Accurate Eigenvalues, Eigenvectors(?) and SVDs of Totally Nonnegative Matrices

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Theory and Numerics of Matrix Eigenvalue Problems
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Outline

- What are TN matrices
- Goals: Accurate Eigenvalues, SVDs, etc.
- Problems
- It's all a matter of correct representation + avoiding subtractions
- Reduce BOTH eigenvalue and SVD problem to SVD(bidiagonal)
- Eigenvalues, SVD – OK, Accurate Eigenvectors?

Totally Nonnegative Matrices

- All minors ≥ 0
- In particular $a_{ij} \geq 0$
- $\lambda_i > 0$, real, distinct
- Generally nonsymmetric
- Examples include notoriously ill conditioned matrices:

$$\text{Vandermonde} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 4 & 4^2 & 4^3 \end{bmatrix}$$

$$\text{Hilbert} = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/3 & 1/4 & 1/5 \\ 1/3 & 1/4 & 1/5 & 1/6 \\ 1/4 & 1/5 & 1/6 & 1/7 \end{bmatrix} = \left[\frac{1}{i+j-1}\right]$$

Goals

In $O(n^3)$ time: All λ_i and all σ_i to high relative accuracy, meaning correct sign and leading digits

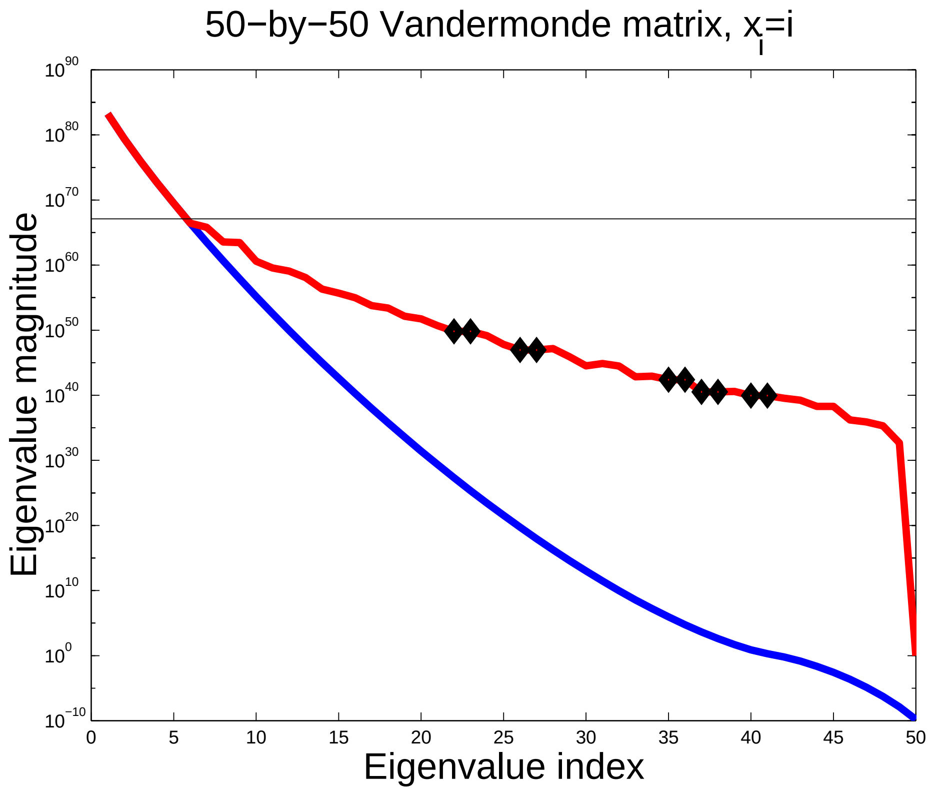
$$|\hat{\lambda}_i - \lambda_i| \leq O(\epsilon)|\lambda_i|, \qquad |\hat{\sigma}_i - \sigma_i| \leq O(\epsilon)\sigma_i,$$

- Contrast: Traditional algorithms

$$|\hat{\lambda}_i - \lambda_i| \leq O(\epsilon)\frac{||A||}{|y^*x|}, \qquad |\hat{\sigma}_i - \sigma_i| \leq O(\epsilon)\sigma_1$$

- Despite nonsymmetry!
- Also:
  - Eigenvectors?, Singular Vectors

Example: 50 × 50 Vandermonde Matrix $V = i^{j-1}$



How do we lose relative accuracy in floating point?

- **MODEL:** $\mathrm{fl}(a \otimes b) = (a \otimes b)(1 + \delta)$, no under/overflow
- **ACCURATE:**
  - Products, Quotients, Sums of positive numbers
    Proof: $1 + \delta$ factors can be factored out

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- **ACCURATE:**
  - Products, Quotients, Sums of positive numbers
    Proof: $1+\delta$ factors can be factored out
- **POSSIBLE LOSS OF ACCURACY:**
  - Subtractive Cancellation when subtracting approximate results:

```
   .12345xxx
 - .12345yyy
 -----------
   .00000zzz
```

Representation of TN matrices – I

- **Matrix elements are a poor choice**
- **ϵ perturbation in $(2,2)$ entry**

$$\begin{bmatrix} 1 & 1 \\ 1 & 1+\epsilon \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1+2\epsilon \end{bmatrix}$$

Representation of TN matrices – II

- **Matrix elements are a poor choice**
- **ϵ perturbation in $(2,2)$ entry**

$$\begin{bmatrix} 1 & 1 \\ 1 & 1+\epsilon \end{bmatrix} = \begin{bmatrix} 1 & \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & \\ & \epsilon \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1+2\epsilon \end{bmatrix} = \begin{bmatrix} 1 & \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & \\ & 2\epsilon \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}$$

- **Leads to 100% relative perturbation in $\sigma_{\min}$**
- **Parlett and Dhillon in their symmetric tridiagonal eigenvalue algorithm discard A and keep LDL^T**

Representation of TN matrices – III

- **Bidiagonals are again the answer**

$$\mathcal{BD}\left(\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix}\right) =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ + & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix}\begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{bmatrix}\begin{bmatrix} 1 & + & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix}$$

Neville Elimination – We can choose the pattern

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 0 & 1 & 7 & 37 \end{bmatrix}$$

Neville Elimination – We can choose the pattern

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 0 & 1 & 5 & 19 \\ 0 & 1 & 7 & 37 \end{bmatrix}$$

Neville Elimination – We can choose the pattern

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & \color{red}{1} & 1 \end{bmatrix} \begin{bmatrix} \color{red}{1} & \color{red}{1} & \color{red}{1} & \color{red}{1} \\ \color{red}{1} & \color{red}{2} & \color{red}{4} & \color{red}{8} \\ 0 & 1 & 5 & 19 \\ 0 & 0 & 2 & 18 \end{bmatrix}$$

Neville Elimination – We can choose the pattern

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ \color{red}{1} & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & \color{red}{1} & \color{red}{3} & \color{red}{7} \\ 0 & \color{red}{1} & \color{red}{5} & \color{red}{19} \\ 0 & 0 & 2 & 18 \end{bmatrix}$$

Neville Elimination – We can choose the pattern

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & \color{red}{1} & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & \color{red}{2} & \color{red}{12} \\ 0 & 0 & \color{red}{2} & \color{red}{18} \end{bmatrix}$$

Neville Elimination – We can choose the pattern

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & 1 & 1 & \\ & & \color{red}{1} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 2 & 12 \\ 0 & 0 & 0 & \color{red}{6} \end{bmatrix}$$

Neville Elimination – We can choose the pattern

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 2 & \\ & & & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Neville Elimination – We can choose the pattern

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 2 & \\ & & & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}$$

Neville Elimination – We can choose the pattern

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} =$$

$$\underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 2 & \\ & & & 6 \end{bmatrix}}_{D}\underbrace{\begin{bmatrix} 1 & 1 & & \\ & 1 & 2 & \\ & & 1 & 3 \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & 1 & \\ & & 1 & 2 \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}}_{U}$$

Notice: positive bidiagonal decomposition

$$U_{i,i+1}^{(k)} = \frac{\det(A(1:k, i-k+2:i+1))}{\det(A(1:k-1, i-k+2:i))} \cdot \frac{\det(A(1:k-1, i-k+1:i-1))}{\det(A(1:k, i-k+1:i))}.$$

Representation of TN matrices – II

$$\mathcal{BD}\left(\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix}\right) =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ + & 1 & & \\ & + & 1 & \\ & & + & 1 \end{bmatrix}\begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{bmatrix}\begin{bmatrix} 1 & + & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & + \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & + \\ & & & 1 \end{bmatrix}$$

- **Entries of BD determine all λ_i, σ_i accurately**

  Proof:

  – **Cauchy–Binet – all minors are determined accurately.**

  – **Entries of the kth compound matrix are all kth order minors of A, thus ≥ 0 and determined accurately.**

  – **Thus so is the Perron root $\lambda_1 \ldots \lambda_k$ and two-norm $\sigma_1 \ldots \sigma_k$.**

Ideas

- **Reduce BOTH eigenvalue problem and SVD to SVD(bidiagonal)**
- **SVD(bidiagonal) is accurate (Demmel-Kahan 1991)**
- **Only**
  - **Subtract a multiple of row/column from next to make a 0**
  - **Add a multiple of a row/column to previous**
- **NEVER actually subtract by working on $\mathcal{BD}(A)$**

Subtract a row from next to make a 0

$$\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{6} \\ \color{blue}{\mathbf{3}} & \color{blue}{\mathbf{10}} & \color{blue}{\mathbf{50}} \\ \color{red}{\mathbf{21}} & \color{blue}{\mathbf{102}} & \color{blue}{\mathbf{615}} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{1} & & \\ & \mathbf{1} & \\ & \mathbf{7} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{1} & & \\ \mathbf{3} & \mathbf{1} & \\ & \mathbf{8} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{1} & & \\ & \mathbf{4} & \\ & & \mathbf{9} \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{2} & \\ & \mathbf{1} & \mathbf{5} \\ & & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{1} & & \\ & \mathbf{1} & \mathbf{3} \\ & & \mathbf{1} \end{bmatrix}$$

Subtract a row from next to make a 0

$$
\begin{aligned}
&\begin{bmatrix} 1 & & \\ & 1 & \\ & -7 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 2 & 6 \\ 3 & 10 & 50 \\ 21 & 102 & 615 \end{bmatrix} \\
&= \begin{bmatrix} 1 & & \\ & 1 & \\ & -7 & 1 \end{bmatrix}
\begin{bmatrix} 1 & & \\ & 1 & \\ & 7 & 1 \end{bmatrix}
\begin{bmatrix} 1 & & \\ 3 & 1 & \\ & 8 & 1 \end{bmatrix}
\begin{bmatrix} 1 & & \\ & 4 & \\ & & 9 \end{bmatrix}
\begin{bmatrix} 1 & 2 & \\ & 1 & 5 \\ & & 1 \end{bmatrix}
\begin{bmatrix} 1 & & \\ & 1 & 3 \\ & & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & & \\ & 1 & \\ & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & & \\ 3 & 1 & \\ & 8 & 1 \end{bmatrix}
\begin{bmatrix} 1 & & \\ & 4 & \\ & & 9 \end{bmatrix}
\begin{bmatrix} 1 & 2 & \\ & 1 & 5 \\ & & 1 \end{bmatrix}
\begin{bmatrix} 1 & & \\ & 1 & 3 \\ & & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2 & 6 \\ 3 & 10 & 50 \\ 0 & 32 & 265 \end{bmatrix}
\end{aligned}
$$

Adding a column to previous one – I

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & a & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ b & 1 & \\ & c & 1 \end{bmatrix} \begin{bmatrix} d & & \\ & e & \\ & & f \end{bmatrix} \begin{bmatrix} 1 & g & \\ & 1 & h \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & k \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & y & \\ & x & z \end{bmatrix}$$

Chasing the “blue” bulge to the left

Adding a column to previous one – II

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & a & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ b & 1 & \\ & c & 1 \end{bmatrix} \begin{bmatrix} d & & \\ & e & \\ & & f \end{bmatrix} \begin{bmatrix} 1 & g & \\ & 1 & h \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & y' & \\ & x' & z' \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & k' \\ & & 1 \end{bmatrix}$$

$$\begin{aligned} x' &= x \\ y' &= y + kx \\ z' &= 1/y' \\ k' &= kz/y_1 \end{aligned}$$

... it's all dqd

Adding a column to previous one – III

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & a & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ b & 1 & \\ & c & 1 \end{bmatrix} \begin{bmatrix} d & & \\ & e & \\ & & f \end{bmatrix} \begin{bmatrix} 1 & & \\ & y'' & \\ & x'' & z'' \end{bmatrix} \begin{bmatrix} 1 & g' & \\ & 1 & h' \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & k' \\ & & 1 \end{bmatrix}$$

Adding a column to previous one – IV

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & a & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ b & 1 & \\ & c & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & x''' & 1 \end{bmatrix} \begin{bmatrix} d & & \\ & e' & \\ & & f' \end{bmatrix} \begin{bmatrix} 1 & g' & \\ & 1 & h' \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & k' \\ & & 1 \end{bmatrix}$$

Adding a column to previous one – V

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & a & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ b & 1 & \\ & c + x''' & 1 \end{bmatrix} \begin{bmatrix} d & & \\ & e' & \\ & & f' \end{bmatrix} \begin{bmatrix} 1 & g' & \\ & 1 & h' \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & k' \\ & & 1 \end{bmatrix}$$

DONE!

Reduction of a nonsymmetric matrix to tridiagonal form – 1

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \color{red}{+} & + & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 2

$$\begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & \mathbf{-} & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ \color{red}{+} & \color{red}{+} & \color{red}{+} & \color{red}{+} \\ \color{red}{+} & \color{red}{+} & \color{red}{+} & \color{red}{+} \end{bmatrix}$$

$$= \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 3

$$\begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & - & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & {\color{red}+} & {\color{red}+} \\ + & + & {\color{red}+} & {\color{red}+} \\ + & + & {\color{red}+} & {\color{red}+} \\ + & + & {\color{red}+} & {\color{red}+} \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & + & \mathbf{1} \end{bmatrix}$$

$$= \begin{bmatrix} + & + & {\color{red}+} & {\color{red}+} \\ + & + & {\color{red}+} & {\color{red}+} \\ + & + & {\color{red}+} & {\color{red}+} \\ \mathbf{0} & + & {\color{red}+} & {\color{red}+} \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & + & \mathbf{1} \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 4

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & - & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}$$

$$= \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}$$

$$= \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 5

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ \color{red}{+} & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 6

$$\begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & - & \mathbf{1} & \\ & & & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & + & + & + \\ {\color{red}+} & {\color{red}+} & {\color{red}+} & {\color{red}+} \\ {\color{red}+} & {\color{red}+} & {\color{red}+} & {\color{red}+} \\ \mathbf{0} & + & + & + \end{bmatrix}$$

$$= \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ {\color{red}\mathbf{0}} & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 7

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & - & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ \color{red}{+} & \color{red}{+} & \color{red}{+} & \color{red}{+} \\ \color{red}{+} & \color{red}{+} & \color{red}{+} & \color{red}{+} \\ 0 & + & + & + \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} + & \color{red}{+} & \color{red}{+} & + \\ + & \color{red}{+} & \color{red}{+} & + \\ 0 & \color{red}{+} & \color{red}{+} & + \\ 0 & \color{red}{+} & \color{red}{+} & + \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & & 1 \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 8

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & - & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ \color{red}{+} & \color{red}{+} & \color{red}{+} & \color{red}{+} \\ \color{red}{+} & \color{red}{+} & \color{red}{+} & \color{red}{+} \\ 0 & + & + & + \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} + & \color{red}{+} & \color{red}{+} & + \\ + & \color{red}{+} & \color{red}{+} & + \\ 0 & \color{red}{+} & \color{red}{+} & + \\ 0 & \color{red}{+} & \color{red}{+} & + \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 9

$$\begin{bmatrix} + & + & + & \color{red}{+} \\ + & + & + & + \\ \mathbf{0} & + & + & + \\ \mathbf{0} & \mathbf{0} & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 10

$$\begin{bmatrix} \mathbf{1} & - & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & & \mathbf{1} \end{bmatrix} \begin{bmatrix} \color{red}{+} & \color{red}{+} & \color{red}{+} & \color{red}{+} \\ \color{red}{+} & \color{red}{+} & \color{red}{+} & \color{red}{+} \\ \mathbf{0} & + & + & + \\ \mathbf{0} & \mathbf{0} & + & + \end{bmatrix}$$

$$= \begin{bmatrix} + & + & + & \color{red}{\mathbf{0}} \\ + & + & + & + \\ \mathbf{0} & + & + & + \\ \mathbf{0} & \mathbf{0} & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 11

$$\begin{bmatrix} 1 & - & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \end{bmatrix} \begin{bmatrix} 1 & + & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \color{red}{+} & \color{red}{+} & + & 0 \\ \color{red}{+} & \color{red}{+} & + & + \\ \color{red}{0} & \color{red}{+} & + & + \\ \color{red}{0} & \color{red}{0} & + & + \end{bmatrix} \begin{bmatrix} 1 & + & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 12

$$\begin{bmatrix} 1 & - & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \end{bmatrix} \begin{bmatrix} 1 & + & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} {\color{red}+} & {\color{red}+} & + & 0 \\ {\color{red}+} & {\color{red}+} & + & + \\ {\color{red}0} & {\color{red}+} & + & + \\ {\color{red}0} & {\color{red}0} & + & + \end{bmatrix} \begin{bmatrix} 1 & + & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} + & + & + & 0 \\ + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 13

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \Rightarrow \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix} \Rightarrow \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix} \Rightarrow \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \\ \mathbf{0} & \mathbf{0} & + & + \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} + & + & + & \mathbf{0} \\ + & + & + & + \\ \mathbf{0} & + & + & + \\ \mathbf{0} & \mathbf{0} & + & + \end{bmatrix} \Rightarrow \begin{bmatrix} + & + & + & \mathbf{0} \\ + & + & + & \mathbf{0} \\ \mathbf{0} & + & + & + \\ \mathbf{0} & \mathbf{0} & + & + \end{bmatrix} \Rightarrow \begin{bmatrix} + & + & \mathbf{0} & \mathbf{0} \\ + & + & + & \mathbf{0} \\ \mathbf{0} & + & + & + \\ \mathbf{0} & \mathbf{0} & + & + \end{bmatrix}$$

Reduction of a nonsymmetric matrix to tridiagonal form – 14

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \Rightarrow \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix} \Rightarrow \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix} \Rightarrow \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \\ \mathbf{0} & \mathbf{0} & + & + \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} + & + & + & \mathbf{0} \\ + & + & + & + \\ \mathbf{0} & + & + & + \\ \mathbf{0} & \mathbf{0} & + & + \end{bmatrix} \Rightarrow \begin{bmatrix} + & + & + & \mathbf{0} \\ + & + & + & \mathbf{0} \\ \mathbf{0} & + & + & + \\ \mathbf{0} & \mathbf{0} & + & + \end{bmatrix} \Rightarrow \begin{bmatrix} + & + & \mathbf{0} & \mathbf{0} \\ + & + & + & \mathbf{0} \\ \mathbf{0} & + & + & + \\ \mathbf{0} & \mathbf{0} & + & + \end{bmatrix}$$

$$(\mathrm{TN})^{-1} \cdot \boldsymbol{A} \cdot (\mathrm{TN}) = \text{Tridiagonal}$$

... Obtained in factored form! $\mathcal{BD}$**(TN Tridiagonal)=**

$$\begin{bmatrix} + & + & \mathbf{0} & \mathbf{0} \\ + & + & + & \mathbf{0} \\ \mathbf{0} & + & + & + \\ \mathbf{0} & \mathbf{0} & + & + \end{bmatrix} = \begin{bmatrix} \mathbf{1} & & & \\ + & \mathbf{1} & & \\ & + & \mathbf{1} & \\ & & + & \mathbf{1} \end{bmatrix} \begin{bmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{bmatrix} \begin{bmatrix} \mathbf{1} & + & & \\ & \mathbf{1} & + & \\ & & \mathbf{1} & + \\ & & & \mathbf{1} \end{bmatrix}$$

Accurate Eigenvalues of Tridiagonals

The eigenvalues of

$$\begin{bmatrix} 1 & & & & \\ b_1 & 1 & & & \\ & \ddots & \ddots & & \\ & & b_{n-2} & 1 & \\ & & & b_{n-1} & 1 \end{bmatrix} \begin{bmatrix} d_1 & & & & \\ & d_2 & & & \\ & & \ddots & & \\ & & & d_{n-1} & \\ & & & & d_n \end{bmatrix} \begin{bmatrix} 1 & c_1 & & & \\ & 1 & c_2 & & \\ & & \ddots & \ddots & \\ & & & 1 & c_{n-1} \\ & & & & 1 \end{bmatrix}$$

are the squares of the singular values of

$$\begin{bmatrix} \sqrt{d_1} & \sqrt{d_1 b_1 c_1} & & & \\ & \sqrt{d_2} & \sqrt{d_2 b_2 c_2} & & \\ & & \ddots & \ddots & \\ & & & \sqrt{d_{n-1}} & \sqrt{d_{n-1} b_{n-1} c_{n-1}} \\ & & & & \sqrt{d_n} \end{bmatrix}$$

- Forward stable
- Total cost $= 7n^3$.

The SVD Problem

- Givens

$$\begin{bmatrix} c & s \\ -s & c \end{bmatrix} = \begin{bmatrix} + & + \\ 0 & + \end{bmatrix} \begin{bmatrix} 1 & 0 \\ - & 1 \end{bmatrix}$$

- Equals:
  - Subtract a row from next to make a 0
  - Add multiple of next to current; Scale
- TN→BIDIAGONAL→SVD
- Again $7n^3$

Eigenvectors

- $(TN)^{-1} \cdot A \cdot (TN) = LDL^T = Q\Lambda Q^{-1}$
- What is the accuracy in the eigenvector matrix $V = (TN) \cdot Q$?
- Perron vector q_1 of LDL^T is accurate and positive
  Because: LDL^T determines each entry of q_1 accurately
- $v_1 = (TP) \cdot q_1$ – also determined and computed accurately
- Smallest eigenvector OK for same reason

Smallest Eigenvector

- $\mathcal{BD}(A^{-1})$ trivial to obtain from $\mathcal{BD}(A)$:

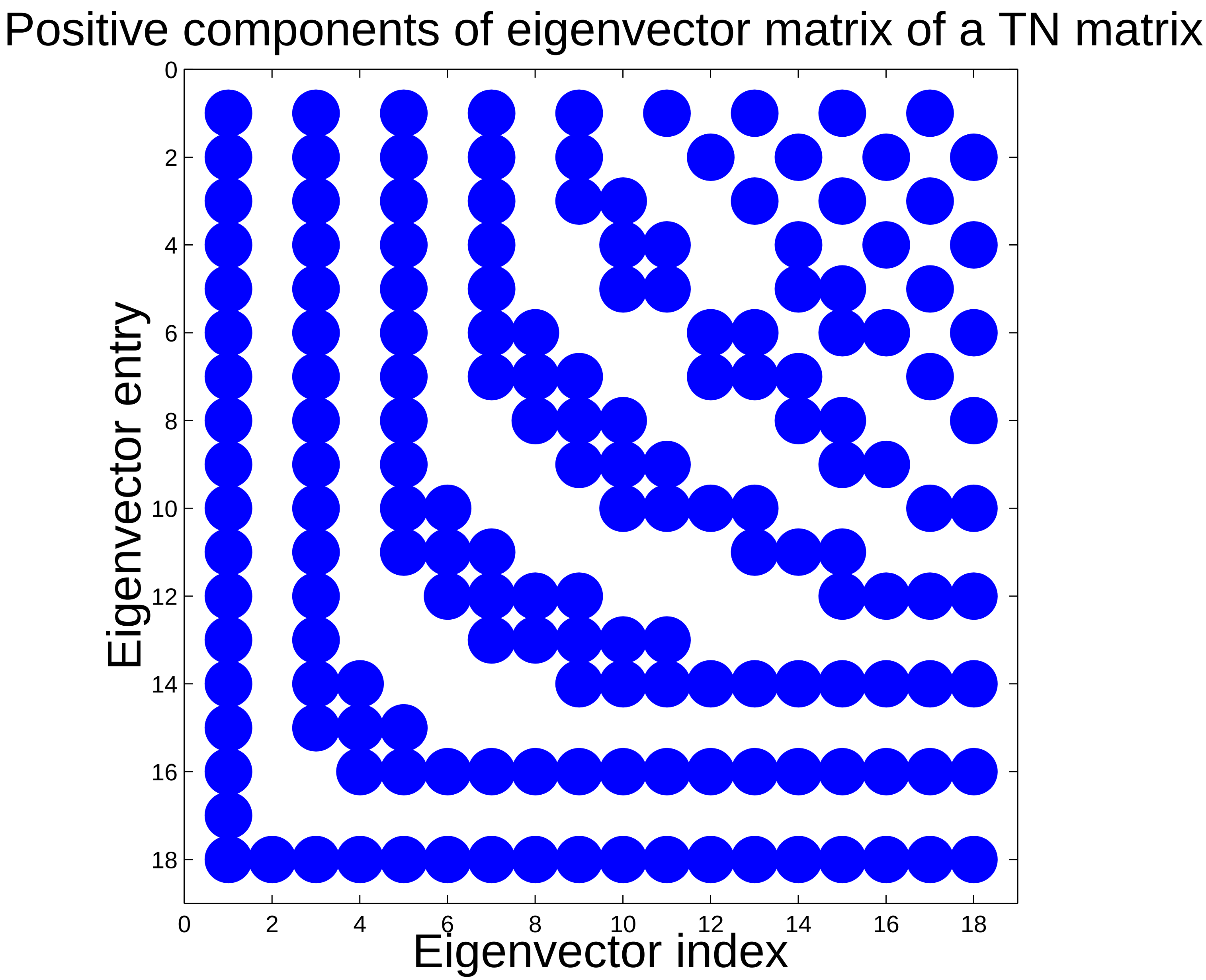
$$(D^{-1})_{ii} = 1/D_{ii}$$

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & x & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & & \\ & 1 & \\ & -x & 1 \end{bmatrix}$$

- $A^{-1} = \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$ is called sign-regular

- $\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix} A^{-1} \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix} = TN$

- "Perron vector" of A^{-1} also determined accurately by data


Positive components of eigenvector matrix of a TN matrix
Eigenvector entry
Eigenvector index
0
2
4
6
8
10
12
14
16
18

More known about eigenvectors

- All minors of $V\begin{pmatrix} i_1 & i_2 & \dots & i_k \\ 1 & 2 & \dots & k \end{pmatrix}$ determined accurately.
- These are the components of the Perron vector of the kth compound matrix

Conclusions

- **$O(n^3)$ algorithms for the eigenvalues and the SVD of (*unsymmetric*) TN Matrices to high relative accuracy**
- **Accurate linear algebra with TN matrices closed under same operations as TN**
- **Applies to:**
  - **Oscillatory**
  - **Totally Nonnegative**
  - **Sign Regular (Inverses of TN)**
- **Questions about accuracy in eigenvectors + algorithms remain?**
- **This talk:** `math.mit.edu/~plamen`