Accurate Eigenvalues of the Laplacian

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Main result

- The most common way to compute eigenvalues of the weighted Laplacian is via finite element discretization.
- This yields a symmetric generalized eigenvalue problem of the form $K\mathbf{x} = \lambda M\mathbf{x}$.
- We argue that this system has special structure allowing high relative precision calculation of all eigenvalues including the smallest ones.

Membrane motion

- Consider a moving two-dimensional membrane defined by bounded set $\Omega \subset \mathbf{R}^2$ whose boundaries are clamped.
- Assume the stiffness varies over the membrane and is given by a coefficient field c. Assume the displacement is small and all motion is elastic.
- The governing equation is a two-dimensional wave equation: $u_{tt} = \nabla \cdot (c \nabla u)$ on Ω and $u = 0$ on $\partial\Omega$.

Standing wave

- A standing wave solution to this problem has the form $u(x,t) = e^{i\lambda t} u_0(x)$.
- Substituting this formula into the PDE yields the continuum eigenvalue problem $\nabla \cdot (c \nabla u_0) + \lambda^2 u_0 = 0$.

Finite element discretization (piecewise linear)

- Assume $\mathcal{T}$ is a *finite element mesh* for the domain Ω, that is, a simplicial subdivision into r triangles.
- Let $w_1, \ldots, w_n$ be the mesh nodes not on $\partial\Omega$.
- Let V_h denote the set of piecewise linear continuous functions u on this triangulation satisfying $u|_{\partial\Omega} = 0$. Note: $\dim(V_h) = n$.

Discrete linear equations

- Obtain weak form of PDE: multiply PDE by a test function q satisfying $q|_{\partial\Omega} = 0$; integrate by parts:

$$\int_\Omega \nabla q \cdot c \nabla u = \int_\Omega \lambda q u$$

- Discrete FE equations: find eigenpairs $u \in V_h, \lambda \in \mathbf{R}$ such that the weak form holds for all $q \in V_h$.

Representation with vectors

- Functions $q \in V_h$ are in 1-1 correspondence with vectors $\mathbf{q} \in \mathbf{R}^n$ according to $q_i = q(w_i)$ for $i = 1 : n$ (homeomorphism of vector spaces).
- Let $\mathbf{u} \in \mathbf{R}^n$ be the vector corresponding to FE solution u. Then $\mathbf{u}$ satisfies: for all $\mathbf{q} \in \mathbf{R}^n$, $\mathbf{q}^T K\mathbf{u} = \mathbf{q}^T \lambda M\mathbf{u}$ for matrix K called the *stiffness matrix*, and matrix M, called the *mass matrix*. Equivalent to $K\mathbf{u} = \lambda M\mathbf{u}$.

Stiffness matrix

- Closed-form expressions for entries of K are obtained by considering $\mathbf{u}$ of the form $[0; 0; \cdots ; 0; 1; 0; \cdots 0]$ and similarly for $\mathbf{q}$ and evaluating the weak form for the corresponding u and q. Expressions also available for M.
- Matrix K so determined is $n \times n$ symmetric positive definite. Matrix M is $n \times n$ is symmetric and strongly positive definite.

A test case

- Consider the unit square domain with a border of width η that surrounds an inner square of width $1 - 2\eta$.
- Assign a very high stiffness s to the border and a constant stiffness of 1 to the inner square.
- It can be proved using classical minimax arguments that as $s \rightarrow \infty$, the smaller eigenpairs of this domain tend to eigenpairs of the geometry of the subsquare.

Experiments with this test case

- In exact arithmetic, one expects that as s gets larger, the smallest eigenvalue of this border problem converges to the smallest eigenvalue of the inner domain.
- In the presence of roundoff, convergence is noted up to a certain threshhold value s^*; after this point, the solution diverges because roundoff error prevents accurate computation of the small eigenvalue.

Why is roundoff error a problem

- Roundoff error corrupts the solution to the problem described above because as $s \longrightarrow \infty$, we have that $\|K\| \rightarrow \infty$ proportionally.
- Thus, the largest entries of K as well as the larger eigenvalues increase without bound. Under this circumstance, conventional eigenvalue algorithms cannot recover the smaller eigenvalues accurately.

Our Approach

- Solving $Ku = \lambda M u$, where $K = A^T D J^T J D A$
- A is DSTU, D is diagonal, J is well conditioned; assume M is diagonal
- Equivalent to SVD of $J \underbrace{DAM^{1/2}}_{\text{DSTU}}$
- DSTU problem solved by Demmel; also Pelaez–Moro
- One more time for SVD of $JU\Sigma V^T$