Accurate Eigenvalues of Sign Regular Matrices

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New Result

- New $\mathcal{O}(n^3)$ eigenvalue algorithm for certain sign regular matrices
- All eigenvalues computed to high relative accuracy:

$$|\lambda_i - \hat{\lambda}_i| = \mathcal{O}(\varepsilon)|\lambda_i|$$

  as opposed to

$$|\lambda_i - \hat{\lambda}_i| = \mathcal{O}(\varepsilon)|\lambda_{\max}|\frac{1}{y_i^T x_i}$$

- Uses only working precision
- Accuracy unaffected by angle between left and right eigenvectors

- First example of:
  $\mathcal{O}(n^3)$ accurate eigenvalue algorithm for a nonsymmetric matrix with both positive and negative eigenvalues

What is a Sign-Regular Matrix?

"A matrix whose kth order minors have sign $(-1)^{k(k-1)/2}$"

- Equivalently:

"Totally positive matrix with reversed columns"

- Recall: "Totally Positive" means "all minors > 0", e.g.,

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 4 & 4^2 & 4^3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/3 & 1/4 & 1/5 \\ 1/3 & 1/4 & 1/5 & 1/6 \\ 1/4 & 1/5 & 1/6 & 1/7 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$$

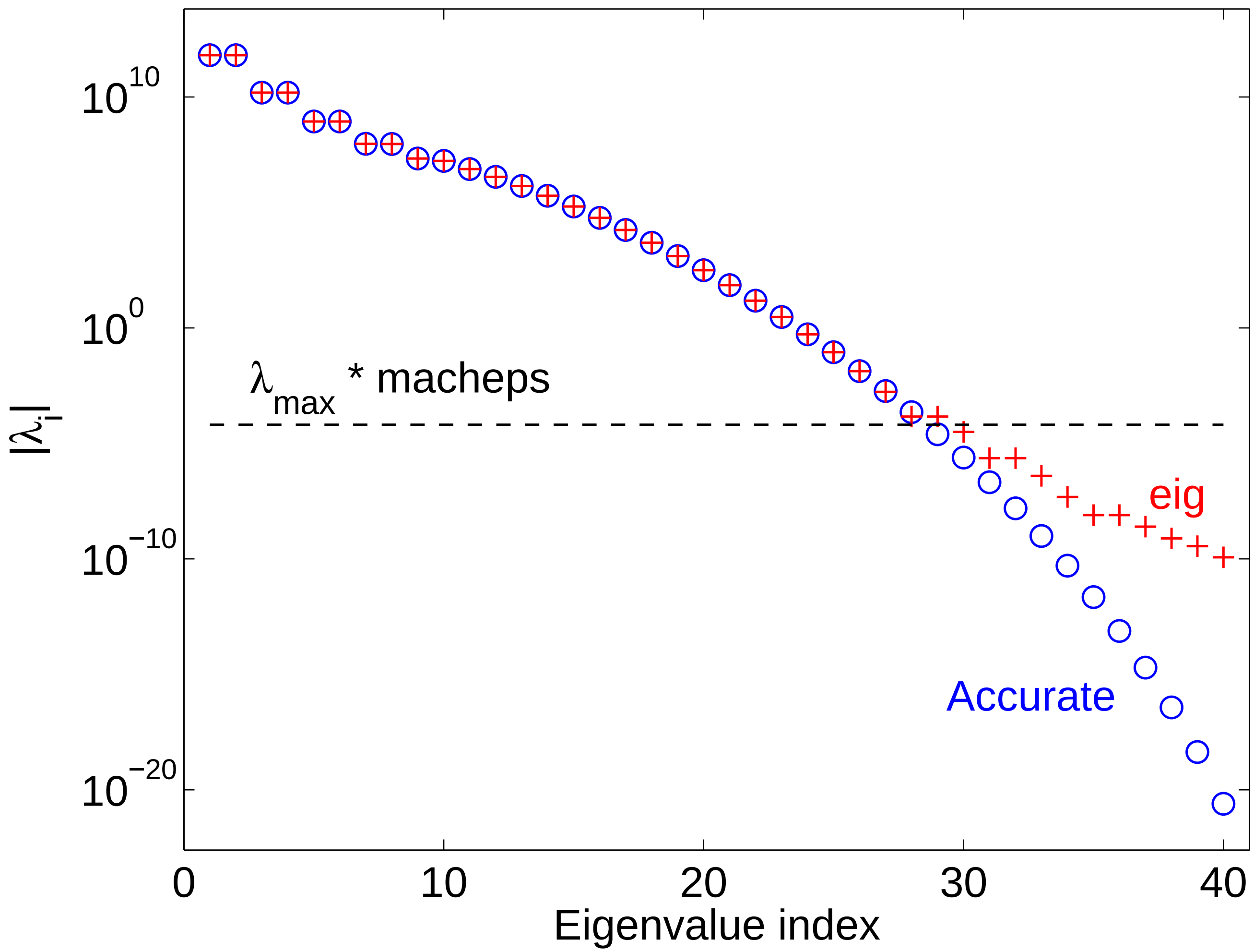
Vandermonde — Hilbert — Pascal

- I.e., we consider

$$\text{(Totally positive)} \times \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

- Generally unsymmetric; NOT similar to TP, but just as ill conditioned

|Eigenvalues| of a 40×40 Sign Regular Vandermonde



$$V = \left[x_i^{j-1} \right]_{i,j=1}^{40}, \; x = (4.0, 3.9, \ldots, 0.1)$$

- **Theory:** $\lambda_i \in \mathbb{R}, \; \text{sign}(\lambda_i) = (-1)^{i-1}$

Why Sign Regular Matrices?

Axel Ruhe comment in 2002:

"... a class of matrices that I have never heard of
... using algorithms I have never heard of
... by people I have never heard of"

Why Sign Regular Matrices?

- **Eigenvalues determined accurately!**
- **The next logical item on the list of accurate algorithms:**
  - **Vandermonde**
  - **Cauchy**
  - **polynomial Vandermonde**
  - **Generalized Vandermonde**
  - **M-matrices**
  - **Elliptic PDEs/Finite elements**
  - **Totally nonnegative**
  - **Sign regular**

The business of λ_i being determined accurately

- We will be computing eigenvalues of

$$(\text{Totally positive}) \times \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

- Start with TP
- Key fact: Demmel–Kahan 1990: σ_i of

$$\begin{bmatrix} a_1 & b_1 & & \\ & a_2 & b_2 & \\ & & \ddots & \ddots \\ & & & a_n \end{bmatrix}$$

determined accurately by a_i, b_i.

The business of λ_i being determined accurately

- We will be computing eigenvalues of

$$\text{(Totally positive)} \times \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

- Start with TP
- Key fact: Demmel–Kahan 1990: λ_i of

$$\begin{bmatrix} 1 & & & \\ b_{21} & 1 & & \\ & b_{32} & 1 & \\ & & b_{43} & 1 \end{bmatrix} \begin{bmatrix} b_{11} & & & \\ & b_{22} & & \\ & & b_{33} & \\ & & & b_{44} \end{bmatrix} \begin{bmatrix} 1 & b_{12} & & \\ & 1 & b_{23} & \\ & & 1 & b_{34} \\ & & & 1 \end{bmatrix}$$

determined accurately by $b_{ij} > 0$.

The business of λ_i being determined accurately

- We will be computing eigenvalues of

$$\text{(Totally positive)} \times \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

- Start with TP
- Key fact: Koev 2005: λ_i of

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & b_{41} & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & b_{31} & 1 & \\ & & b_{42} & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ b_{21} & 1 & & \\ & b_{32} & 1 & \\ & & b_{43} & 1 \end{bmatrix} \begin{bmatrix} b_{11} & & & \\ & b_{22} & & \\ & & b_{33} & \\ & & & b_{44} \end{bmatrix} \begin{bmatrix} 1 & b_{12} & & \\ & 1 & b_{23} & \\ & & 1 & b_{34} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & b_{13} & \\ & & 1 & b_{24} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & b_{14} \\ & & & 1 \end{bmatrix}$$

determined accurately by $b_{ij} > 0$.

The business of λ_i being determined accurately

- We will be computing eigenvalues of

$$\text{(Totally positive)} \times \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

- Start with TP
- New result for SR matrices: λ_i of

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & b_{41} & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & b_{31} & 1 & \\ & & b_{42} & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ b_{21} & 1 & & \\ & b_{32} & 1 & \\ & & b_{43} & 1 \end{bmatrix} \begin{bmatrix} b_{11} & & & \\ & b_{22} & & \\ & & b_{33} & \\ & & & b_{44} \end{bmatrix} \begin{bmatrix} 1 & b_{12} & & \\ & 1 & b_{23} & \\ & & 1 & b_{34} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & b_{13} & \\ & & 1 & b_{24} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & b_{14} \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

determined accurately by $b_{ij} > 0$.

TP Eigenvalue algorithm

- Using only
  - Subtract a multiple of row from next to make a zero
  - Add a positive multiple of a row to next/previous
- Both preserve TP structure
- When applied implicitly, no subtractions are required

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & b_{31} & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ b_{21} & 1 & \\ & b_{32} & 1 \end{bmatrix} \begin{bmatrix} b_{11} & & \\ & b_{22} & \\ & & b_{33} \end{bmatrix} \begin{bmatrix} 1 & b_{12} & \\ & 1 & b_{23} \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & b_{13} \\ & & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & & \\ b'_{21} & 1 & \\ & b'_{32} & 1 \end{bmatrix} \begin{bmatrix} b'_{11} & & \\ & b'_{22} & \\ & & b'_{33} \end{bmatrix} \begin{bmatrix} 1 & b'_{12} & \\ & 1 & b'_{23} \\ & & 1 \end{bmatrix}$$

Why No Subtractive Cancellation?

- Relative accuracy preserved in $\times, +, /$
  Proof: $(1+\delta)$ factors accumulate multiplicatively
- Subtractions of approximate quantities dangerous:

$$\begin{array}{rl} & .123456789xxx \\ - & .123456789yyy \\ \hline & .000000000zzz \end{array}$$

- Initial data is OK to subtract: $(x_i - y_j)$
- Reduction to anti-bidiagonal involves no subtractions

Algorithm Outline

- SR similar to a symmetric anti-bidiagonal; (eigs then easy)

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \longrightarrow \begin{bmatrix} & & + & + \\ & + & + & \\ + & + & & \\ + & & & \end{bmatrix}$$

Sign regular $\longrightarrow$ anti-bidiagonal

- But done by transforming the TP factor:

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \cdot \begin{bmatrix} & & & \mathbf{1} \\ & & \mathbf{1} & \\ & \mathbf{1} & & \\ \mathbf{1} & & & \end{bmatrix} \longrightarrow \begin{bmatrix} + & + & & \\ & + & + & \\ & & + & + \\ & & & + \end{bmatrix} \cdot \begin{bmatrix} & & & \mathbf{1} \\ & & \mathbf{1} & \\ & \mathbf{1} & & \\ \mathbf{1} & & & \end{bmatrix}$$

Totally Positive $\cdot$ J $\longrightarrow$ Bidiagonal $\cdot$ J

- using TP/Accuracy-preserving subtraction-free transformations

Similarity Reduction to an Anti-Bidiagonal

- First reduce to (Upper Triangular) $\times J$

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \color{red}{+} & + & + & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

Similarity Reduction to an Anti-Bidiagonal 2

- **First reduce to (Upper Triangular)** $\times J$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -1 & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

Similarity Reduction to an Anti-Bidiagonal 3

- **First reduce to (Upper Triangular)** $\times J$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & - & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}$$

Similarity Reduction to an Anti-Bidiagonal 4

- **First reduce to (Upper Triangular) $\times J$**

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & - & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}$$

Similarity Reduction to an Anti-Bidiagonal 5

- **First reduce to (Upper Triangular) $\times J$**

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \color{red}{0} & + & + & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}$$

Similarity Reduction to an Anti-Bidiagonal 6

- **First reduce to (Upper Triangular) $\times J$**

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \color{red}{\mathbf{0}} & + & + & + \end{bmatrix} \begin{bmatrix} & & & \mathbf{1} \\ & & \mathbf{1} & \\ & \mathbf{1} & & \\ \mathbf{1} & & & \end{bmatrix} \begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & + & \mathbf{1} \end{bmatrix} \color{blue}{\begin{bmatrix} & & & \mathbf{1} \\ & & \mathbf{1} & \\ & \mathbf{1} & & \\ \mathbf{1} & & & \end{bmatrix}} \color{blue}{\begin{bmatrix} & & & \mathbf{1} \\ & & \mathbf{1} & \\ & \mathbf{1} & & \\ \mathbf{1} & & & \end{bmatrix}}$$

Similarity Reduction to an Anti-Bidiagonal 7

- **First reduce to (Upper Triangular)** $\times J$

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \color{red}{\mathbf{0}} & + & + & + \end{bmatrix} \color{blue}{\begin{bmatrix} & & & \mathbf{1} \\ & & \mathbf{1} & \\ & \mathbf{1} & & \\ \mathbf{1} & & & \end{bmatrix}} \color{blue}{\begin{bmatrix} \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & + & \mathbf{1} \end{bmatrix}} \color{blue}{\begin{bmatrix} & & & \mathbf{1} \\ & & \mathbf{1} & \\ & \mathbf{1} & & \\ \mathbf{1} & & & \end{bmatrix}} \begin{bmatrix} & & & \mathbf{1} \\ & & \mathbf{1} & \\ & \mathbf{1} & & \\ \mathbf{1} & & & \end{bmatrix}$$

Similarity Reduction to an Anti-Bidiagonal 8

- First reduce to (Upper Triangular) $\times J$

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \color{red}{\mathbf{0}} & + & + & + \end{bmatrix} \qquad \color{blue}{\begin{bmatrix} \mathbf{1} & + & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & & \mathbf{1} \end{bmatrix}} \qquad \begin{bmatrix} & & & \mathbf{1} \\ & & \mathbf{1} & \\ & \mathbf{1} & & \\ \mathbf{1} & & & \end{bmatrix}$$

Similarity Reduction to an Anti-Bidiagonal 9

- First reduce to (Upper Triangular) $\times J$

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix} \qquad \begin{bmatrix} \mathbf{1} & + & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & & \mathbf{1} \end{bmatrix} \qquad \begin{bmatrix} & & & \mathbf{1} \\ & & \mathbf{1} & \\ & \mathbf{1} & & \\ \mathbf{1} & & & \end{bmatrix}$$

Similarity Reduction to an Anti-Bidiagonal 10

- **First reduce to (Upper Triangular) $\times J$**

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix} \qquad \begin{bmatrix} \mathbf{1} & + & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & & \mathbf{1} \end{bmatrix} \qquad \begin{bmatrix} & & & \mathbf{1} \\ & & \mathbf{1} & \\ & \mathbf{1} & & \\ \mathbf{1} & & & \end{bmatrix}$$

Similarity Reduction to an Anti-Bidiagonal 11

- **First reduce to (Upper Triangular)** $\times J$

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

Similarity Reduction to an Anti-Bidiagonal 12

- **Making more zeros analogous**

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ \color{red}{+} & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix} \begin{bmatrix} & & & \mathbf{1} \\ & & \mathbf{1} & \\ & \mathbf{1} & & \\ \mathbf{1} & & & \end{bmatrix}$$

Similarity Reduction to an Anti-Bidiagonal 13

- **Making more zeros analogous**

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ \color{red}{0} & + & + & + \\ 0 & + & + & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

Similarity Reduction to an Anti-Bidiagonal 14

- **Making more zeros analogous**

$$\begin{bmatrix} + & + & + & + \\ \color{red}{0} & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

- **Making more zeros analogous**

$$\begin{bmatrix} + & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \\ 0 & \color{red}{0} & + & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

Similarity Reduction to an Anti-Bidiagonal 16

- **Making more zeros analogous**

$$\begin{bmatrix} + & + & + & + \\ 0 & + & + & + \\ 0 & \color{red}{0} & + & + \\ 0 & 0 & + & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

- **Making more zeros analogous**

$$\begin{bmatrix} + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & \color{red}{0} & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

Similarity Reduction to an Anti-Bidiagonal 18

- **(Upper Triangular)**$\times J \longrightarrow$ **(Upper bidiagonal)** $\times\ J$

$$\begin{bmatrix} + & + & + & \color{red}{+} \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

Similarity Reduction to an Anti-Bidiagonal 19

- **(Upper Triangular)** $\times J \longrightarrow$ **(Upper bidiagonal)** $\times J$

$$
\begin{bmatrix} 1 & - & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}
\begin{bmatrix} + & + & + & {\color{red}+} \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
\begin{bmatrix} 1 & + & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
$$

Similarity Reduction to an Anti-Bidiagonal 20

- **(Upper Triangular)** $\times J \longrightarrow$ **(Upper bidiagonal)** $\times J$

$$
\begin{bmatrix} 1 & - & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}
\begin{bmatrix} + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
\begin{bmatrix} 1 & + & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
$$

$$
\begin{bmatrix} + & + & + & 0 \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix}
\qquad
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}
\qquad
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
$$

Similarity Reduction to an Anti-Bidiagonal 21

- **(Upper Triangular)**$\times J \longrightarrow$ **(Upper bidiagonal)** $\times\ J$

$$
\begin{bmatrix} 1 & - & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}
\begin{bmatrix} + & + & + & \color{red}{+} \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
\begin{bmatrix} 1 & + & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
$$

$$
\begin{bmatrix} + & + & \color{red}{+} & \color{red}{0} \\ 0 & + & \color{red}{+} & \color{red}{+} \\ 0 & 0 & \color{red}{+} & \color{red}{+} \\ 0 & 0 & \color{red}{0} & \color{red}{+} \end{bmatrix}
\qquad
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}
\qquad
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
$$

$$
\begin{bmatrix} + & + & + & 0 \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & \color{red}{+} & + \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
$$

Similarity Reduction to an Anti-Bidiagonal 22

- **(Upper Triangular)** $\times J \longrightarrow$ **(Upper bidiagonal)** $\times J$

$$\begin{bmatrix} 1 & - & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} 1 & + & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

$$\begin{bmatrix} + & + & + & 0 \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix} \qquad \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix} \qquad \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

$$\begin{bmatrix} + & + & + & 0 \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

Similarity Reduction to an Anti-Bidiagonal 23

- **(Upper Triangular)**$\times J \longrightarrow$ **(Upper bidiagonal)** $\times J$

$$\begin{bmatrix} 1 & - & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} 1 & + & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

$$\begin{bmatrix} + & + & + & 0 \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix} \qquad \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix} \qquad \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

$$\begin{bmatrix} + & + & + & 0 \\ 0 & + & + & \color{red}{+} \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

Similarity Reduction to an Anti-Bidiagonal 24

- **(Upper Triangular)**$\times J \longrightarrow$ **(Upper bidiagonal)** $\times J$

$$
\begin{bmatrix} 1 & - & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}
\begin{bmatrix} + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
\begin{bmatrix} 1 & + & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
$$

$$
\begin{bmatrix} + & + & + & 0 \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix}
\qquad
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}
\qquad
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
$$

$$
\begin{bmatrix} + & + & + & 0 \\ 0 & + & + & 0 \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
$$

Similarity Reduction to an Anti-Bidiagonal 25

- **(Upper Triangular)** $\times J \longrightarrow$ **(Upper bidiagonal)** $\times J$

$$
\begin{bmatrix} 1 & - & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}
\begin{bmatrix} + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
\begin{bmatrix} 1 & + & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
$$

$$
\begin{bmatrix} + & + & + & 0 \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix}
\qquad
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}
\qquad
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
$$

$$
\begin{bmatrix} + & + & 0 & 0 \\ 0 & + & + & 0 \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix}
\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
$$

Similarity Reduction to an Anti-Bidiagonal 26

- **(Upper Triangular)**$\times J \longrightarrow$ **(Upper bidiagonal)** $\times\ J$

$$\begin{bmatrix} 1 & - & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} 1 & + & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

$$\begin{bmatrix} + & + & + & 0 \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix} \qquad \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix} \qquad \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

$$\begin{bmatrix} + & + & 0 & 0 \\ 0 & + & + & 0 \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} = \begin{bmatrix} & & + & + \\ & + & + & \\ + & + & & \\ + & & & \end{bmatrix}$$

- **Finally, symmetrizing:**

$$\begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & d_4 \end{bmatrix} \begin{bmatrix} & & b_1 & a_1 \\ & b_2 & a_2 & \\ b_3 & a_3 & & \\ a_4 & & & \end{bmatrix} \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & d_4 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} & & \sqrt{b_1 b_3} & \sqrt{a_1 a_4} \\ & b_2 & \sqrt{a_2 a_3} & \\ \sqrt{b_1 b_3} & \sqrt{a_2 a_3} & & \\ \sqrt{a_1 a_4} & & & \end{bmatrix}$$

Eigenvalues of a Sign Regular Matrix

$$\text{eig(Sign Regular)} = \text{eig}\underbrace{\left(\begin{bmatrix} & & & b_1 & a_1 \\ & & b_2 & a_2 & \\ & b_2 & a_3 & & \\ b_1 & a_2 & & & \\ a_1 & & & & \end{bmatrix}\right)}_{\text{symmetric } \Rightarrow |\lambda_i| = \sigma_i}$$

- sign $\lambda_i = (-1)^{i-1}$ (from theory)
- Thus suffices to compute σ_i, but that's easy:

$$\sigma_i\left(\begin{bmatrix} & & & b_1 & a_1 \\ & & b_2 & a_2 & \\ & b_2 & a_3 & & \\ b_1 & a_2 & & & \\ a_1 & & & & \end{bmatrix}\right) = \sigma_i\left(\begin{bmatrix} a_1 & b_1 & & & \\ & a_1 & b_2 & & \\ & & a_3 & b_2 & \\ & & & a_2 & b_1 \\ & & & & a_1 \end{bmatrix}\right)$$

—the latter solved by Demmel and Kahan 1990

Conclusions

- New $O(n^3)$ algorithm
- Computes all eigenvalues of a SR matrix to high relative accuracy
- First example of negative eigenvalues of a nonsymmetric matrix computed to high relative accuracy

Open problems

- Sign regular matrices with other signatures—no parameterization known, let alone algorithms
- Eigenvectors: Products of accurate factors, but what does accuracy mean?

- This talk, papers, software: `http://math.mit.edu/~plamen`

Perturbation Theory

$$C = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & b_{41} & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & b_{31} & 1 & \\ & & b_{42} & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ b_{21} & 1 & & \\ & b_{32} & 1 & \\ & & b_{43} & 1 \end{bmatrix} \begin{bmatrix} b_{11} & & & \\ & b_{22} & & \\ & & b_{33} & \\ & & & b_{44} \end{bmatrix} \begin{bmatrix} 1 & b_{12} & & \\ & 1 & b_{23} & \\ & & 1 & b_{34} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & b_{13} & \\ & & 1 & b_{24} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & b_{14} \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

- **Fact: Any minor of C is a linear function in any b_{ij} with coefficients of same sign**

  $\Rightarrow$ All minors are accurately determined by b_{ij}

- **In particular, c_{ij} are accurately determined, and in turn, the Perron root, λ_1, is accurately determined**
- **Next, consider the second compound matrix, $\mathcal{C}_2$, —the $\binom{n}{2} \times \binom{n}{2}$ matrix with entries all 2×2 minors of C**
- **The entries of $\mathcal{C}_2$ are negative and accurately determined by b_{ij}**
- **The Perron root of $\mathcal{C}_2$, $\lambda_1\lambda_2$, is also accurately determined**
- **Thus, λ_2 is accurately determined**

Accurate $\mathcal{BD}$ of Vandermonde and Cauchy

- $V = \left[x_i^{j-1}\right]_{i,j=1}^{n}$ (TP if $0 < x_1 < x_2 < \cdots < x_n$)

$$D_{ii} = \prod_{j=1}^{i-1}(x_i - x_j), \qquad L_{i+1,i}^{(k)} = \prod_{j=n-k}^{i-1} \frac{x_{i+1} - x_{j+1}}{x_i - x_j}, \qquad U_{i,i+1}^{(k)} = x_{i+n-k}$$

- $C = \left[\frac{1}{x_i + y_j}\right]_{i,j=1}^{n}$ (TP if $0 < x_1 < \cdots < x_n,\ 0 < y_1 < \cdots < y_n$)

$$D_{ii} = \prod_{k=1}^{i-1} \frac{(x_i - x_k)(y_i - y_k)}{(x_i + y_k)(y_i + x_k)}$$

$$L_{i,i+1}^{(k)} = \frac{x_{n-k} + y_{i-n+k+1}}{x_i + y_{i-n+k+1}} \prod_{l=n-k}^{i-1} \frac{x_{i+1} - x_{l+1}}{x_i - x_l} \cdot \prod_{l=1}^{i-n+k-1} \frac{x_i + y_l}{x_{i+1} + y_l}$$

$$U_{i+1,i}^{(k)} = \frac{y_{n-k} + x_{i-n+k+1}}{y_i + x_{i-n+k+1}} \prod_{l=n-k}^{i-1} \frac{y_{i+1} - y_{l+1}}{y_i - y_l} \cdot \prod_{l=1}^{i-n+k-1} \frac{y_i + x_l}{y_{i+1} + x_l}$$

- Similar formulas for Cauchy–Vandermonde, confluent (José Javier Martínez)
- No subtractive cancellation $\Rightarrow$ accurate

Applying EETs on $\mathcal{BD}(A)$ – I

- **Subtracting a row from next to make a zero**

$$\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{6} \\ \color{blue}{\mathbf{3}} & \color{blue}{\mathbf{10}} & \color{blue}{\mathbf{50}} \\ \color{red}{\mathbf{21}} & \color{blue}{\mathbf{102}} & \color{blue}{\mathbf{615}} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{1} & & \\ & \mathbf{1} & \\ & \mathbf{7} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{1} & & \\ \mathbf{3} & \mathbf{1} & \\ & \mathbf{8} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{1} & & \\ & \mathbf{4} & \\ & & \mathbf{9} \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{2} & \\ & \mathbf{1} & \mathbf{5} \\ & & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{1} & & \\ & \mathbf{1} & \mathbf{3} \\ & & \mathbf{1} \end{bmatrix}$$

Applying EETs on $\mathcal{BD}(A)$ – II

- **Subtracting a row from next to make a zero**

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & -7 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 6 \\ 3 & 10 & 50 \\ 21 & 102 & 615 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & -7 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & 7 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 3 & 1 & \\ & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 4 & \\ & & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 & \\ & 1 & 5 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & 3 \\ & & 1 \end{bmatrix}$$

Applying EETs on $\mathcal{BD}(A)$ – III

- **Subtracting a row from next to make a zero**

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & -7 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 6 \\ 3 & 10 & 50 \\ 21 & 102 & 615 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & -7 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & 7 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 3 & 1 & \\ & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 4 & \\ & & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 & \\ & 1 & 5 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & 3 \\ & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 3 & 1 & \\ & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 4 & \\ & & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 & \\ & 1 & 5 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & 3 \\ & & 1 \end{bmatrix}$$

- **Is equivalent to setting an entry of $\mathcal{BD}(A)$ to zero and performing no arithmetic.**

Adding a multiple of a column to previous – I

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & a & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ b & 1 & \\ & c & 1 \end{bmatrix} \begin{bmatrix} d & & \\ & e & \\ & & f \end{bmatrix} \begin{bmatrix} 1 & g & \\ & 1 & h \\ & & 1 \end{bmatrix} \underline{\begin{bmatrix} 1 & & \\ & 1 & k \\ & & 1 \end{bmatrix} \color{red}{\begin{bmatrix} 1 & & \\ & y & \\ & x & z \end{bmatrix}}}$$

Adding a multiple of a column to previous – II

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & a & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ b & 1 & \\ & c & 1 \end{bmatrix} \begin{bmatrix} d & & \\ & e & \\ & & f \end{bmatrix} \begin{bmatrix} 1 & g & \\ & 1 & h \\ & & 1 \end{bmatrix} \underline{\begin{bmatrix} 1 & & \\ & 1 & k \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & y & \\ & x & z \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & a & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ b & 1 & \\ & c & 1 \end{bmatrix} \begin{bmatrix} d & & \\ & e & \\ & & f \end{bmatrix} \begin{bmatrix} 1 & g & \\ & 1 & h \\ & & 1 \end{bmatrix} \underline{\begin{bmatrix} 1 & & \\ & y' & \\ & x' & z' \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & k' \\ & & 1 \end{bmatrix}}$$

$$\begin{aligned} x' &= x \\ y' &= y + kx \\ z' &= 1/y' \\ k' &= kz/y_1 \end{aligned}$$

... it's all qd recurrences

Adding a multiple of a column to previous – III

$$\begin{bmatrix}1&&\\&1&\\&a&1\end{bmatrix}\begin{bmatrix}1&&\\b&1&\\&c&1\end{bmatrix}\begin{bmatrix}d&&\\&e&\\&&f\end{bmatrix}\begin{bmatrix}1&g&\\&1&h\\&&1\end{bmatrix}\begin{bmatrix}1&&\\&1&k\\&&1\end{bmatrix}\begin{bmatrix}1&&\\&y&\\&x&z\end{bmatrix}$$

$$=\begin{bmatrix}1&&\\&1&\\&a&1\end{bmatrix}\begin{bmatrix}1&&\\b&1&\\&c&1\end{bmatrix}\begin{bmatrix}d&&\\&e&\\&&f\end{bmatrix}\underline{\begin{bmatrix}1&g&\\&1&h\\&&1\end{bmatrix}\begin{bmatrix}1&&\\&y'&\\&x'&z'\end{bmatrix}}\begin{bmatrix}1&&\\&1&k'\\&&1\end{bmatrix}$$

$$=\begin{bmatrix}1&&\\&1&\\&a&1\end{bmatrix}\begin{bmatrix}1&&\\b&1&\\&c&1\end{bmatrix}\begin{bmatrix}d&&\\&e&\\&&f\end{bmatrix}\underline{\begin{bmatrix}1&&\\&y''&\\&x''&z''\end{bmatrix}\begin{bmatrix}1&g'&\\&1&h'\\&&1\end{bmatrix}}\begin{bmatrix}1&&\\&1&k'\\&&1\end{bmatrix}$$

Adding a multiple of a column to previous – IV

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & a & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ b & 1 & \\ & c & 1 \end{bmatrix} \begin{bmatrix} d & & \\ & e & \\ & & f \end{bmatrix} \begin{bmatrix} 1 & g & \\ & 1 & h \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & k \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & y & \\ & x & z \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & a & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ b & 1 & \\ & c & 1 \end{bmatrix} \begin{bmatrix} d & & \\ & e & \\ & & f \end{bmatrix} \begin{bmatrix} 1 & g & \\ & 1 & h \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & y' & \\ & x' & z' \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & k' \\ & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & a & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ b & 1 & \\ & c & 1 \end{bmatrix} \underline{\begin{bmatrix} d & & \\ & e & \\ & & f \end{bmatrix} \begin{bmatrix} 1 & & \\ & y'' & \\ & x'' & z'' \end{bmatrix}} \begin{bmatrix} 1 & g' & \\ & 1 & h' \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & k' \\ & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & a & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ b & 1 & \\ & c & 1 \end{bmatrix} \underline{\begin{bmatrix} 1 & & \\ & 1 & \\ & x''' & 1 \end{bmatrix} \begin{bmatrix} d & & \\ & e' & \\ & & f' \end{bmatrix}} \begin{bmatrix} 1 & g' & \\ & 1 & h' \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & k' \\ & & 1 \end{bmatrix}$$

Adding a multiple of a column to previous – V

$$\begin{bmatrix}1&&\\&1&\\&a&1\end{bmatrix}\begin{bmatrix}1&&\\b&1&\\&c&1\end{bmatrix}\begin{bmatrix}d&&\\&e&\\&&f\end{bmatrix}\begin{bmatrix}1&g&\\&1&h\\&&1\end{bmatrix}\begin{bmatrix}1&&\\&1&k\\&&1\end{bmatrix}\begin{bmatrix}1&&\\&y&\\&x&z\end{bmatrix}$$

$$=\begin{bmatrix}1&&\\&1&\\&a&1\end{bmatrix}\begin{bmatrix}1&&\\b&1&\\&c&1\end{bmatrix}\begin{bmatrix}d&&\\&e&\\&&f\end{bmatrix}\begin{bmatrix}1&g&\\&1&h\\&&1\end{bmatrix}\begin{bmatrix}1&&\\&y'&\\&x'&z'\end{bmatrix}\begin{bmatrix}1&&\\&1&k'\\&&1\end{bmatrix}$$

$$=\begin{bmatrix}1&&\\&1&\\&a&1\end{bmatrix}\begin{bmatrix}1&&\\b&1&\\&c&1\end{bmatrix}\begin{bmatrix}d&&\\&e&\\&&f\end{bmatrix}\begin{bmatrix}1&&\\&y''&\\&x''&z''\end{bmatrix}\begin{bmatrix}1&g'&\\&1&h'\\&&1\end{bmatrix}\begin{bmatrix}1&&\\&1&k'\\&&1\end{bmatrix}$$

$$=\begin{bmatrix}1&&\\&1&\\&a&1\end{bmatrix}\underline{\begin{bmatrix}1&&\\b&1&\\&c&1\end{bmatrix}\begin{bmatrix}1&&\\&1&\\&x'''&1\end{bmatrix}}\begin{bmatrix}d&&\\&e'&\\&&f'\end{bmatrix}\begin{bmatrix}1&g'&\\&1&h'\\&&1\end{bmatrix}\begin{bmatrix}1&&\\&1&k'\\&&1\end{bmatrix}$$

$$=\begin{bmatrix}1&&\\&1&\\&a&1\end{bmatrix}\underline{\begin{bmatrix}1&&\\b&1&\\&c+x'''&1\end{bmatrix}\begin{bmatrix}d&&\\&e'&\\&&f'\end{bmatrix}}\begin{bmatrix}1&g'&\\&1&h'\\&&1\end{bmatrix}\begin{bmatrix}1&&\\&1&k'\\&&1\end{bmatrix}$$

Done.