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Accurate Computations with Totally Nonnegative Matrices

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Is it possible to perform numerical linear algebra with structured matrices to high relative accuracy at a reasonable cost?

In our talk we answer this question affirmatively for a class of structured matrices whose applications range from approximation theory to combinatorics to multivariate statistical analysis [1, 2, 4]—the *Totally Nonnegative* (TN) matrices, i.e. matrices all of whose minors are nonnegative.

The trick in performing accurate computations with TN matrices is to choose a representation that reveals their TN structure. The matrix entries are a poor choice of parameters and it is difficult to tell from them if all 4^n minors are nonnegative (and thus the matrix is TN).

Instead, we represent any TN matrix A as a product of nonnegative bidiagonal matrices:

$$A = L^{(1)} L^{(2)} \cdots L^{(n-1)} D U^{(n-1)} U^{(n-2)} \cdots U^{(1)}, \tag{1}$$

where $L^{(i)}$ and $U^{(i)}$ are lower and upper unit bidiagonal matrices and D is diagonal. A total of n^2 entries on the sub/super diagonals of $L^{(i)}$ and $U^{(i)}$ and on the diagonal of D can be nonzero and these *independent* nonnegative entries parameterize the set of *all* TN matrices. These parameters immediately reveal the TN structure and determine

- The entries of the inverse;
- The solution to a linear system (if the right hand side has alternating sign pattern);
- The LDU decomposition;
- The eigenvalues and
- The SVD

to high relative accuracy.

Algorithms for computing the eigenvalues and the SVD of a TN matrix to high relative accuracy were proposed recently in [6], and algorithms for computing accurate inverses, solutions and LDU decompositions date back to the 1980s [3].

The total nonnegativity is preserved under a number of matrix operations, including

- Taking a product
- Schur complementation
- Taking a submatrix
- Forming the LDU decomposition (L,D, and U are TN)
- Forming the QR decomposition (R is TN)
- Forming the J-inverse $((-1)^{i+j}a_{ij})^{-1}$
- Forming the converse $(a_{n+1-i,n+1-j})$

Can the high relative accuracy be preserved under all operations, listed above, that preserve the total nonegativity?

The answer is "yes". Starting with the bidiagonal decompositions (1) of the input TN matrices, one can compute the bidiagonal decomposition of the output TN matrices (and then compute accurate inverses, eigenvalues and SVDs). This computation is subtraction free and therefore accurate [5]. The complexity is comparable to that of performing the same operations directly on the matrix and never exceeds $O(n^3)$.

In particular, one can compute accurate eigenvalues, SVDs and inverses of products, Schur complements, etc. of TN matrices that are notoriously ill conditioned in the traditional sense, such as the Hilbert, Pascal, Vandermonde, etc. We demonstrate that from a totally nonnegative point of view these matrices are perfectly conditioned with respect to perturbations in the entries that define them.

We will also present progress on open problems including:

1. Accurate Eigenvectors and Accurate Invariant Subspaces of TN matrices
2. Accurate Computations with singular TN matrices

References

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