Accurate Eigenvalues and SVDs of Totally Nonnegative Matrices

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RESULTS

- New $O(n^3)$ algorithms for Eigenvalues and SVD
- of (unsymmetric) Totally Nonnegative (TN) Matrices
- to High relative accuracy

$$|\hat{\lambda}_i - \lambda_i| \le O(\epsilon)|\lambda_i|, \qquad |\hat{\sigma}_i - \sigma_i| \le O(\epsilon)\sigma_i,$$

  – Contrast: Traditional

$$|\hat{\lambda}_i - \lambda_i| \le O(\epsilon)\frac{||A||}{|y^*x|}, \qquad |\hat{\sigma}_i - \sigma_i| \le O(\epsilon)\sigma_1$$

- Motivation:
  – Eigenvalues and SVD well determined
  – Right answer costs $O(n^3)$, same as the wrong

Totally Nonnegative Matrices

- All minors ≥ 0
- In particular $a_{ij} \geq 0$
- $\lambda_i > 0$, real, distinct
- Generally nonsymmetric
- Examples include notoriously ill conditioned matrices:

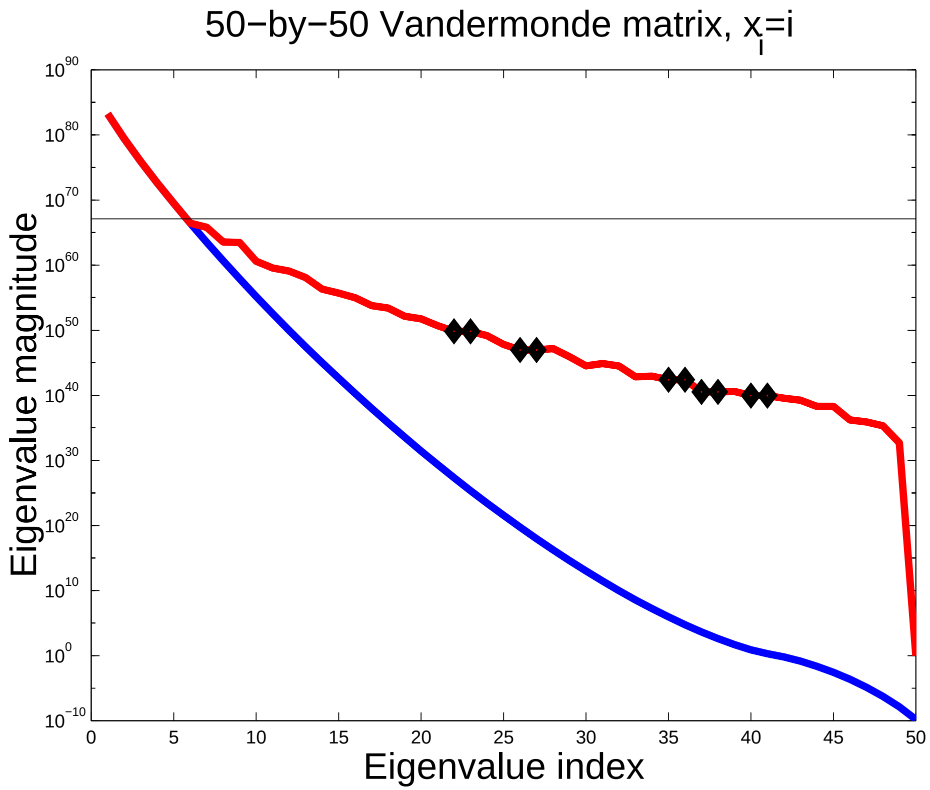
$$\text{Vandermonde} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 4 & 4^2 & 4^3 \end{bmatrix}$$

$$\text{Hilbert} = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/3 & 1/4 & 1/5 \\ 1/3 & 1/4 & 1/5 & 1/6 \\ 1/4 & 1/5 & 1/6 & 1/7 \end{bmatrix} = \left[\frac{1}{i+j-1} \right]$$

  Also Cauchy, Generalized Vandermonde, etc.

- Books by Karlin, Gantmacher, Krein

Example: 50×50 Vandermonde Matrix $V = i^{j-1}$



More properties of TN matrices

- **A positive bidiagonal matrix is TN**
- **An s.p.d. positive tridiagonal is TN**
- **If A, B are TN, then so is**
  - **Any minor of A**
  - **The Schur complement of A**
  - $\mathrm{diag}(\pm 1) \cdot A^{-1} \cdot \mathrm{diag}(\pm 1)$
  - $A \cdot B$
- **Other new results (not in this talk):**
  - **Accurate linear algebra (solutions, eig.vals, SVD, A^{-1}) with TN matrices**
  - **e.g. eigenvalues and SVD of products, minors and Schur complements of TN matrices**
  - **in particular: accurate eigenvalues of**

$$\text{Cauchy} \times \text{Vandermonde} \times \text{Bidiagonal}$$

OUTLINE

- **Subtractions are treacherous**
- **Choose a representation of TN matrices**
  - **matrix entries are bad**
- **Reduce BOTH eigenvalue and SVD problems:**
  - **to bidiagonal SVD Problem (solved: Demmel, Kahan '91)**
  - **by performing no subtractions**

Why are subtractions bad?

- if $\hat{a} \approx a > 0$ and $\hat{b} \approx b > 0$ to (say) 9 digits, then

$$\left.\begin{array}{rcl} \hat{a}\cdot\hat{b} & \approx & a\cdot b \\ \hat{a}+\hat{b} & \approx & a+b \\ \hat{a}/\hat{b} & \approx & a/b \end{array}\right\} \quad \text{to about 9 digits}$$

BUT

$$\hat{a} - \hat{b}$$

may have no correct digits if $a \approx b$:

$$\begin{array}{rr} & .123456789xxx \\ - & .123456789yyy \\ \hline & .000000000zzz \end{array}$$

- So: NO SUBTRACTIONS of approximate quantities
- Initial data is OK to subtract: $(x_i - y_j)$

Representation of TN matrices – I

- **Matrix entries are bad**
- ε **perturbation in (2, 2) entry**

$$\begin{bmatrix} 1 & 1 \\ 1 & 1+\varepsilon \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1+2\varepsilon \end{bmatrix}$$

Representation of TN matrices – II

- **Matrix elements are a poor choice**
- **ϵ perturbation in $(2, 2)$ entry**

$$\begin{bmatrix} 1 & 1 \\ 1 & 1+\varepsilon \end{bmatrix} = \begin{bmatrix} 1 & \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & \\ & \varepsilon \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1+2\varepsilon \end{bmatrix} = \begin{bmatrix} 1 & \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & \\ & 2\varepsilon \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}$$

- **Leads to 100% relative perturbation in $\lambda_{\min}$**
- **We need a structure revealing representation**

Representation of TN matrices – III

- **A product of (TN) nonnegative bidiagonals is TN**
- **Any TN is a (unique) product of nonnegative bidiagonals**

$$\mathcal{BD}\left(\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix}\right) =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ + & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix}\begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{bmatrix}\begin{bmatrix} 1 & + & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix}$$

Neville Elimination – eliminate using adjacent rows

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ \color{red}{1} & \color{red}{3} & \color{red}{9} & \color{red}{27} \\ \color{red}{1} & \color{red}{4} & \color{red}{16} & \color{red}{64} \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix}$$

Neville Elimination

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & \color{red}{1} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ \color{red}{1} & \color{red}{2} & \color{red}{4} & \color{red}{8} \\ \color{red}{1} & \color{red}{3} & \color{red}{9} & \color{red}{27} \\ 0 & 1 & 7 & 37 \end{bmatrix}$$

Neville Elimination

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & {\color{red}1} & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 0 & {\color{red}1} & {\color{red}5} & {\color{red}19} \\ 0 & {\color{red}1} & {\color{red}7} & {\color{red}37} \end{bmatrix}$$

Neville Elimination

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & {\color{red}1} & 1 \end{bmatrix} \begin{bmatrix} {\color{red}1} & {\color{red}1} & {\color{red}1} & {\color{red}1} \\ {\color{red}1} & {\color{red}2} & {\color{red}4} & {\color{red}8} \\ 0 & 1 & 5 & 19 \\ 0 & 0 & 2 & 18 \end{bmatrix}$$

Neville Elimination – We can choose the pattern

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ \color{red}{1} & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & \color{red}{1} & \color{red}{3} & \color{red}{7} \\ 0 & \color{red}{1} & \color{red}{5} & \color{red}{19} \\ 0 & 0 & 2 & 18 \end{bmatrix}$$

Neville Elimination

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & \color{red}{1} & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & \color{red}{2} & \color{red}{12} \\ 0 & 0 & \color{red}{2} & \color{red}{18} \end{bmatrix}$$

Neville Elimination

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & 1 & 1 & \\ & & \color{red}{1} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 2 & 12 \\ 0 & 0 & 0 & \color{red}{6} \end{bmatrix}$$

Neville Elimination

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 2 & \\ & & & 6 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Neville Elimination

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 2 & \\ & & & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}$$

Neville Elimination

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 2 & \\ & & & 6 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 1 & & \\ & 1 & 2 & \\ & & 1 & 3 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & 1 & \\ & & 1 & 2 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}}_{U}$$

Notice: positive bidiagonal decomposition

$$U_{i,i+1}^{(k)} = \frac{\det(A(1:k, i-k+2:i+1))}{\det(A(1:k-1, i-k+2:i))} \cdot \frac{\det(A(1:k-1, i-k+1:i-1))}{\det(A(1:k, i-k+1:i))}.$$

Our Unique Input Representation of a TN matrices – Product of Bidiagonals

$$\mathcal{BD}\left(\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix}\right) =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ + & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{bmatrix} \begin{bmatrix} 1 & + & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix}$$

- Reveals the TN structure
- Determines λ_i, σ_i to high relative accuracy (next slide)
- TN matrices fill an octant in n^2 space
- NASC for $\mathcal{BD}(A)$ to be computable accurately:
  Initial minors must be computable accurately
  (initial = contiguous, include first row or column)
  Formulas:

$$U_{i,i+1}^{(k)} = \frac{\det(A(1:k, i-k+2:i+1))}{\det(A(1:k-1, i-k+2:i))} \cdot \frac{\det(A(1:k-1, i-k+1:i-1))}{\det(A(1:k, i-k+1:i))}.$$

similarly for $D_{ii}, L_{i+1,i}^{(k)}$.

Can we get $\mathcal{BD}(A)$ given A?

- **TN Vandermonde** $(0 < x_1 < x_2 < \cdots < x_n)$

$$V = \left[x_i^{j-1}\right]_{i,j=1}^{n}.$$

$$D_{ii} = \prod_{j=1}^{i-1}(x_i - x_j), \quad L_{i+1,i}^{(k)} = \prod_{j=n-k}^{i-1} \frac{x_{i+1} - x_{j+1}}{x_i - x_j}, \quad U_{i,i+1}^{(k)} = x_{i+n-k}$$

- **TN Cauchy** $(x_1 < \cdots < x_n,\ y_1 < \cdots < y_n\ , x_1 + y_1 > 0)$

$$C = \left[\frac{1}{x_i + y_j}\right]_{i,j=1}^{n}$$

$$\begin{aligned}
D_{ii} &= \prod_{k=1}^{i-1} \frac{(x_i - x_k)(y_i - y_k)}{(x_i + y_k)(y_i + x_k)} \\
L_{i,i+1}^{(k)} &= \frac{x_{n-k} + y_{i-n+k+1}}{x_i + y_{i-n+k+1}} \prod_{l=n-k}^{i-1} \frac{x_{i+1} - x_{l+1}}{x_i - x_l} \cdot \prod_{l=1}^{i-n+k-1} \frac{x_i + y_l}{x_{i+1} + y_l} \\
U_{i+1,i}^{(k)} &= \frac{y_{n-k} + x_{i-n+k+1}}{y_i + x_{i-n+k+1}} \prod_{l=n-k}^{i-1} \frac{y_{i+1} - y_{l+1}}{y_i - y_l} \cdot \prod_{l=1}^{i-n+k-1} \frac{y_i + x_l}{y_{i+1} + x_l}.
\end{aligned}$$

Thm (K.): $\mathcal{BD}(A)$ determines λ_i, σ_i to high relative accuracy

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Proof:

- **Cauchy–Binet – all minors are determined accurately.**
- **Entries of the kth compound matrix are all kth order minors of A, thus ≥ 0 and determined accurately.**
- **Thus so is the Perron root $\lambda_1 \ldots \lambda_k$ and two-norm $\sigma_1 \ldots \sigma_k$.**

Idea for Eigenvalue Problem

- Reduce $\mathcal{BD}(A)$ to $\mathcal{BD}$(Tridiagonal)

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ + & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix}\begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{bmatrix}\begin{bmatrix} 1 & + & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & & & \\ * & 1 & & \\ & * & 1 & \\ & & * & 1 \end{bmatrix}\begin{bmatrix} * & & & \\ & * & & \\ & & * & \\ & & & * \end{bmatrix}\begin{bmatrix} 1 & * & & \\ & 1 & * & \\ & & 1 & * \\ & & & 1 \end{bmatrix}$$

- By transforming $\mathcal{BD}(A)$ (and never forming A)
- NEVER subtracting

Recall: Reduction of a nonsymmetric matrix to tridiagonal form

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 2

$$\begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & - & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ {\color{red}+} & {\color{red}+} & {\color{red}+} & {\color{red}+} \\ {\color{red}+} & {\color{red}+} & {\color{red}+} & {\color{red}+} \end{bmatrix}$$

$$= \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 3

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & - & 1 \end{bmatrix} \begin{bmatrix} + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}$$

$$= \begin{bmatrix} + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ \mathbf{0} & + & \color{red}{+} & \color{red}{+} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 4

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & - & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}$$

$$= \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}$$

$$= \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 5

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ \color{red}{+} & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 6

$$\begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & - & \mathbf{1} & \\ & & & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & + & + \\ {\color{red}+} & {\color{red}+} & {\color{red}+} & {\color{red}+} \\ {\color{red}+} & {\color{red}+} & {\color{red}+} & {\color{red}+} \\ \mathbf{0} & + & + & + \end{bmatrix}$$

$$= \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ {\color{red}\mathbf{0}} & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 7

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & - & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ {\color{red}+} & {\color{red}+} & {\color{red}+} & {\color{red}+} \\ {\color{red}+} & {\color{red}+} & {\color{red}+} & {\color{red}+} \\ 0 & + & + & + \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} + & {\color{red}+} & {\color{red}+} & + \\ + & {\color{red}+} & {\color{red}+} & + \\ 0 & {\color{red}+} & {\color{red}+} & + \\ 0 & {\color{red}+} & {\color{red}+} & + \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & & 1 \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 8

$$\begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & - & \mathbf{1} & \\ & & & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & + & + \\ {\color{red}+} & {\color{red}+} & {\color{red}+} & {\color{red}+} \\ {\color{red}+} & {\color{red}+} & {\color{red}+} & {\color{red}+} \\ \mathbf{0} & + & + & + \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & + & \mathbf{1} & \\ & & & \mathbf{1} \end{bmatrix}$$

$$= \begin{bmatrix} + & {\color{red}+} & {\color{red}+} & + \\ + & {\color{red}+} & {\color{red}+} & + \\ \mathbf{0} & {\color{red}+} & {\color{red}+} & + \\ \mathbf{0} & {\color{red}+} & {\color{red}+} & + \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & + & \mathbf{1} & \\ & & & \mathbf{1} \end{bmatrix}$$

$$= \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 9

$$\begin{bmatrix} + & + & + & \color{red}{+} \\ + & + & + & + \\ \mathbf{0} & + & + & + \\ \mathbf{0} & \mathbf{0} & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 10

$$\begin{bmatrix} + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ \mathbf{0} & + & \color{red}{+} & \color{red}{+} \\ \mathbf{0} & \mathbf{0} & \color{red}{+} & \color{red}{+} \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & - \\ & & & \mathbf{1} \end{bmatrix}$$

$$= \begin{bmatrix} + & + & + & \color{red}{\mathbf{0}} \\ + & + & + & + \\ \mathbf{0} & + & + & + \\ \mathbf{0} & \mathbf{0} & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 11

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & - \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & 0 \\ + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 12

$$\begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & + \\ & & & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \\ \mathbf{0} & \mathbf{0} & + & + \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & - \\ & & & \mathbf{1} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & + \\ & & & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & + & \mathbf{0} \\ + & + & + & + \\ \mathbf{0} & + & + & + \\ \mathbf{0} & \mathbf{0} & + & + \end{bmatrix}$$

$$= \begin{bmatrix} + & + & + & \mathbf{0} \\ + & + & + & + \\ \mathbf{0} & + & + & + \\ \mathbf{0} & \mathbf{0} & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 13

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ {\color{red}+} & + & + & + \end{bmatrix} \Rightarrow \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ {\color{red}+} & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix} \Rightarrow \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \\ \mathbf{0} & {\color{red}+} & + & + \end{bmatrix} \Rightarrow \begin{bmatrix} + & + & + & {\color{red}+} \\ + & + & + & + \\ \mathbf{0} & + & + & + \\ \mathbf{0} & \mathbf{0} & + & + \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} + & + & {\color{red}+} & \mathbf{0} \\ + & + & + & + \\ \mathbf{0} & + & + & + \\ \mathbf{0} & \mathbf{0} & + & + \end{bmatrix} \Rightarrow \begin{bmatrix} + & + & \mathbf{0} & \mathbf{0} \\ + & + & + & {\color{red}+} \\ \mathbf{0} & + & + & + \\ \mathbf{0} & \mathbf{0} & + & + \end{bmatrix} \Rightarrow \begin{bmatrix} + & + & \mathbf{0} & \mathbf{0} \\ + & + & + & \mathbf{0} \\ \mathbf{0} & + & + & + \\ \mathbf{0} & \mathbf{0} & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 14

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \Rightarrow \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix} \Rightarrow \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix} \Rightarrow \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \\ \mathbf{0} & \mathbf{0} & + & + \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} + & + & + & \mathbf{0} \\ + & + & + & + \\ \mathbf{0} & + & + & + \\ \mathbf{0} & \mathbf{0} & + & + \end{bmatrix} \Rightarrow \begin{bmatrix} + & + & \mathbf{0} & \mathbf{0} \\ + & + & + & + \\ \mathbf{0} & + & + & + \\ \mathbf{0} & \mathbf{0} & + & + \end{bmatrix} \Rightarrow \begin{bmatrix} + & + & \mathbf{0} & \mathbf{0} \\ + & + & + & \mathbf{0} \\ \mathbf{0} & + & + & + \\ \mathbf{0} & \mathbf{0} & + & + \end{bmatrix}$$

Instead we implicitly transform $\mathcal{BD}(A)$

$$\mathcal{BD}\left(\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix}\right) \Rightarrow \mathcal{BD}\left(\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix}\right) \Rightarrow \mathcal{BD}\left(\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix}\right) \Rightarrow \mathcal{BD}\left(\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \\ \mathbf{0} & \mathbf{0} & + & + \end{bmatrix}\right) \Rightarrow$$

$$\mathcal{BD}\left(\begin{bmatrix} + & + & + & \mathbf{0} \\ + & + & + & + \\ \mathbf{0} & + & + & + \\ \mathbf{0} & \mathbf{0} & + & + \end{bmatrix}\right) \Rightarrow \mathcal{BD}\left(\begin{bmatrix} + & + & \mathbf{0} & \mathbf{0} \\ + & + & + & + \\ \mathbf{0} & + & + & + \\ \mathbf{0} & \mathbf{0} & + & + \end{bmatrix}\right) \Rightarrow \mathcal{BD}\left(\begin{bmatrix} + & + & \mathbf{0} & \mathbf{0} \\ + & + & + & \mathbf{0} \\ \mathbf{0} & + & + & + \\ \mathbf{0} & \mathbf{0} & + & + \end{bmatrix}\right)$$

Reduction of a nonsymmetric matrix to tridiagonal form – 15

$$\mathcal{BD}\left(\begin{bmatrix} + & + & & \\ + & + & + & \\ & + & + & + \\ & & + & + \end{bmatrix}\right) = \begin{bmatrix} 1 & & & \\ l_1 & 1 & & \\ & l_2 & 1 & \\ & & l_3 & 1 \end{bmatrix} \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & d_4 \end{bmatrix} \begin{bmatrix} 1 & u_1 & & \\ & 1 & u_2 & \\ & & 1 & u_3 \\ & & & 1 \end{bmatrix}$$

and the eigenvalues are the squares of the singular values of

$$\begin{bmatrix} \sqrt{d_1} & \sqrt{d_1 l_1 u_1} & & \\ & \sqrt{d_2} & \sqrt{d_2 l_2 u_2} & \\ & & \sqrt{d_3} & \sqrt{d_3 b_3 c_3} \\ & & & \sqrt{d_4} \end{bmatrix}$$

Problem solved by Demmel and Kahan in 1991.
(LAPACK routine `DLASQ1`**)**

Applying a similarity transformation to $\mathcal{BD}(A) - 1$

$$\begin{bmatrix} 1 & 2 & 6 \\ 3 & 10 & 50 \\ 21 & 102 & 615 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & 7 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 3 & 1 & \\ & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 4 & \\ & & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 & \\ & 1 & 5 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & 3 \\ & & 1 \end{bmatrix}$$

Applying a similarity transformation to $\mathcal{BD}(A) - 2$

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & -7 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 6 \\ 3 & 10 & 50 \\ 21 & 102 & 615 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & 7 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & -7 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & 7 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 3 & 1 & \\ & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 4 & \\ & & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 & \\ & 1 & 5 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & 3 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & 7 & 1 \end{bmatrix}$$

Applying a similarity transformation to $\mathcal{BD}(A) - 3$

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & -7 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 6 \\ 3 & 10 & 50 \\ 21 & 102 & 615 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & 7 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & -7 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & 7 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 3 & 1 & \\ & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 4 & \\ & & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 & \\ & 1 & 5 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & 3 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & 7 & 1 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & & \\ & 1 & \\ & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 3 & 1 & \\ & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 4 & \\ & & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 & \\ & 1 & 5 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & 3 \\ & & 1 \end{bmatrix}}_{\mathcal{BD}(\text{matrix})} \underbrace{\begin{bmatrix} 1 & & \\ & 1 & \\ & 7 & 1 \end{bmatrix}}_{\text{bulge}}$$

Chasing the bulge – I

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ b & 1 & \\ & c & 1 \end{bmatrix} \begin{bmatrix} d & & \\ & e & \\ & & f \end{bmatrix} \begin{bmatrix} 1 & g & \\ & 1 & h \\ & & 1 \end{bmatrix} \underline{\begin{bmatrix} 1 & & \\ & 1 & k \\ & & 1 \end{bmatrix} \color{red}{\begin{bmatrix} 1 & & \\ & y & \\ & x & z \end{bmatrix}}}$$

Chasing the bulge – II

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ b & 1 & \\ & c & 1 \end{bmatrix}\begin{bmatrix} d & & \\ & e & \\ & & f \end{bmatrix}\begin{bmatrix} 1 & g & \\ & 1 & h \\ & & 1 \end{bmatrix}\underline{\begin{bmatrix} 1 & & \\ & 1 & k \\ & & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ & y & \\ & x & z \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ b & 1 & \\ & c & 1 \end{bmatrix}\begin{bmatrix} d & & \\ & e & \\ & & f \end{bmatrix}\begin{bmatrix} 1 & g & \\ & 1 & h \\ & & 1 \end{bmatrix}\underline{\begin{bmatrix} 1 & & \\ & y' & \\ & x' & z' \end{bmatrix}\begin{bmatrix} 1 & & \\ & 1 & k' \\ & & 1 \end{bmatrix}}$$

$$\begin{aligned} x' &= x \\ y' &= y + kx \\ z' &= 1/y' \\ k' &= kz/y_1 \end{aligned}$$

... it's all qd recurrences

Chasing the bulge – III

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ b & 1 & \\ & c & 1 \end{bmatrix} \begin{bmatrix} d & & \\ & e & \\ & & f \end{bmatrix} \begin{bmatrix} 1 & g & \\ & 1 & h \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & k \\ & & 1 \end{bmatrix} \color{red}{\begin{bmatrix} 1 & & \\ & y & \\ & x & z \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ b & 1 & \\ & c & 1 \end{bmatrix} \begin{bmatrix} d & & \\ & e & \\ & & f \end{bmatrix} \underline{\begin{bmatrix} 1 & g & \\ & 1 & h \\ & & 1 \end{bmatrix} \color{red}{\begin{bmatrix} 1 & & \\ & y' & \\ & x' & z' \end{bmatrix}}} \begin{bmatrix} 1 & & \\ & 1 & \color{blue}{k'} \\ & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ b & 1 & \\ & c & 1 \end{bmatrix} \begin{bmatrix} d & & \\ & e & \\ & & f \end{bmatrix} \underline{\color{red}{\begin{bmatrix} 1 & & \\ & y'' & \\ & x'' & z'' \end{bmatrix}} \begin{bmatrix} 1 & \color{blue}{g'} & \\ & 1 & \color{blue}{h'} \\ & & 1 \end{bmatrix}} \begin{bmatrix} 1 & & \\ & 1 & \color{blue}{k'} \\ & & 1 \end{bmatrix}$$

Chasing the bulge – IV

$$
\begin{bmatrix} 1 & & \\ & 1 & \\ & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & & \\ b & 1 & \\ & c & 1 \end{bmatrix}
\begin{bmatrix} d & & \\ & e & \\ & & f \end{bmatrix}
\begin{bmatrix} 1 & g & \\ & 1 & h \\ & & 1 \end{bmatrix}
\begin{bmatrix} 1 & & \\ & 1 & k \\ & & 1 \end{bmatrix}
\begin{bmatrix} 1 & & \\ & y & \\ & x & z \end{bmatrix}
$$

$$
= \begin{bmatrix} 1 & & \\ & 1 & \\ & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & & \\ b & 1 & \\ & c & 1 \end{bmatrix}
\begin{bmatrix} d & & \\ & e & \\ & & f \end{bmatrix}
\begin{bmatrix} 1 & g & \\ & 1 & h \\ & & 1 \end{bmatrix}
\begin{bmatrix} 1 & & \\ & y' & \\ & x' & z' \end{bmatrix}
\begin{bmatrix} 1 & & \\ & 1 & k' \\ & & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} 1 & & \\ & 1 & \\ & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & & \\ b & 1 & \\ & c & 1 \end{bmatrix}
\underline{\begin{bmatrix} d & & \\ & e & \\ & & f \end{bmatrix}
\begin{bmatrix} 1 & & \\ & y'' & \\ & x'' & z'' \end{bmatrix}}
\begin{bmatrix} 1 & g' & \\ & 1 & h' \\ & & 1 \end{bmatrix}
\begin{bmatrix} 1 & & \\ & 1 & k' \\ & & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} 1 & & \\ & 1 & \\ & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & & \\ b & 1 & \\ & c & 1 \end{bmatrix}
\underline{\begin{bmatrix} 1 & & \\ & 1 & \\ & x''' & 1 \end{bmatrix}
\begin{bmatrix} d & & \\ & e' & \\ & & f' \end{bmatrix}}
\begin{bmatrix} 1 & g' & \\ & 1 & h' \\ & & 1 \end{bmatrix}
\begin{bmatrix} 1 & & \\ & 1 & k' \\ & & 1 \end{bmatrix}
$$

Chasing the bulge – V

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ b & 1 & \\ & c & 1 \end{bmatrix} \begin{bmatrix} d & & \\ & e & \\ & & f \end{bmatrix} \begin{bmatrix} 1 & g & \\ & 1 & h \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & k \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & y & \\ & x & z \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ b & 1 & \\ & c & 1 \end{bmatrix} \begin{bmatrix} d & & \\ & e & \\ & & f \end{bmatrix} \begin{bmatrix} 1 & g & \\ & 1 & h \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & y' & \\ & x' & z' \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & k' \\ & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ b & 1 & \\ & c & 1 \end{bmatrix} \begin{bmatrix} d & & \\ & e & \\ & & f \end{bmatrix} \begin{bmatrix} 1 & & \\ & y'' & \\ & x'' & z'' \end{bmatrix} \begin{bmatrix} 1 & g' & \\ & 1 & h' \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & k' \\ & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & 0 & 1 \end{bmatrix} \underline{\begin{bmatrix} 1 & & \\ b & 1 & \\ & c & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & x''' & 1 \end{bmatrix}} \begin{bmatrix} d & & \\ & e' & \\ & & f' \end{bmatrix} \begin{bmatrix} 1 & g' & \\ & 1 & h' \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & k' \\ & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & 0 & 1 \end{bmatrix} \underline{\begin{bmatrix} 1 & & \\ b & 1 & \\ & c + x''' & 1 \end{bmatrix} \begin{bmatrix} d & & \\ & e' & \\ & & f' \end{bmatrix}} \begin{bmatrix} 1 & g' & \\ & 1 & h' \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & k' \\ & & 1 \end{bmatrix}$$

DONE!

- $7n^3$ flops; no subtractions; forward stable

The SVD Problem

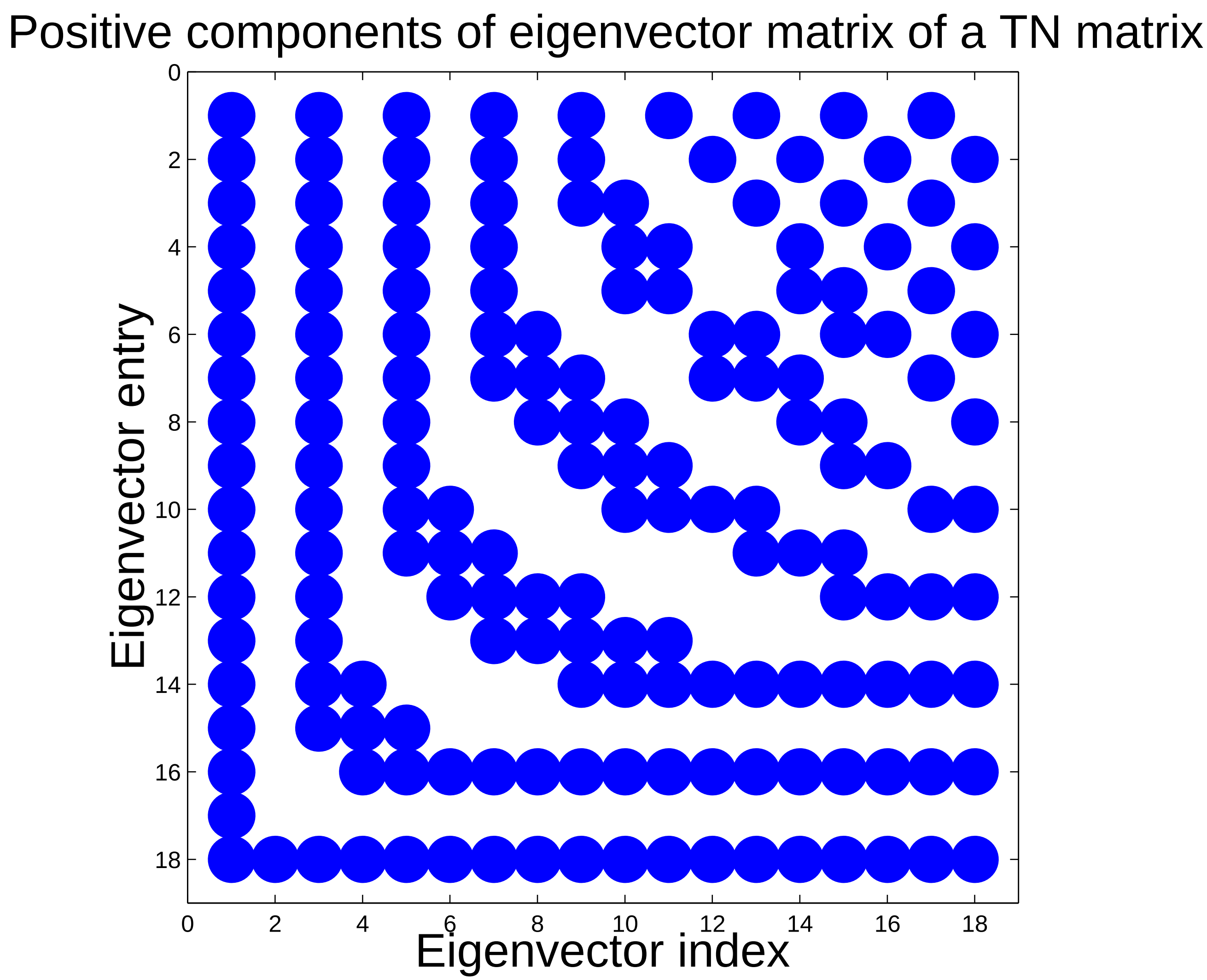
- Givens

$$\begin{bmatrix} c & s \\ -s & c \end{bmatrix} = \begin{bmatrix} + & + \\ 0 & + \end{bmatrix} \begin{bmatrix} 1 & 0 \\ - & 1 \end{bmatrix}$$

- Equals:
  - Subtract a row from next to make a 0
  - Add multiple of next to current; Scale
- TN→BIDIAGONAL→SVD
- Again $7n^3$

Eigenvectors?

- **Eigenvalues are determined accurately, are eigenvectors?**



Eigenvectors?

- **Eigenvector matrix of a TN matrix is a "γ-matrix"**

$$V = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ + & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{bmatrix} \begin{bmatrix} 1 & - & & \\ & 1 & - & \\ & & 1 & - \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & - & \\ & & 1 & - \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & - \\ & & & 1 \end{bmatrix}$$

- **Eigenvectors can be assembled as products of accurate bidiagonal matrices.**
- **Largest and smallest eigenvectors suffer no cancellation**
- **Middle eigenvectors suffer cancellation**
- **A lot of structure in the eigenvectors**
- **Let $P^{-1}AP = T$ —tridiagonal and $Q^TTQ = \Lambda$**
  Q eigenvector matrix of T; $V = PQ$ eigenvector matrix of A

(A) jth column of Q, V has $j-1$ sign changes

(B) Q and V are γ-matrices:

$$Q = LU, \quad V = L'U'$$

where L, L', U^{-1} and $(U')^{-1}$ are TN.

- **A $\not\Longleftrightarrow$ B**
- **How are eigenvectors parameterized? What is accuracy?**
- **Are the entries of the Perron vector of a TN tridiagonal matrix determined accurately by the Cholesky factor.**

Conclusions

- **$O(n^3)$ algorithms for the eigenvalues and the SVD of (*unsymmetric*) TN Matrices to high relative accuracy**
- **Accurate linear algebra with TN matrices closed under same operations as TN**
- **Applies to:**
  - **Oscillatory**
  - **Totally Nonnegative**
  - **Sign Regular (Inverses of TN)**
- **Questions about eigenvectors remain.**
- **This talk, software and paper (to appear in SIMAX):** `math.mit.edu/~plame`