Oscillating Vectors and Totally Positive Matrices

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Supported by NSF

ILAS Meeting, Regina, Canada, June 2005

$$\boldsymbol{A} \textbf{ is TP} \Longleftrightarrow \textbf{All minors} > \mathbf{0}$$

SOME APPLICATIONS OF TN MATRICES

- **Vibrations of mechanical systems (Gantmacher, Krein)**
- **Corner cutting algorithms (Cutting corners from polytopes)**
- **Electrical Impedance Tomography (Y. Chen)**
- **Stochastic analysis**

- **Books by Karlin, Gantmacher–Krein, Gasca–Micchelli**
- **Motivated by work of Whitney, Björck, Pereyra, Higham, Gasca, Peña, Kahan, C. Johnson, Fallat**

EXAMPLES

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 4 & 4^2 & 4^3 \end{bmatrix}$$

Vandermonde

$$\begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/3 & 1/4 & 1/5 \\ 1/3 & 1/4 & 1/5 & 1/6 \\ 1/4 & 1/5 & 1/6 & 1/7 \end{bmatrix}$$

Hilbert (Cauchy)

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$$

Pascal

Interested in oscillating properties of TP matrices

MATLAB Example - Variation Diminishing + Oscillating vectors

OSCILLATING PROPERTIES

- **Variation diminishing property**
- **jth eigenvector has $j-1$ changes of sign**
- **New: q_j in the $A = QR$ has $j-1$ changes of sign**

OUTLINE

- **Bidiagonal Decompositions (BD) of TP matrices**
- **New results:**
  - **Connection between oscillating vectors and BD**
  - **If A is TP and $A = QR$, then the columns of Q are oscillating**
- **MATLAB toolbox available**

BIDIAGONAL DECOMPOSITIONS OF TP MATRICES

$$\textbf{PASCAL}(4) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}}_{D}\underbrace{\begin{bmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & 1 & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}}_{U}$$

- **Reveals TN structure**
- **Obtained by elimination using adjacent rows (Neville)**

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- **Reveals TN structure**
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$$\textbf{Vandermonde} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 4 & 4^2 & 4^3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 2 & \\ & & & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 & & \\ & 1 & 2 & \\ & & 1 & 3 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & 1 & \\ & & 1 & 2 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}$$

ANY TP = PRODUCT OF NONNEGATIVE BIDIAGONALS

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & l_{41} & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & l_{31} & 1 & \\ & & l_{42} & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ l_{21} & 1 & & \\ & l_{32} & 1 & \\ & & l_{43} & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & d_4 \end{bmatrix}}_{D}\underbrace{\begin{bmatrix} 1 & u_{12} & & \\ & 1 & u_{23} & \\ & & 1 & u_{34} \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & u_{13} & \\ & & 1 & u_{24} \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & u_{14} \\ & & & 1 \end{bmatrix}}_{U}$$

Theorem: A is TP $\iff l_{ij}, d_i, u_{ij} > 0$

Proof:

$\Leftarrow$ (TP)$\times$(TP) = TP (Cauchy–Binet)

$\Rightarrow$ Red entries $= \dfrac{\det A(1{:}k, i-k+2{:}i+1)}{\det A(1{:}k-1, i-k+2{:}i)} \cdot \dfrac{\det A(1{:}k-1, i-k+1{:}i-1)}{\det A(1{:}k, i-k+1{:}i)}$

ANY TP = PRODUCT OF NONNEGATIVE BIDIAGONALS

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

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$$= \quad \Pi \; E_i(l_{ij}) \quad \cdot D \cdot \quad \Pi \; E_i^T(u_{ij})$$

$$E_i(x) = \begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & x & 1 & & \\ & & & & & 1 & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{bmatrix}$$

VARIATION DIMINISHING PROPERTY

If A is TP, then $S(Az) \leq S(z)$

Suffices to prove $S(E_i(x) \cdot z) \leq S(z)$:

$$E_i(x) \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & & \\ x & 1 & \\ & & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ b + ax \\ c \end{bmatrix},$$

$$x > 0, \qquad \text{WLOG } a > 0.$$

$$\begin{matrix} + & & + \\ + & \longrightarrow & + \\ + & & + \end{matrix} \qquad \begin{matrix} + & & + \\ * & \longrightarrow & * \\ - & & - \end{matrix} \qquad \begin{matrix} + & & + \\ - & \longrightarrow & * \\ + & & + \end{matrix}$$

$$0 \longrightarrow 0 \qquad 1 \longrightarrow 1 \qquad 2 \longrightarrow 0, 2$$

$$\Rightarrow \qquad S(Az) \leq S(z)$$

BD(Eigenvector Matrix)

Let A be TP, $A = V \cdot \Lambda \cdot V^{-1}$, then

$$V = \underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ + & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{bmatrix}}_{D}\underbrace{\begin{bmatrix} 1 & - & & \\ & 1 & - & \\ & & 1 & - \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & - & \\ & & 1 & - \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & - \\ & & & 1 \end{bmatrix}}_{U}$$

. . . suggests the study of Lowerly TP matrices

Lowerly Totally Positive (LTP) Matrices

- $$V = \underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ + & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} * & & & \\ & * & & \\ & & * & \\ & & & * \end{bmatrix}}_{D}\underbrace{\begin{bmatrix} 1 & * & & \\ & 1 & * & \\ & & 1 & * \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & * & \\ & & 1 & * \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & * \\ & & & 1 \end{bmatrix}}_{U}$$
- L is TN, nontrivial minors > 0
- Variation Diminishing Property $\Rightarrow S(v_j) \leq j - 1$
- We want to prove $S(v_j) = j - 1 \ \Rightarrow$ LTP not enough
- V^{-T} eigenvectors of A^T—also TP
- Now V^{-T} is LTP $\Rightarrow S(v_j) \geq j - 1$
- Finally, $S(v_j) = j - 1$

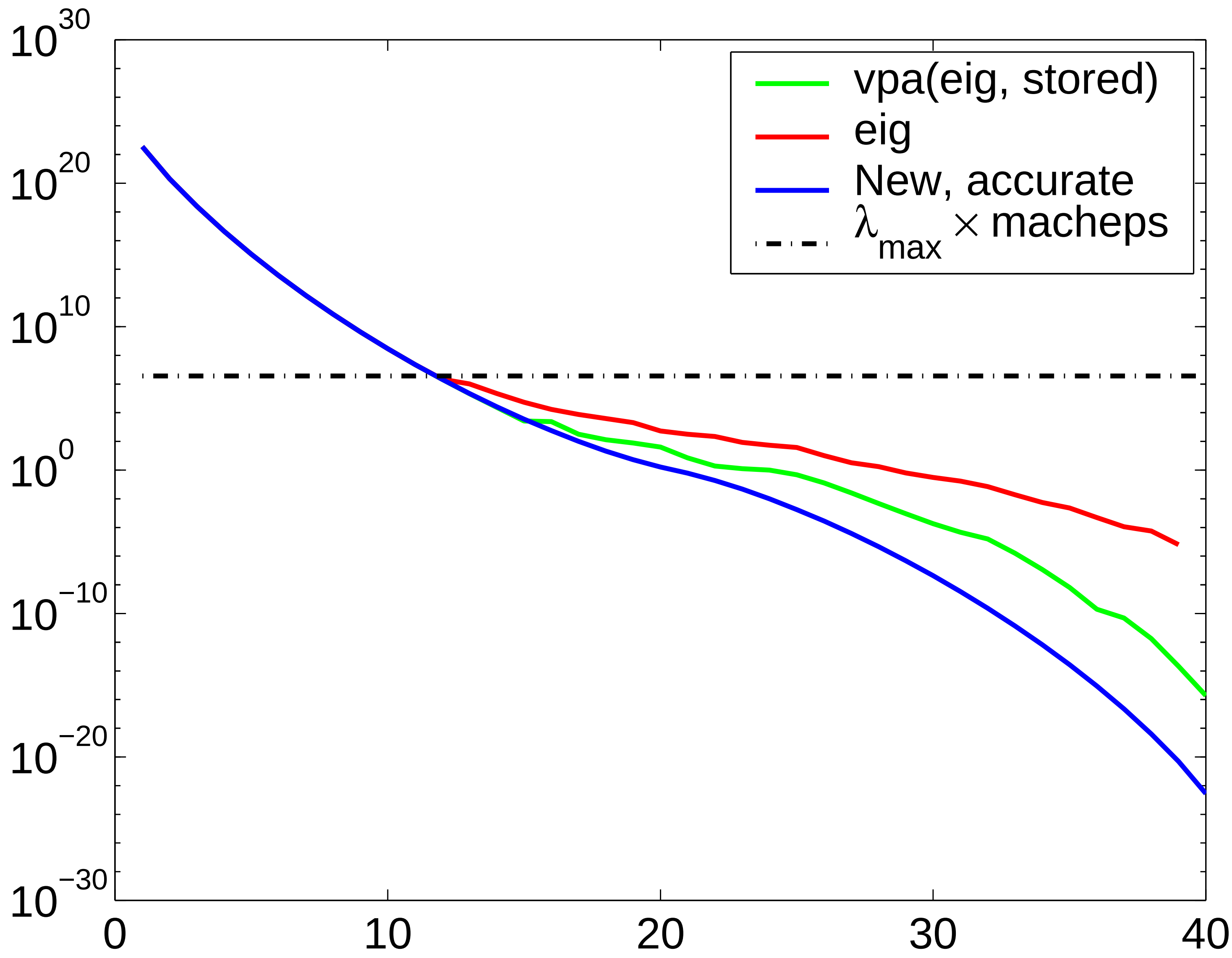
Lowerly Totally Positive (LTP) Matrices

- $V = \underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ + & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} * & & & \\ & * & & \\ & & * & \\ & & & * \end{bmatrix}}_{D}\underbrace{\begin{bmatrix} 1 & * & & \\ & 1 & * & \\ & & 1 & * \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & * & \\ & & 1 & * \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & * \\ & & & 1 \end{bmatrix}}_{U}$
- L is TN, nontrivial minors > 0
- Variation Diminishing Property $\Rightarrow S(v_j) \leq j-1$
- We want to prove $S(v_j) = j-1 \;\Rightarrow$ LTP not enough
- V^{-T} eigenvectors of A^T—also TP
- Now V^{-T} is LTP $\Rightarrow S(v_j) \geq j-1$
- Finally, $S(v_j) = j-1$
- … suggests we study DOUBLY LOWERLY TP matrices

$\mathbf{LTP^2}$ MATRICES

- **Def: V is $\mathbf{LTP^2}$ if V and V^{-T} are LTP**
- **Then (new results):**
  - $S(v_j) = j - 1$
  - **If A is TP, and $A = QR$, then Q is LTP**
    - **But $Q^{-T} = Q$, so Q is $\mathbf{LTP^2}$**
    - **In particular, $S(q_j) = j - 1$**
- **Also works numerically**
- **TP toolbox available**

Computed Eigenvalues of Pascal(40)



CONCLUSIONS

- **Understand what makes matrices have oscillating columns:**

$$\boldsymbol{V} \text{ and } \boldsymbol{V}^{-T} \text{ must be LTP}$$

- **Any such matrix is the eigenvector matrix of some TP matrix**
- **Key: Bidiagonal Decompositions**
- **Slides, papers, software:**

`http://math.mit.edu/~plamen` **(= Google(Plamen Koev))**

CAN WE GET $\mathcal{BD}(A)$ GIVEN A?

- **TN Vandermonde** $(0 < x_1 < x_2 < \cdots < x_n)$

$$V = \left[x_i^{j-1}\right]_{i,j=1}^{n}$$

$$D_{ii} = \prod_{j=1}^{i-1}(x_i - x_j), \quad L_{i+1,i}^{(k)} = \prod_{j=n-k}^{i-1} \frac{x_{i+1} - x_{j+1}}{x_i - x_j}, \quad U_{i,i+1}^{(k)} = x_{i+n-k}$$

- **TN Cauchy** $(x_1 < \cdots < x_n,\ y_1 < \cdots < y_n\ , x_1 + y_1 > 0)$

$$C = \left[\frac{1}{x_i + y_j}\right]_{i,j=1}^{n}$$

$$D_{ii} = \prod_{k=1}^{i-1} \frac{(x_i - x_k)(y_i - y_k)}{(x_i + y_k)(y_i + x_k)}$$

$$L_{i,i+1}^{(k)} = \frac{x_{n-k} + y_{i-n+k+1}}{x_i + y_{i-n+k+1}} \prod_{l=n-k}^{i-1} \frac{x_{i+1} - x_{l+1}}{x_i - x_l} \cdot \prod_{l=1}^{i-n+k-1} \frac{x_i + y_l}{x_{i+1} + y_l}$$

$$U_{i+1,i}^{(k)} = \frac{y_{n-k} + x_{i-n+k+1}}{y_i + x_{i-n+k+1}} \prod_{l=n-k}^{i-1} \frac{y_{i+1} - y_{l+1}}{y_i - y_l} \cdot \prod_{l=1}^{i-n+k-1} \frac{y_i + x_l}{y_{i+1} + x_l}$$