Accurate Jordan Structures of Irreducible Totally Nonnegative Matrices

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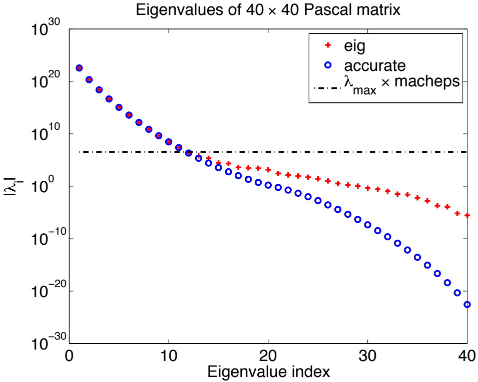
Goal

- Compute the Jordan structure of IrTN matrix in floating point arithmetic to high relative accuracy:
  - all eigenavlues will have correct sign and leading digits (including tiniest ones)
  - the Jordan blocks will be correctly computed

Background

- Spectral structure well understood (Fallat/Johnson/Gekhtman):
  - Positive eigenvalues are distinct
  - Zero eigenvalues can have Jordan blocks
- Computationally hard: TN matrices can be ill conditioned, so accuracy in tiny eigenvalues lost in floating point

Example: 40 $\times$ 40 Pascal matrix, $P = \binom{n}{k}$



$\mathrm{cond}(P) \approx 10^{45}$; eigenvalues $< \lambda_{\max} * 10^{-16}$ lost

Reason accuracy is lost in floating point arithmetic

- Relative accuracy preserved in $\times, +, /$
  Proof: $(1 + \delta)$ factors accumulate multiplicatively
- Subtractions of approximate quantities dangerous:

$$\begin{array}{rl} & .123456789\textit{xxx} \\ - & .123456789\textit{yyy} \\ \hline & .000000000\textit{zzz} \end{array}$$

- Thus, if we avoid subtractions, we get accuracy

Previous results

- All linear algebra with *nonsingular* TN matrices possible accurately (K., '05)
- Trick: represent a matrix by the entries of its bidiagonal decomposition (BD)

$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & \color{red}{1} & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ \color{red}{1} & 1 & \\ & \color{red}{1} & 1 \end{bmatrix} \begin{bmatrix} \color{red}{1} & & \\ & \color{red}{1} & \\ & & \color{red}{2} \end{bmatrix} \begin{bmatrix} 1 & \color{red}{2} & \\ & 1 & \color{red}{3} \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \color{red}{2} \\ & & 1 \end{bmatrix}$$

- Then operate on those entries, NOT on the matrix entries!
- TN-preserving operations require no subtractions $\Rightarrow$ accuracy

Basic Operations

- Subtracting a multiple of one row from next to create a 0 is equivalent to setting an entry of the BD to 0

$$\begin{bmatrix}1&2&4\\1&3&9\\1&4&16\end{bmatrix} = \begin{bmatrix}1&&\\&1&\\&1&1\end{bmatrix}\begin{bmatrix}1&&\\1&1&\\&1&1\end{bmatrix}\begin{bmatrix}1&&\\&1&\\&&2\end{bmatrix}\begin{bmatrix}1&2&\\&1&3\\&&1\end{bmatrix}\begin{bmatrix}1&&\\&1&2\\&&1\end{bmatrix}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\begin{bmatrix}1&2&4\\1&3&9\\0&1&7\end{bmatrix} = \begin{bmatrix}1&&\\&1&\\&0&1\end{bmatrix}\begin{bmatrix}1&&\\1&1&\\&1&1\end{bmatrix}\begin{bmatrix}1&&\\&1&\\&&2\end{bmatrix}\begin{bmatrix}1&2&\\&1&3\\&&1\end{bmatrix}\begin{bmatrix}1&&\\&1&2\\&&1\end{bmatrix}$$

- No subtractions $\Rightarrow$ accuracy
- New matrix still TN

Basic Operations

- Adding a multiple of one row/col to next/previous is done by changing the entries of the BD only

$$\begin{bmatrix}1&2&4\\1&3&9\\0&1&7\end{bmatrix}=\begin{bmatrix}1&&\\&1&\\&0&1\end{bmatrix}\begin{bmatrix}1&&\\1&1&\\&1&1\end{bmatrix}\begin{bmatrix}1&&\\&1&\\&&2\end{bmatrix}\begin{bmatrix}1&2&\\&1&3\\&&1\end{bmatrix}\begin{bmatrix}1&&\\&1&2\\&&1\end{bmatrix}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\begin{bmatrix}1&6&4\\1&12&9\\0&8&7\end{bmatrix}=\begin{bmatrix}1&&\\&1&\\&0&1\end{bmatrix}\begin{bmatrix}1&&\\2&1&\\&\frac{3}{2}&1\end{bmatrix}\begin{bmatrix}1&&\\&6&\\&&2\end{bmatrix}\begin{bmatrix}1&6&\\&1&\frac{4}{3}\\&&1\end{bmatrix}\begin{bmatrix}1&&\\&1&2\\&&1\end{bmatrix}$$

- New entries are rational functions with >0 coefficients
- Again, no subtractions $\Rightarrow$ accuracy
- New matrix is still TN (Cauchy–Binet)
- All TN linear algebra is a combination of above 2 ops (plus scaling, which is trivial)

All Nonsingular TN linear algebra possible accurately

- All TN-preserving ops are multiplications by an EB matrix

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & -2 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & & \\ & 1 & \\ & 3 & 1 \end{bmatrix}$$

- The following can be computed accurately
  - Product
  - LU
  - submatrix
  - R factor of QR (still TN)
  - Converse, ...
- Next: How eigenvalue computation works

Eigenvalues of Nonsingular TN matrices

- Reduction to tridiagonal form using above similarities
- To create a 0 in position (3,1) of

$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}$$

- We use similarity

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 0 & 1 & 7 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 4 \\ 1 & 12 & 9 \\ 0 & 8 & 7 \end{bmatrix}$$

Eigenvalues of Nonsingular TN matrices

- Reduction to tridiagonal form possible using standard approach (Cryer '76)

$$\begin{bmatrix}+&+&+&+\\+&+&+&+\\+&+&+&+\\+&+&+&+\end{bmatrix}\rightarrow\begin{bmatrix}+&+&+&+\\+&+&+&+\\+&+&+&+\\0&+&+&+\end{bmatrix}\rightarrow\begin{bmatrix}+&+&+&+\\+&+&+&+\\0&+&+&+\\0&+&+&+\end{bmatrix}\rightarrow\begin{bmatrix}+&+&+&+\\+&+&+&+\\0&+&+&+\\0&0&+&+\end{bmatrix}\rightarrow$$

$$\begin{bmatrix}+&+&+&0\\+&+&+&+\\0&+&+&+\\0&0&+&+\end{bmatrix}\rightarrow\begin{bmatrix}+&+&0&0\\+&+&+&+\\0&+&+&+\\0&0&+&+\end{bmatrix}\rightarrow\begin{bmatrix}+&+&0&0\\+&+&+&0\\0&+&+&+\\0&0&+&+\end{bmatrix}$$

Irreducible (singular) TN matrices

- At the end we have an (irreducible TN) tridiagonal in factored form

$$\begin{bmatrix} + & + & 0 & 0 \\ + & + & + & 0 \\ 0 & + & + & + \\ 0 & 0 & + & + \end{bmatrix} = \begin{bmatrix} 1 & & & \\ l_1 & 1 & & \\ & l_2 & 1 & \\ & & l_3 & 1 \end{bmatrix} \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & d_4 \end{bmatrix} \begin{bmatrix} 1 & l_1 & & \\ & 1 & l_2 & \\ & & 1 & l_3 \\ & & & 1 \end{bmatrix}$$

- Eigenvalues readily computable accurately as singular values of bidiagonal factor (Demmel–Kahan, 1990)

Irreducible (singular) TN matrices

- Bidiagonal decompositions exist, but not unique
- Some bidiagonal factors may have zeros:

$$
\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}
$$

$$
\begin{aligned}
&= \begin{bmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 1 & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 1 & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 0 & \\ & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 1 & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 0 & \\ & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & 3 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{bmatrix}
\end{aligned}
$$

Tridiagonal reduction breaks

- All TN linear algebra still possible accurately (unaffected by the new zeros)
- But tridiagonal reduction can no longer be done with EB matrices only:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & 4 \end{bmatrix}$$

- To kill the (3,1) entry we need to form

$$\begin{bmatrix} 1 & & \\ & 1 & \\ -1 & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ 1 & & 1 \end{bmatrix}$$

—the similarity is not NOT TN!!!

Preserve the nonzero eigenvalues

- We can erase zero rows and columns (TN preserving operations)

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}$$

  which preserves the nonzero eigenvalues
- This yields accurate nonzero eigenvalues; how about the zero ones (and those Jordan blocks)?

Jordan blocks corresponding to zero eigenvalues

- $n - \mathrm{rank}(A) =$# zero eigenvalues
- $\mathrm{rank}(A) - \mathrm{rank}(A^2) =$# of Jordan blocks of size 2
- ...
- The $\mathrm{rank}(A)$ readily obtainable from its BD
- A^2 is TN (as a product of TN) and its BD is a TN-preserving op, thus BD accurate
- need to form BD of $A^2, \ldots, A^n$, a potential $O(n^4)$ algorithm