Total Positivity, Bidiagonal Decompositions, and Variation–Diminishing Properties

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Focus of talk: Totally Positive (TP) matrices

Def: TP $\Leftrightarrow$ <u>all</u> minors > 0

Ref: Ando, Gantmacher–Krein, Fallat, Johnson, Karlin, Neumann, Peña, Zelevinsky, ...

Appl: Combinatorics, approximation theory, stochastic analysis, CG, vibrations, ...

Examples:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 4 & 4^2 & 4^3 \end{bmatrix} \quad \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/3 & 1/4 & 1/5 \\ 1/3 & 1/4 & 1/5 & 1/6 \\ 1/4 & 1/5 & 1/6 & 1/7 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$$

Vandermonde **Hilbert** **Pascal**

Properties:

Example of a TP Vandermonde matrix

```
>> A=vander([1 2 3 4]); A=A(:,end:-1:1)
```

Example of a TP Vandermonde matrix

```
>> A=vander([1 2 3 4]); A=A(:,end:-1:1)
A =
     1     1     1     1
     1     2     4     8
     1     3     9    27
     1     4    16    64
```

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>> [V,D]=eig(A)
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>> [V,D]=eig(A)
V =
    0.0203   -0.4017   -0.8945   -0.5273
    0.1276   -0.6737   -0.0872    0.7817
    0.3982   -0.5850    0.4295   -0.3305
    0.9082    0.2063   -0.0889    0.0420
D =
   71.5987         0         0         0
         0    3.6199         0         0
         0         0    0.7168         0
         0         0         0    0.0646
```

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         0         0         0    0.0646
```

- A **is TP. How do we know?**
- **Observe: (1)** $\lambda_i > 0$, **real!** **(2)** $\mathcal{S}(j$**th eigenvector**$) = j - 1$

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Examples:

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 4 & 4^2 & 4^3 \end{bmatrix}}_{\text{Vandermonde}} \quad \underbrace{\begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/3 & 1/4 & 1/5 \\ 1/3 & 1/4 & 1/5 & 1/6 \\ 1/4 & 1/5 & 1/6 & 1/7 \end{bmatrix}}_{\text{Hilbert}} \quad \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}}_{\text{Pascal}}$$

Properties:

- > 0 (real!) eigenvalues
- TP inherited by product, Schur complementation, $|A^{-1}|$, ...
- $\mathcal{S}(j\text{th eigenvector}) = j - 1$
- Variation–diminishing property: $\mathcal{S}(Ax) \le \mathcal{S}(x)$
- Closure = totally nonnegative (TN) matrices

Questions:

- How does one tell if a matrix is TP? (4^n minors to check)
- Elementary proofs of above properties?
- Characterization of eigenvector matrices?

The Idea of Using Bidiagonal Decompositions

- **Whitney '50s, Gasca–Peña '90s, Fallat–Johnson–Neumann '00s**
- **Main idea:**
  - n^2 **matrix entries are an unfortunate (conventional) choice of parameters (TP = complicated inequalities)**
  - **another set of** n^2 **parameters much better (TP = positivity of new parameters)**
- **New parameters are the multipliers and pivots in "Neville elimination"**
- **Once the matrix is reparameterized, most properties are easy to prove**
- **Next: Numerics—another reason not to use matrix entries as parameters**

TP Matrices are a Numerical Nightmare, Unless ...

- Consider computing the eigenvalues of the Pascal matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & \\ 1 & 2 & 3 & 4 & \\ 1 & 3 & 6 & 10 & \\ 1 & 4 & 10 & 20 & \\ & & & & \ddots \end{bmatrix}$$

- They come in reciprocal pairs:

$$\lambda_i \lambda_{n-i+1} = 1$$

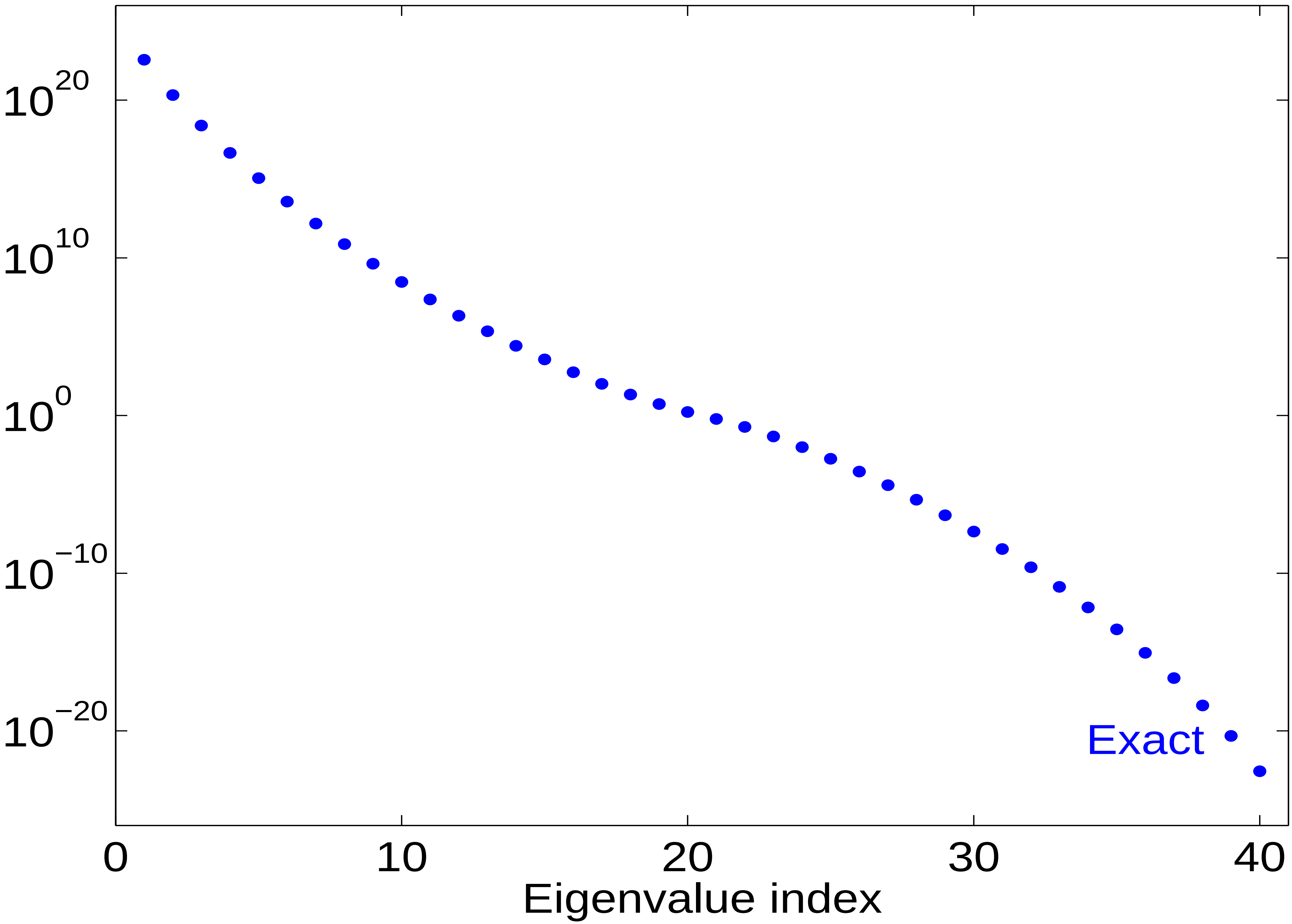
because

$$A^{-1} = \begin{bmatrix} 1 & -1 & 1 & -1 & \\ -1 & 2 & -3 & 4 & \\ 1 & -3 & 6 & -10 & \\ -1 & 4 & -10 & 20 & \\ & & & & \ddots \end{bmatrix} = JAJ^{-1}$$

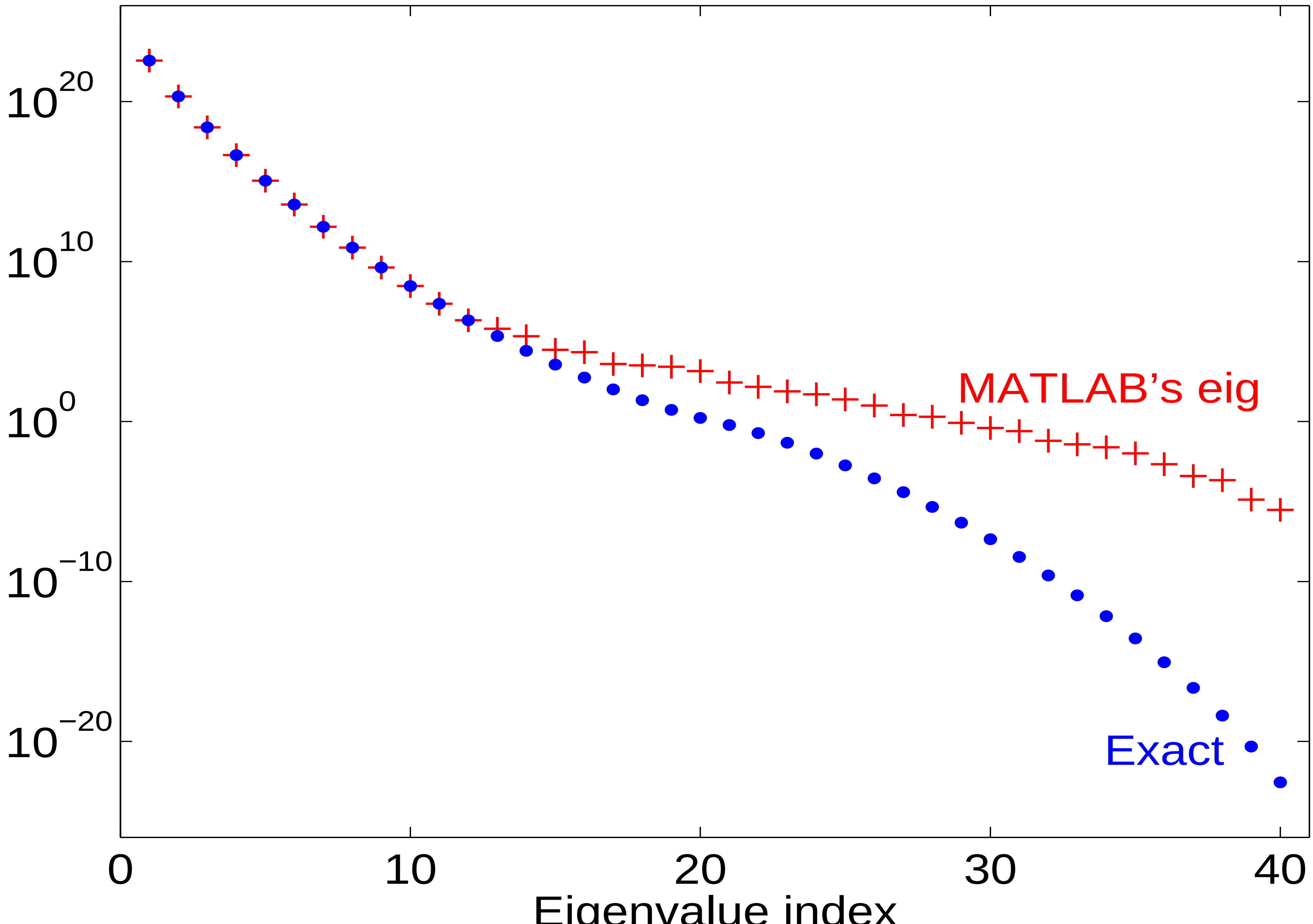
where $J = \mathrm{diag}(1, -1, 1, -1, \ldots, \pm 1)$

- In practice?

Eigenvalues of 40×40 Pascal Matrix



Eigenvalues of 40×40 Pascal Matrix



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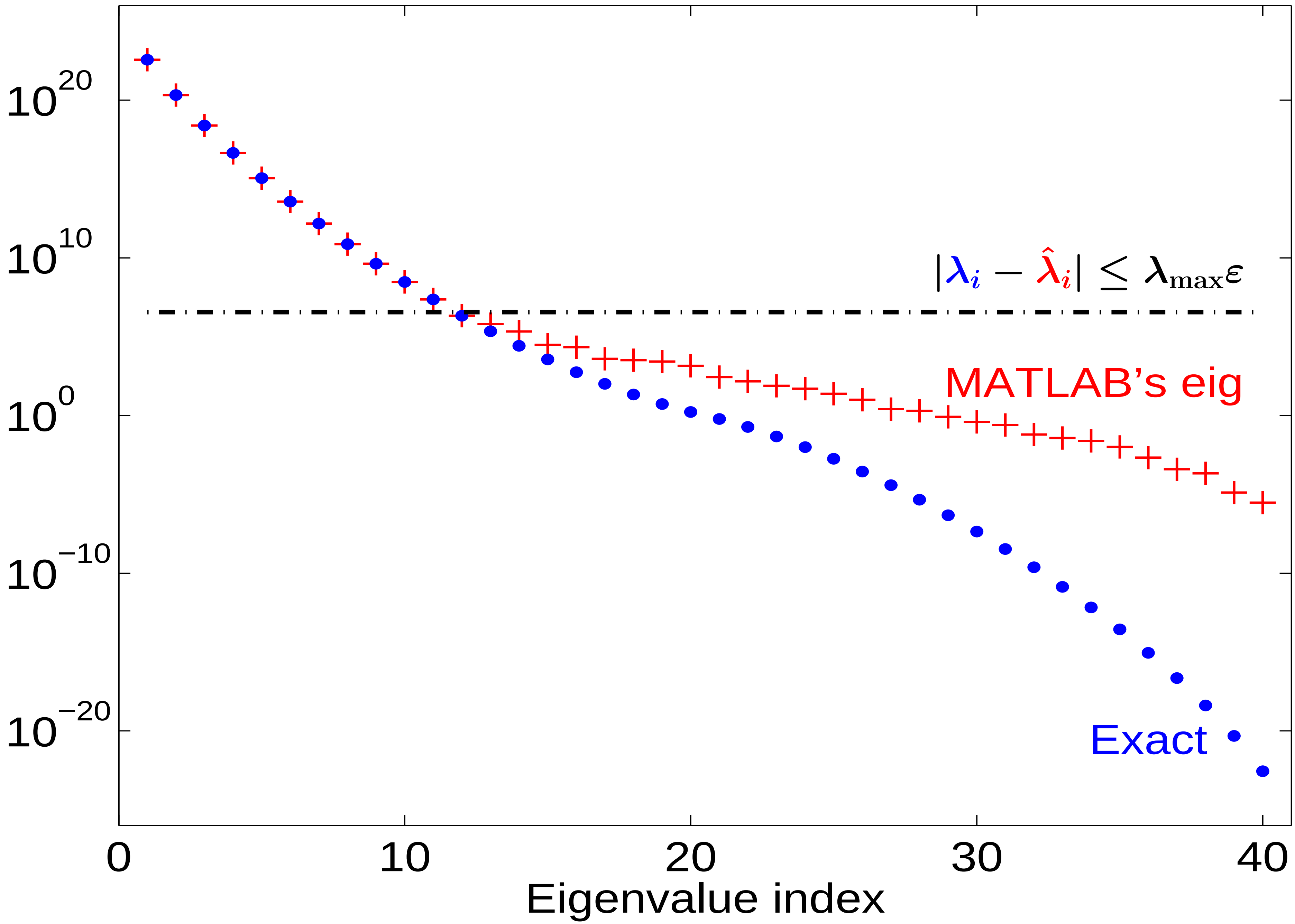
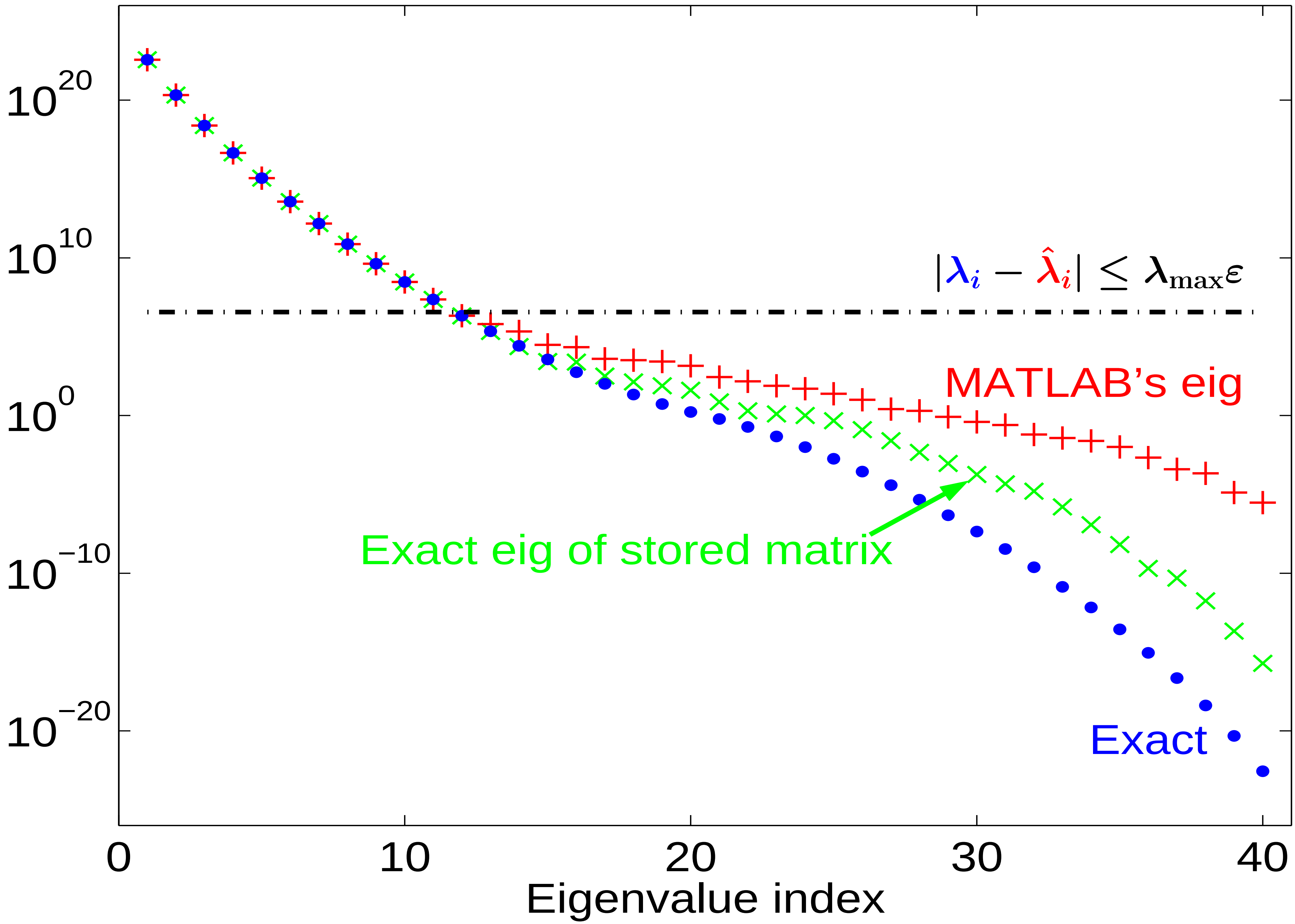


Eigenvalues of 40×40 Pascal Matrix



How is accuracy lost in floating point?

- $\times$, $+$, $/$ preserve relative accuracy
  Proof: $1+\delta$ factors can be factor out
- Subtractions are dangerous:

$$\begin{array}{rl} & .123456789xxx \\ - & .123456789yyy \\ \hline & .000000000zzz \end{array}$$

- Virtually all linear algebra with TP matrices possible subtraction-free

Outline of talk

- **Neville elimination and bidiagonal decompositions**
- **Immediate consequences:**
  - A **is TP** $\Rightarrow |A^{-1}|$ **is TP**
  - **Variation-diminishing property:** $\mathcal{S}(Az) \leq \mathcal{S}(z)$
- **The eigenvalues are real and distinct (+ accurate algorithm free of charge)**

New results:

- **Characterization of eigenvector matrix in terms of bidiagonal decompositions**
  Immediately: Oscillating properties
- **The** Q **matrix in the QR decomposition**

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Neville Elimination

Main idea:

- **Eliminate a matrix using only adjacent rows and columns**

Neville Elimination – 1

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{4} & \mathbf{8} \\ \color{red}{\mathbf{1}} & \color{red}{\mathbf{3}} & \color{red}{\mathbf{9}} & \color{red}{\mathbf{27}} \\ \color{red}{\mathbf{1}} & \color{red}{\mathbf{4}} & \color{red}{\mathbf{16}} & \color{red}{\mathbf{64}} \end{bmatrix} =$$

$$\begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{4} & \mathbf{8} \\ \mathbf{1} & \mathbf{3} & \mathbf{9} & \mathbf{27} \\ \mathbf{1} & \mathbf{4} & \mathbf{16} & \mathbf{64} \end{bmatrix}$$

Neville Elimination – 2

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & \color{red}{1} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ \color{red}{1} & \color{red}{2} & \color{red}{4} & \color{red}{8} \\ \color{red}{1} & \color{red}{3} & \color{red}{9} & \color{red}{27} \\ 0 & 1 & 7 & 37 \end{bmatrix}$$

Neville Elimination – 3

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 0 & 1 & 5 & 19 \\ 0 & 1 & 7 & 37 \end{bmatrix}$$

Neville Elimination – 4

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & {\color{red}1} & 1 \end{bmatrix} \begin{bmatrix} {\color{red}1} & {\color{red}1} & {\color{red}1} & {\color{red}1} \\ {\color{red}1} & {\color{red}2} & {\color{red}4} & {\color{red}8} \\ 0 & 1 & 5 & 19 \\ 0 & 0 & 2 & 18 \end{bmatrix}$$

Neville Elimination – 5

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & \color{red}{1} & 1 \end{bmatrix} \begin{bmatrix} \color{red}{1} & \color{red}{1} & \color{red}{1} & \color{red}{1} \\ \color{red}{1} & \color{red}{2} & \color{red}{4} & \color{red}{8} \\ 0 & 1 & 5 & 19 \\ 0 & 0 & 2 & 18 \end{bmatrix}$$

Neville Elimination – 6

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ \color{red}{1} & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & \color{red}{1} & \color{red}{3} & \color{red}{7} \\ 0 & \color{red}{1} & \color{red}{5} & \color{red}{19} \\ 0 & 0 & 2 & 18 \end{bmatrix}$$

Neville Elimination – 7

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & {\color{red}1} & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & {\color{red}2} & {\color{red}12} \\ 0 & 0 & {\color{red}2} & {\color{red}18} \end{bmatrix}$$

Neville Elimination – 8

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & 1 & 1 & \\ & & {\color{red}1} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 2 & 12 \\ 0 & 0 & 0 & {\color{red}6} \end{bmatrix}$$

Neville Elimination – 9

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 2 & \\ & & & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Neville Elimination – 10

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 2 & \\ & & & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}$$

Neville Elimination – 11

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 2 & \\ & & & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 & & \\ & 1 & 2 & \\ & & 1 & 3 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & 1 & \\ & & 1 & 2 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}$$

Neville Elimination – 12

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 2 & \\ & & & 6 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 1 & & \\ & 1 & 2 & \\ & & 1 & 3 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & 1 & \\ & & 1 & 2 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}}_{U}$$

Bidiagonal Decomposition of a TP Matrix

$$A = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & l_{41} & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & l_{31} & 1 & \\ & & l_{42} & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ l_{21} & 1 & & \\ & l_{32} & 1 & \\ & & l_{43} & 1 \end{bmatrix} \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & d_4 \end{bmatrix} \begin{bmatrix} 1 & u_{12} & & \\ & 1 & u_{23} & \\ & & 1 & u_{34} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & u_{13} & \\ & & 1 & u_{24} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & u_{14} \\ & & & 1 \end{bmatrix}$$

- **Theorem (Gasca–Peña '96):**

  A is TP $\Leftrightarrow$ Red entries > 0

  $\longrightarrow$ Red entries $= \frac{\text{minor}_1(A)}{\text{minor}_2(A)} \cdot \frac{\text{minor}_3(A)}{\text{minor}_4(A)}$

  $\longrightarrow$ Cauchy–Binet: TP $\times$ TP $=$ TP

- TP $=$ octant in n^2 space

- Reveals the TP structure

- **Idea:** Discard matrix entries, work only with entries of bidiagonal decomposition

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New results:

- Characterization of eigenvector matrix in terms of bidiagonal decompositions
  Immediately: Oscillating properties
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The Inverse of a TP Matrix

$$A = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & l_{41} & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & l_{31} & 1 & \\ & & l_{42} & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ l_{21} & 1 & & \\ & l_{32} & 1 & \\ & & l_{43} & 1 \end{bmatrix} \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & d_4 \end{bmatrix} \begin{bmatrix} 1 & u_{12} & & \\ & 1 & u_{23} & \\ & & 1 & u_{34} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & u_{13} & \\ & & 1 & u_{24} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & u_{14} \\ & & & 1 \end{bmatrix}$$

- **Inverse?**

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & x & 1 & \\ & & & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & -x & 1 & \\ & & & 1 \end{bmatrix}$$

- **Therefore**

$$A^{-1} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -\bar{l}_{41} & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & -\bar{l}_{31} & 1 & \\ & & -\bar{l}_{42} & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ -\bar{l}_{21} & 1 & & \\ & -\bar{l}_{32} & 1 & \\ & & -\bar{l}_{43} & 1 \end{bmatrix} \begin{bmatrix} \bar{d}_1 & & & \\ & \bar{d}_2 & & \\ & & \bar{d}_3 & \\ & & & \bar{d}_4 \end{bmatrix} \begin{bmatrix} 1 & -\bar{u}_{12} & & \\ & 1 & -\bar{u}_{23} & \\ & & 1 & -\bar{u}_{34} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & -\bar{u}_{13} & \\ & & 1 & -\bar{u}_{24} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & -\bar{u}_{14} \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \bar{a}_{11} & -\bar{a}_{21} & \bar{a}_{31} & -\bar{a}_{41} \\ -\bar{a}_{12} & \bar{a}_{22} & -\bar{a}_{32} & \bar{a}_{42} \\ \bar{a}_{13} & -\bar{a}_{23} & \bar{a}_{33} & -\bar{a}_{43} \\ -\bar{a}_{14} & \bar{a}_{24} & -\bar{a}_{34} & \bar{a}_{44} \end{bmatrix}$$

Re-signing and Taking Converse

- **Consider the following operations**

$$A^* = JAJ, \; J = \mathrm{diag}(1, -1, 1, -1, \ldots, \pm 1)$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{21} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}^* = \begin{bmatrix} a_{11} & -a_{12} & a_{13} & -a_{14} \\ -a_{21} & a_{22} & -a_{23} & a_{24} \\ a_{21} & -a_{32} & a_{33} & -a_{34} \\ -a_{41} & a_{42} & -a_{43} & a_{44} \end{bmatrix}$$

$$A^{\#} = \mathcal{I} A \mathcal{I}, \; \mathcal{I} = (\delta_{n-i+1,i}) \text{ “reverse identity”}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{21} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}^{\#} = \begin{bmatrix} a_{44} & a_{43} & a_{42} & a_{41} \\ a_{34} & a_{33} & a_{32} & a_{31} \\ a_{24} & a_{23} & a_{22} & a_{21} \\ a_{14} & a_{13} & a_{12} & a_{11} \end{bmatrix}$$

Operations that Preserve TP

- **Fact:** $C = AB \Rightarrow C^* = A^* B^*$

  Proof: $C = JABJ = (JAJ)(JAJ) = A^* B^*$

- A **is TP** $\Leftrightarrow$ A^{-*} **and** $A^{\#}$ **are TP.**

  Proof:
$$A^{-*} = \begin{bmatrix} 1 & & \\ & 1 & \\ & -\bar{l}_{31} & 1 \end{bmatrix}^* \begin{bmatrix} 1 & & \\ -\bar{l}_{21} & 1 & \\ & -\bar{l}_{32} & 1 \end{bmatrix}^* \begin{bmatrix} \bar{d}_1 & & \\ & \bar{d}_2 & \\ & & \bar{d}_3 \end{bmatrix}^* \begin{bmatrix} 1 & -\bar{u}_{12} & \\ & 1 & -\bar{u}_{23} \\ & & 1 \end{bmatrix}^* \begin{bmatrix} 1 & & \\ & 1 & -\bar{u}_{13} \\ & & 1 \end{bmatrix}^*$$
$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & \bar{l}_{31} & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ \bar{l}_{21} & 1 & \\ & \bar{l}_{32} & 1 \end{bmatrix} \begin{bmatrix} \bar{d}_1 & & \\ & \bar{d}_2 & \\ & & \bar{d}_3 \end{bmatrix} \begin{bmatrix} 1 & \bar{u}_{12} & \\ & 1 & \bar{u}_{23} \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \bar{u}_{13} \\ & & 1 \end{bmatrix}$$

Outline of talk

- **Neville elimination and bidiagonal decompositions**
- **Immediate consequences:**
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  - **Variation-diminishing property:** $\mathcal{S}(Az) \leq \mathcal{S}(z)$
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Variation–Diminishing Property

$$A - \text{TP},\ z - \text{vector} \Rightarrow \mathcal{S}(Az) \le \mathcal{S}(z)$$

- Suffices to prove for

$$\begin{bmatrix} 1 & & & & \\ & \cdot & & & \\ & x & 1 & & \\ & & & \cdot & \\ & & & & 1 \end{bmatrix} \begin{bmatrix} \vdots \\ a \\ b \\ c \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ a \\ b + ax \\ c \\ \vdots \end{bmatrix}, \quad x > 0$$

- Can assume $a > 0, b < 0, c < 0$ (ignore $b = 0$ or $c = 0$ possibilities)
- Then $\mathcal{S}(Az) = \mathcal{S}(z)$, regardless of sign of $b + ax$.

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- **Neville elimination and bidiagonal decompositions**
- **Immediate consequences:**
  - A **is TP** $\Rightarrow |A^{-1}|$ **is TP**
  - **Variation-diminishing property:** $\mathcal{S}(Az) \leq \mathcal{S}(z)$
- **The eigenvalues are real and distinct (+ accurate algorithm free of charge)**

New results:

- **Characterization of eigenvector matrix in terms of bidiagonal decompositions**
  Immediately: Oscillating properties
- **The** Q **matrix in the QR decomposition**

The Eigenvalues are Real and Distinct

- Any TP matrix is similar to an irreducible, positive, s.p.d. tridiagonal
- Similarity known, but unused since unstable on non-TP matrices
- Involves only 2 elementary TP–preserving transformations:
  - Subtracting a row from next to create a zero
  - Adding a positive multiple of a column to previous

- Next: How these operations change the bidiagonal decomposition

Subtracting a Row from Next To Make a Zero

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 1 & 1 & \\ & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & \\ & 1 & 2 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{bmatrix}$$

Subtracting a Row from Next To Make a Zero

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}=\begin{bmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ 1 & 1 & \\ & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & \\ & 1 & 2 \\ & & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{bmatrix}$$

Subtracting a Row from Next To Make a Zero

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 1 & 1 & \\ & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & \\ & 1 & 2 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 0 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 1 & 1 & \\ & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & \\ & 1 & 2 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{bmatrix}$$

- Equivalent to setting an l_{ij} to 0
- In practice: No arithmetic required

Adding a Positive Multiple of a Column to Next

$$A = \begin{bmatrix} 1 & & \\ & 1 & \\ & l_{31} & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ l_{21} & 1 & \\ & l_{32} & 1 \end{bmatrix} \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix} \begin{bmatrix} 1 & u_{12} & \\ & 1 & u_{23} \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & u_{13} \\ & & 1 \end{bmatrix}$$

Adding a Positive Multiple of a Column to Next

$$A\begin{bmatrix}1&&\\&1&\\&x&1\end{bmatrix}=\begin{bmatrix}1&&\\&1&\\&l_{31}&1\end{bmatrix}\begin{bmatrix}1&&\\l_{21}&1&\\&l_{32}&1\end{bmatrix}\begin{bmatrix}d_1&&\\&d_2&\\&&d_3\end{bmatrix}\begin{bmatrix}1&u_{12}&\\&1&u_{23}\\&&1\end{bmatrix}\begin{bmatrix}1&&\\&1&u_{13}\\&&1\end{bmatrix}\begin{bmatrix}1&&\\&1&\\&x&1\end{bmatrix}$$

Adding a Positive Multiple of a Column to Next

$$A\begin{bmatrix}1 & & \\ & 1 & \\ & x & 1\end{bmatrix} = \begin{bmatrix}1 & & \\ & 1 & \\ & l_{31} & 1\end{bmatrix}\begin{bmatrix}1 & & \\ l_{21} & 1 & \\ & l_{32} & 1\end{bmatrix}\begin{bmatrix}d_1 & & \\ & d_2 & \\ & & d_3\end{bmatrix}\begin{bmatrix}1 & u_{12} & \\ & 1 & u_{23} \\ & & 1\end{bmatrix}\begin{bmatrix}1 & & \\ & 1 & u_{13} \\ & & 1\end{bmatrix}\begin{bmatrix}1 & & \\ & 1 & \\ & x & 1\end{bmatrix}$$

$$= \begin{bmatrix}1 & & \\ & 1 & \\ & l'_{31} & 1\end{bmatrix}\begin{bmatrix}1 & & \\ l'_{21} & 1 & \\ & l'_{32} & 1\end{bmatrix}\begin{bmatrix}d'_1 & & \\ & d'_2 & \\ & & d'_3\end{bmatrix}\begin{bmatrix}1 & u'_{12} & \\ & 1 & u'_{23} \\ & & 1\end{bmatrix}\begin{bmatrix}1 & & \\ & 1 & u'_{13} \\ & & 1\end{bmatrix}$$

Adding a Positive Multiple of a Column to Next

$$A\begin{bmatrix}1&&\\&1&\\&x&1\end{bmatrix}=\begin{bmatrix}1&&\\&1&\\&l_{31}&1\end{bmatrix}\begin{bmatrix}1&&\\l_{21}&1&\\&l_{32}&1\end{bmatrix}\begin{bmatrix}d_1&&\\&d_2&\\&&d_3\end{bmatrix}\begin{bmatrix}1&u_{12}&\\&1&u_{23}\\&&1\end{bmatrix}\begin{bmatrix}1&&\\&1&u_{13}\\&&1\end{bmatrix}\begin{bmatrix}1&&\\&1&\\&x&1\end{bmatrix}$$

$$=\begin{bmatrix}1&&\\&1&\\&l'_{31}&1\end{bmatrix}\begin{bmatrix}1&&\\l'_{21}&1&\\&l'_{32}&1\end{bmatrix}\begin{bmatrix}d'_1&&\\&d'_2&\\&&d'_3\end{bmatrix}\begin{bmatrix}1&u'_{12}&\\&1&u'_{23}\\&&1\end{bmatrix}\begin{bmatrix}1&&\\&1&u'_{13}\\&&1\end{bmatrix}$$

$$\begin{aligned}u'_{i-1,j-1}&=u_{i-1,j-1}/(1+u_{i-1,j-1}x)\\u'_{ij}&=u_{ij}(1+u_{i-1,j-1}x)\\d'_i&=d_i/(1+u_{n-i,n-i+1}x)\\d'_{i+1}&=d_{i+1}(1+u_{n-i,n-i+1}x)\\l'_{i-1,j-1}&=l_{i-1,j-1}+x\\l'_{ij}&=l_{ij}l_{i-1,j-1}/l'_{i-1,j-1}\end{aligned}$$

- **Note: No subtractions**

Reduction of a TP Matrix to a Tridiagonal

- Using only:
  - Subtracting a row from next to create a zero
  - Adding a positive multiple of a column to previous

  similarity reduce a TP matrix to tridiagonal form

- Ref: BR algorithm, Geist–Howell-Watkins, SIMAX '99

Reduction of a TP Matrix to a Tridiagonal – 1

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ {\color{red}+} & + & + & + \end{bmatrix}$$

Reduction of a TP Matrix to a Tridiagonal – 2

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & - & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ \color{red}{+} & \color{red}{+} & \color{red}{+} & \color{red}{+} \\ \color{red}{+} & \color{red}{+} & \color{red}{+} & \color{red}{+} \end{bmatrix}$$

$$= \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix}$$

Reduction of a TP Matrix to a Tridiagonal – 3

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & - & 1 \end{bmatrix} \begin{bmatrix} + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}$$

$$= \begin{bmatrix} + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ \mathbf{0} & + & \color{red}{+} & \color{red}{+} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}$$

Reduction of a TP Matrix to a Tridiagonal – 4

$$\begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & - & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & + & \mathbf{1} \end{bmatrix}$$

$$= \begin{bmatrix} + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ \mathbf{0} & + & \color{red}{+} & \color{red}{+} \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & + & \mathbf{1} \end{bmatrix}$$

$$= \begin{bmatrix} + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ + & + & \color{red}{+} & \color{red}{+} \\ \mathbf{0} & + & \color{red}{+} & \color{red}{+} \end{bmatrix}$$

Reduction of a TP Matrix to a Tridiagonal – 5

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ \color{red}{+} & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix}$$

Reduction of a TP Matrix to a Tridiagonal – 6

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & -1 & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \end{bmatrix}$$

$$= \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \end{bmatrix}$$

Reduction of a TP Matrix to a Tridiagonal – 7

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & - & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & & 1 \end{bmatrix}$$

Reduction of a TP Matrix to a Tridiagonal – 8

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & - & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ {\color{red}+} & {\color{red}+} & {\color{red}+} & {\color{red}+} \\ {\color{red}+} & {\color{red}+} & {\color{red}+} & {\color{red}+} \\ 0 & + & + & + \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} + & {\color{red}+} & {\color{red}+} & + \\ + & {\color{red}+} & {\color{red}+} & + \\ 0 & {\color{red}+} & {\color{red}+} & + \\ 0 & {\color{red}+} & {\color{red}+} & + \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \end{bmatrix}$$

Reduction of a TP Matrix to a Tridiagonal – 9

$$\begin{bmatrix} + & + & + & {\color{red}+} \\ + & + & + & + \\ \mathbf{0} & + & + & + \\ \mathbf{0} & \mathbf{0} & + & + \end{bmatrix}$$

Reduction of a TP Matrix to a Tridiagonal – 10

$$\begin{bmatrix} 1 & - & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \color{red}{+} & \color{red}{+} & \color{red}{+} & \color{red}{+} \\ \color{red}{+} & \color{red}{+} & \color{red}{+} & \color{red}{+} \\ 0 & + & + & + \\ 0 & 0 & + & + \end{bmatrix}$$

$$= \begin{bmatrix} + & + & + & \color{red}{0} \\ + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \end{bmatrix}$$

Reduction of a TP Matrix to a Tridiagonal – 11

$$\begin{bmatrix} 1 & - & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \color{red}{+} & \color{red}{+} & \color{red}{+} & \color{red}{+} \\ \color{red}{+} & \color{red}{+} & \color{red}{+} & \color{red}{+} \\ 0 & + & + & + \\ 0 & 0 & + & + \end{bmatrix} \begin{bmatrix} 1 & + & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \color{red}{+} & \color{red}{+} & + & 0 \\ \color{red}{+} & \color{red}{+} & + & + \\ \color{red}{0} & \color{red}{+} & + & + \\ \color{red}{0} & \color{red}{0} & + & + \end{bmatrix} \begin{bmatrix} 1 & + & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

Reduction of a TP Matrix to a Tridiagonal – 12

$$\begin{bmatrix} 1 & - & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \end{bmatrix} \begin{bmatrix} 1 & + & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} + & + & + & 0 \\ + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \end{bmatrix} \begin{bmatrix} 1 & + & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} + & + & + & 0 \\ + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \end{bmatrix}$$

Reduction of a TP Matrix to a Tridiagonal – 13

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ {\color{red}+} & + & + & + \end{bmatrix} \Rightarrow \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ {\color{red}+} & + & + & + \\ {\color{blue}\mathbf{0}} & + & + & + \end{bmatrix} \Rightarrow \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ {\color{blue}\mathbf{0}} & + & + & + \\ {\color{blue}\mathbf{0}} & {\color{red}+} & + & + \end{bmatrix} \Rightarrow \begin{bmatrix} + & + & + & {\color{red}+} \\ + & + & + & + \\ {\color{blue}\mathbf{0}} & + & + & + \\ {\color{blue}\mathbf{0}} & {\color{blue}\mathbf{0}} & + & + \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} + & + & {\color{red}+} & {\color{blue}\mathbf{0}} \\ + & + & + & + \\ {\color{blue}\mathbf{0}} & + & + & + \\ {\color{blue}\mathbf{0}} & {\color{blue}\mathbf{0}} & + & + \end{bmatrix} \Rightarrow \begin{bmatrix} + & + & {\color{blue}\mathbf{0}} & {\color{blue}\mathbf{0}} \\ + & + & + & {\color{red}+} \\ {\color{blue}\mathbf{0}} & + & + & + \\ {\color{blue}\mathbf{0}} & {\color{blue}\mathbf{0}} & + & + \end{bmatrix} \Rightarrow \begin{bmatrix} + & + & {\color{blue}\mathbf{0}} & {\color{blue}\mathbf{0}} \\ + & + & + & {\color{blue}\mathbf{0}} \\ {\color{blue}\mathbf{0}} & + & + & + \\ {\color{blue}\mathbf{0}} & {\color{blue}\mathbf{0}} & + & + \end{bmatrix} \equiv T = ({\color{red}\mathbf{TP}})^{-1} A ({\color{red}\mathbf{TP}})$$

where T is tridiagonal, positive with positive principal minors

- **Next: Eigenvalues of T**

The Eigenvalues of T

$$T = \begin{bmatrix} 1 & & & \\ b_1 & 1 & & \\ & b_2 & 1 & \\ & & b_3 & 1 \end{bmatrix} \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & d_4 \end{bmatrix} \begin{bmatrix} 1 & c_1 & & \\ & 1 & c_2 & \\ & & 1 & c_3 \\ & & & 1 \end{bmatrix}$$

$$\sim \underbrace{\begin{bmatrix} 1 & & & \\ e_1 & 1 & & \\ & e_2 & 1 & \\ & & e_3 & 1 \end{bmatrix} \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & d_4 \end{bmatrix} \begin{bmatrix} 1 & e_1 & & \\ & 1 & e_2 & \\ & & 1 & e_3 \\ & & & 1 \end{bmatrix}}_{\text{s.p.d., tridiagonal}}, \quad e_i = \sqrt{b_i c_i}$$

$\Rightarrow$ $\lambda_i > 0$, distinct

- **Numerically:**
  - No subtractions $\Rightarrow$ accuracy
  - eig(T) via SVD of the bidiagonal factor
    Ref: Demmel–Kahan 1990; LAPACK routine `DLASQ1`

Outline of talk

- **Neville elimination and bidiagonal decompositions**
- **Immediate consequences:**
  - A **is TP** $\Rightarrow |A^{-1}|$ **is TP**
  - **Variation-diminishing property:** $\mathcal{S}(Az) \leq \mathcal{S}(z)$
- **The eigenvalues are real and distinct (+ accurate algorithm free of charge)**

New results:

- **Characterization of eigenvector matrix in terms of bidiagonal decompositions**
  Immediately: Oscillating properties
- **The Q matrix in the QR decomposition**

Structure of the Eigenvector Matrix

- **Goals:**
  - Characterization of eigenvector matrices in terms of bidiagonal factors
  - Prove $\mathcal{S}(j\text{th eigenvector}) = j - 1$

- **Def: LTP Matrix:**
  A nonsingular matrix with $C = LDU$ such that L is "lowerly" TP, i.e.,

$$C = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & l_{41} & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & l_{31} & 1 & \\ & & l_{42} & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ l_{21} & 1 & & \\ & l_{32} & 1 & \\ & & l_{43} & 1 \end{bmatrix} DU$$

- **Next:** If $P^{-1}AP = \Lambda$, then P is LTP

Reduction to Bidiagonal

- **Goal:** P **is LTP, where** $A = P\Lambda P^{-1}$
- **So far:** $A = (\mathbf{TP})T(\mathbf{TP})^{-1}$
- **Next:** $A = (\mathbf{TP})B(\mathbf{TP})^{-1}$ **(Peña, '97), where:**

$$B = \begin{bmatrix} \lambda_1 & \mu_1 & & & \\ & \lambda_2 & \mu_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & \mu_{n-1} \\ & & & & \lambda_n \end{bmatrix}$$

- **This suffices, since the eigenvector matrix of** B **is upper triangular** (U')**:**

$$P = (\mathbf{TP}) \cdot U' = LDUU' = L(DUU')$$

Reduction to Bidiagonal (Peña '97)

- **Suffices to show**

$$T = \begin{bmatrix} a_1 & b_1 & & \\ b_1 & a_2 & b_2 & \\ & b_2 & a_3 & b_3 \\ & & b_3 & a_4 \end{bmatrix} \sim \begin{bmatrix} \lambda_1 & \mu_1 & & & \\ & a'_2 & b'_2 & & \\ & b'_2 & a'_3 & b'_3 & \\ & & b'_3 & a'_4 & \end{bmatrix}$$

- **Can assume Perron vector of T is $e = (1, 1, \ldots, 1)^T$**
  (Otherwise replace T by $D^{-1}TD,\ D = \mathrm{diag}(d)$,
  where $d = (d_1, d_2, \ldots, d_n)$ is the Perron vector)

Reduction to Bidiagonal (Peña '97)

- **Consider the (TP–preserving) similarity:**

$$\begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & -1 & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -1 & 1 \end{bmatrix} \begin{bmatrix} a_1 & c_1 & & \\ b_1 & a_2 & c_2 & \\ & b_2 & a_3 & c_3 \\ & & b_3 & a_4 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

Reduction to Bidiagonal (Peña '97)

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$$\begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & -1 & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -1 & 1 \end{bmatrix} \begin{bmatrix} a_1 & c_1 & & \\ b_1 & a_2 & c_2 & \\ & b_2 & a_3 & c_3 \\ & & b_3 & a_4 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Reduction to Bidiagonal (Peña '97)

- **Consider the (TP–preserving) similarity:**

$$\begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & -1 & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -1 & 1 \end{bmatrix} \begin{bmatrix} a_1 & c_1 & & \\ b_1 & a_2 & c_2 & \\ & b_2 & a_3 & c_3 \\ & & b_3 & a_4 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & -1 & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -1 & 1 \end{bmatrix} \begin{bmatrix} a_1 & c_1 & & \\ b_1 & a_2 & c_2 & \\ & b_2 & a_3 & c_3 \\ & & b_3 & a_4 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & -1 & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & c_1 & & \\ \lambda_1 & a_2 + c_2 & c_2 & \\ \lambda_1 & \lambda_1 & a_3 + c_3 & c_3 \\ \lambda_1 & \lambda_1 & \lambda_1 & a_4 \end{bmatrix}$$

Reduction to Bidiagonal (Peña '97)

- **Consider the (TP–preserving) similarity:**

$$\begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & -1 & 1 & \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -1 & 1 \end{bmatrix}\begin{bmatrix} a_1 & c_1 & & \\ b_1 & a_2 & c_2 & \\ & b_2 & a_3 & c_3 \\ & & b_3 & a_4 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & -1 & 1 & \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -1 & 1 \end{bmatrix}\begin{bmatrix} a_1 & c_1 & & \\ b_1 & a_2 & c_2 & \\ & b_2 & a_3 & c_3 \\ & & b_3 & a_4 \end{bmatrix}\begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & -1 & 1 & \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -1 & 1 \end{bmatrix}\begin{bmatrix} \lambda_1 & c_1 & & \\ \lambda_1 & a_2+c_2 & c_2 & \\ \lambda_1 & \lambda_1 & a_3+c_3 & c_3 \\ \lambda_1 & \lambda_1 & \lambda_1 & a_4 \end{bmatrix}$$

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Reduction to Bidiagonal (Peña '97)

- **Consider the (TP–preserving) similarity:**

$$\begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & -1 & 1 & \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -1 & 1 \end{bmatrix}\begin{bmatrix} a_1 & c_1 & & \\ b_1 & a_2 & c_2 & \\ & b_2 & a_3 & c_3 \\ & & b_3 & a_4 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & -1 & 1 & \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -1 & 1 \end{bmatrix}\begin{bmatrix} a_1 & c_1 & & \\ b_1 & a_2 & c_2 & \\ & b_2 & a_3 & c_3 \\ & & b_3 & a_4 \end{bmatrix}\begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

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$$\begin{bmatrix} 1 &&& \\ -1 & 1 && \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ & 1 && \\ & -1 & 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ && -1 & 1 \end{bmatrix} \begin{bmatrix} a_1 & c_1 && \\ b_1 & a_2 & c_2 & \\ & b_2 & a_3 & c_3 \\ && b_3 & a_4 \end{bmatrix} \begin{bmatrix} 1 &&& \\ 1 & 1 && \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 &&& \\ -1 & 1 && \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ & 1 && \\ & -1 & 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ && -1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & c_1 && \\ \lambda_1 & a_2 + c_2 & c_2 & \\ \lambda_1 & \lambda_1 & a_3 + c_3 & c_3 \\ \lambda_1 & \lambda_1 & \lambda_1 & a_4 \end{bmatrix}$$

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$$\begin{bmatrix} 1 &&& \\ -1 & 1 && \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & c_1 && \\ \lambda_1 & a_2 + c_2 & c_2 & \\ 0 & b_2' & a_3' & c_3 \\ 0 & 0 & b_3' & a_4' \end{bmatrix} = \begin{bmatrix} \lambda_1 & c_1 && \\ 0 & a_2' & c_2 & \\ 0 & b_2' & a_3' & c_3 \\ 0 & 0 & b_3' & a_4' \end{bmatrix}$$

The Eigenvector Matrix is LTP

- **Reduction to tridiagonal:** $A = (\textbf{TP})T(\textbf{TP})^{-1}$
- **Reduction to bidiagonal:** $T = (\textbf{TP})B(\textbf{TP})^{-1}$
- $B = (U')^{-1}\Lambda U'$, **where** U' **is upper triangular**
- **In** $A = P\Lambda P^{-1}$

$$P = (\textbf{TP}) \cdot U' = LDUU' = L(DUU')$$

is LTP

- **Property unaffected by eigenvector normalization**

Outline of talk

- **Neville elimination and bidiagonal decompositions**
- **Immediate consequences:**
  - A **is TP** $\Rightarrow |A^{-1}|$ **is TP**
  - **Variation-diminishing property:** $\mathcal{S}(Az) \leq \mathcal{S}(z)$
- **The eigenvalues are real and distinct (**$+$ **accurate algorithm free of charge)**

New results:

- **Characterization of eigenvector matrix in terms of bidiagonal decompositions**
  Immediately: Oscillating properties
- **The** Q **matrix in the QR decomposition**

Number of Sign Changes in an Eigenvector

- $A = P\Lambda P^{-1}$, where P is LTP

$$P = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & l_{41} & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & l_{31} & 1 & \\ & & l_{42} & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ l_{21} & 1 & & \\ & l_{32} & 1 & \\ & & l_{43} & 1 \end{bmatrix} \begin{bmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{bmatrix}$$

Number of Sign Changes in an Eigenvector – 2

- $A = P\Lambda P^{-1}$, where P is LTP

$$P = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & l_{41} & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & l_{31} & 1 & \\ & & l_{42} & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ l_{21} & 1 & & \\ & l_{32} & 1 & \\ & & l_{43} & 1 \end{bmatrix} \begin{bmatrix} + & - & + & - \\ & + & - & + \\ & & + & - \\ & & & + \end{bmatrix}$$

- Thus $\mathcal{S}(p_j) \leq j - 1$

- Next: $\mathcal{S}(p_j) \geq j - 1$

Goal: $\mathcal{S}(p_j) \geq j - 1$

- **For any vector** z, $\mathcal{S}(z) + \mathcal{S}(Jz) = n - 1, \quad J = \mathrm{diag}(1, -1, \ldots, \pm 1)$
- **Example:**

$$\begin{aligned} z &= \begin{bmatrix} 1 & 2 & -3 & 4 \end{bmatrix}^T \\ Jz &= \begin{bmatrix} 1 & -2 & -3 & -4 \end{bmatrix}^T \end{aligned}$$

$\mathcal{S}(z) + \mathcal{S}(Jz) = 3$

Goal: $\mathcal{S}(p_j) \geq j - 1$, part 2

$$\begin{aligned} A &= P\Lambda P^{-1} \\ \Rightarrow A^{-*\#} &= P^{*\#}\Lambda^{-*\#}P^{-*\#} \text{ is TP} \end{aligned}$$

$\Rightarrow P^{*\#}$ is LTP

- Consider $P^{*\#}$

$$P^{*\#} = \begin{bmatrix} p_{nn} & -p_{n-1,n} & p_{n-2,n} & \cdots & -p_{1n} \\ -p_{n,n-1} & p_{n-1,n-1} & -p_{n-2,n-1} & \cdots & p_{1,n-1} \\ & & & \ddots & \\ p_{n1} & -p_{n-1,1} & p_{n-2,1} & \cdots & p_{11} \end{bmatrix}$$

- $(n - j + 1)$th column of $P^{*\#}$ is re-signed jth column of P in reverse order
- As before $\mathcal{S}(\bar{p}_{n-j+1}) \leq n - j + 1$
- However $\mathcal{S}(\bar{p}_{n-j+1}) + \mathcal{S}(p_j) = n - 1$
- Therefore $\mathcal{S}(p_j) = n - 1 - \mathcal{S}(\bar{p}_{n-j+1}) \geq j - 1$
- Finally: $\mathcal{S}(p_j) = j - 1$

Doubly Lowerly TP Matrices

- A matrix such that P and $P^{*\#}$ are LTP is called LTP^2

- Theorem (Dopico, K., '07):
  - P is LTP^2 $\Rightarrow$ $\mathcal{S}(p_j) = j - 1$
  - P is LTP^2 $\Leftrightarrow$ P is eigenvector matrix of a TP matrix

Outline of talk

- **Neville elimination and bidiagonal decompositions**
- **Immediate consequences:**
  - A **is TP** $\Rightarrow |A^{-1}|$ **is TP**
  - **Variation-diminishing property:** $\mathcal{S}(Az) \leq \mathcal{S}(z)$
- **The eigenvalues are real and distinct (+ accurate algorithm free of charge)**

New results:

- **Characterization of eigenvector matrix in terms of bidiagonal decompositions**
  Immediately: Oscillating properties
- **The Q matrix in the QR decomposition**

The Matrix Q is the QR Decomposition of a TP Matrix

- If $A = QR$ is TP, then Q is LTP (A and Q share L factor)
- Q is also LTP^2
  Proof: If $Q^{*\#} = LU$ and $Q = Q^{-T} = \bar{L}\bar{U}$ then
  $$L^{*\#}U^{*\#} = \bar{L}^{-T}\bar{U}^{-T}$$
  Uniqueness of UL decomposition: $\Rightarrow \bar{L} = L^{-*\#T}, \bar{U} = U^{-*\#T}$

- In particular:
  - Any such Q has the same oscillating properties
  - Q and is an eigenvector of some TP matrix

QR of a Vandermonde

```
>> A=vander([1 2 3 4 5]); A=A(:,end:-1:1)
A =
     1     1     1     1     1
     1     2     4     8    16
     1     3     9    27    81
     1     4    16    64   256
     1     5    25   125   625
>> [Q,R]=qr(A)
Q =
   -0.4472   -0.6325    0.5345   -0.3162   -0.1195
   -0.4472   -0.3162   -0.2673    0.6325    0.4781
   -0.4472    0.0000   -0.5345   -0.0000   -0.7171
   -0.4472    0.3162   -0.2673   -0.6325    0.4781
   -0.4472    0.6325    0.5345    0.3162   -0.1195
R =
   -2.2361   -6.7082  -24.5967 -100.6231 -437.8221
         0    3.1623   18.9737   96.1332  470.5469
         0         0    3.7417   33.6749  218.6197
         0         0         0    3.7947   45.5368
         0         0         0         0   -2.8685
```

Conclusions

- **Bidiagonal decompositions are a powerful tool for study of TP matrices**
- **Through this change in representation, most properties become trivial to prove**
- **Virtually all linear algebra with TP matrices can be performed to high relative accuracy in $\mathcal{O}(n^3)$ time**

- **References:**
  - **Dopico and K., Bidiagonal decompositions of oscillating systems of vectors, submitted to LAA;**
  - **K., Accurate computations with totally nonnegative matrices, SIMAX 29 (2007), 731-751.**

- **This talk, papers, software for TP computations:**

  `http://math.mit.edu/~plamen`

THANK YOU!