&&1\end{bmatrix}$$

$$=\begin{bmatrix}1&&\\&1&\\&a&1\end{bmatrix}\begin{bmatrix}1&&\\b&1&\\&c&1\end{bmatrix}\begin{bmatrix}d&&\\&e&\\&&f\end{bmatrix}\begin{bmatrix}1&&\\&y''&\\&x''&z''\end{bmatrix}\begin{bmatrix}1&g'&\\&1&h'\\&&1\end{bmatrix}\begin{bmatrix}1&&\\&1&k'\\&&1\end{bmatrix}$$

$$=\begin{bmatrix}1&&\\&1&\\&a&1\end{bmatrix}\underline{\begin{bmatrix}1&&\\b&1&\\&c&1\end{bmatrix}\begin{bmatrix}1&&\\&1&\\&x'''&1\end{bmatrix}}\begin{bmatrix}d&&\\&e'&\\&&f'\end{bmatrix}\begin{bmatrix}1&g'&\\&1&h'\\&&1\end{bmatrix}\begin{bmatrix}1&&\\&1&k'\\&&1\end{bmatrix}$$

$$=\begin{bmatrix}1&&\\&1&\\&a&1\end{bmatrix}\underline{\begin{bmatrix}1&&\\b&1&\\&c+x'''&1\end{bmatrix}\begin{bmatrix}d&&\\&e'&\\&&f'\end{bmatrix}}\begin{bmatrix}1&g'&\\&1&h'\\&&1\end{bmatrix}\begin{bmatrix}1&&\\&1&k'\\&&1\end{bmatrix}$$

**Done.**