Lecture #9

Outline:

1) Review
2) Vector Fields
3) The Hairy Ball theorem
4) Zeros and their indices
5) The Poincaré - Hopf Formula

Section 1: Review

Definition: A surface is space that locally looks like $\mathbb{R}^2$

↪ ie, zoom in close it just looks like a "piece of paper."

Definition: A polygonal complex is a space obtained by gluing together polygons, edges, and vertices, where by glue we mean that we identify edges w/ edges and vertices w/ vertices (could glue polygon to self)

Definition: Let X = polygonal complex w/

- $V(X)$ = # of vertices
- $E(X)$ = # of edges
- $F(X)$ = # of faces

The Euler characteristic of X is

$$\chi(X) = V(X) - E(X) + F(X)$$

Proposition: Let X and Y be polygonal complexes that are homeomorphic to the same surface. Then their Euler characteristics agree.

$$\chi(X) = \chi(Y)$$

Definition: The Euler characteristic of a surface Σ is the Euler characteristic of any polygonal cpx that is homeomorphic to Σ.

Remark: To compute $\chi(\Sigma)$, break Σ up into regions and count the # of vertices, edges, and faces.

Examples:

1) $\chi(S^2) = 2$

2) $\chi(T^2) = 0$

3) $\chi(\text{klein bottle}) = 0$

4) $\chi(\text{genus 2 surface}) = -2$

5) $\chi(\text{genus } g \text{ surface}) = 2-2g$

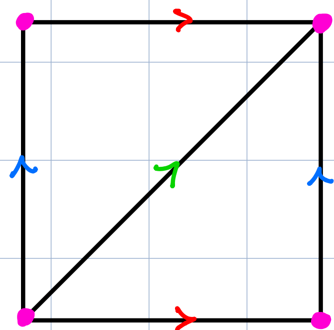
Definition: A <u>triangulation</u> of a surface Σ is a polygonal complex for Σ such that

i) each face is a triangle, and

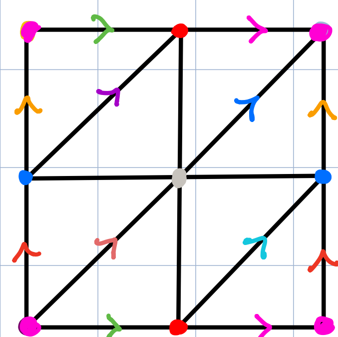
ii) no face is glued to itself.

↪ Once glued each triangle has 3 unique edges and 3 unique vertices.

Example:



$= T^2 =$



non-triangulation

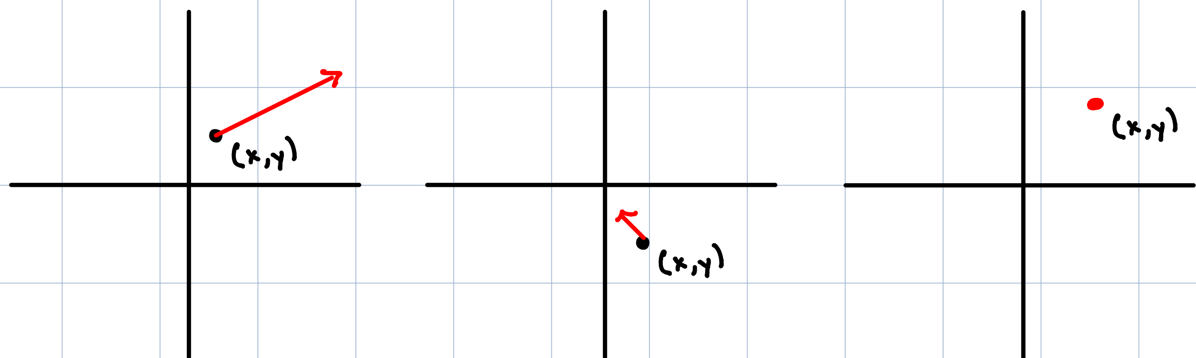
triangulation

Section 2: Vector fields

Definition: A vector at a point (x,y) in $\mathbb{R}^2$ is a choice of direction and magnitude based at (x,y)

↪ arrow at (x,y) that lies in some direction and has some magnitude/length

Picture:



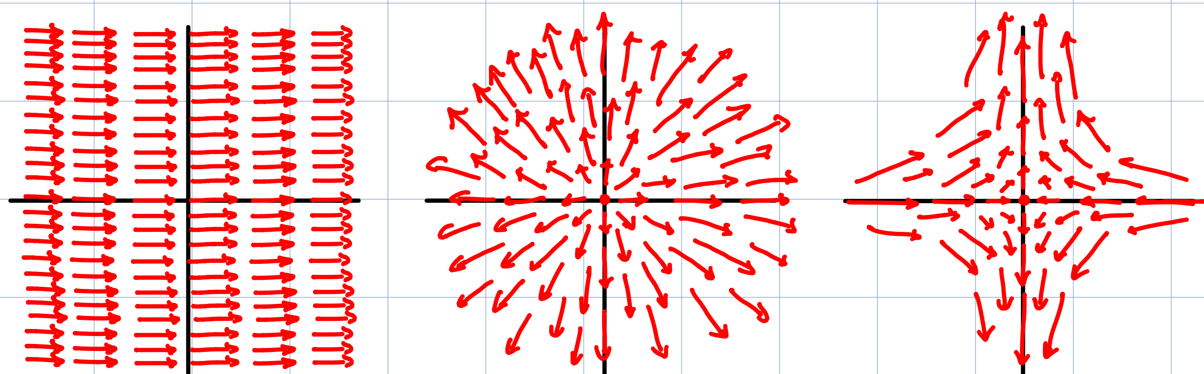
Definition:

A vector field on $\mathbb{R}^2$ is a ① continuous choice of vectors at each point in $\mathbb{R}^2$.

↪ really this should be a "smooth/differentiable" choice.

① Continuous means that if two points are infintesimally close together in $\mathbb{R}^2$, then the vectors at these points have infintesimally close directions and magnitudes.

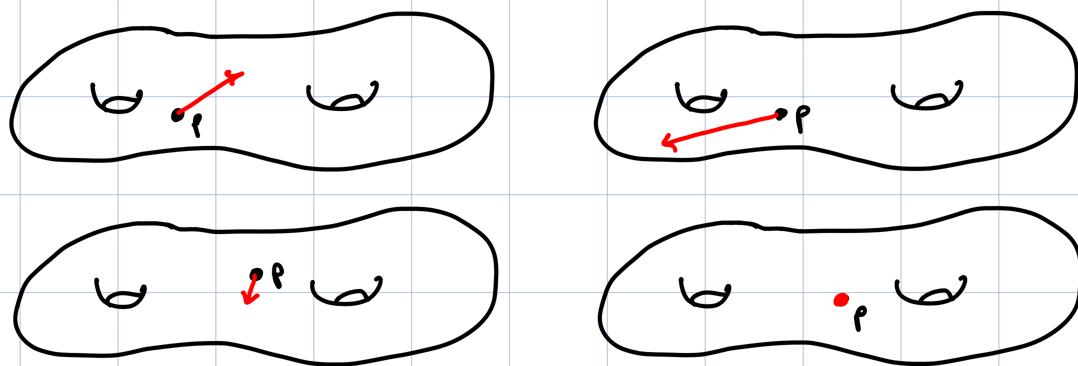
Picture:



Definition: A vector at a point p in a surface Σ is a choice of direction along Σ and magnitude at the point p.

$\hookrightarrow$ arrow at p that lies in some direction along Σ and has some magnitude/length

Picture:



Remark: Locally a vector on Σ just looks like a vector on $\mathbb{R}^2$

Definition: A vector field on a surface Σ is a continuous choice of vectors at each point in Σ.

Remark: Locally a vector field on Σ just looks like a vector field on $\mathbb{R}^2$.

Remark: Intuitively, a vector field on a surface Σ can be described as follows:

Vector fields describe how the wind blows on Σ.

↪ At a location p in Σ, the vector at p gives the direction the wind is blowing and how fast the wind is blowing (magnitude of the vector)

Notation: Let V to denote a vector field on a surface Σ.

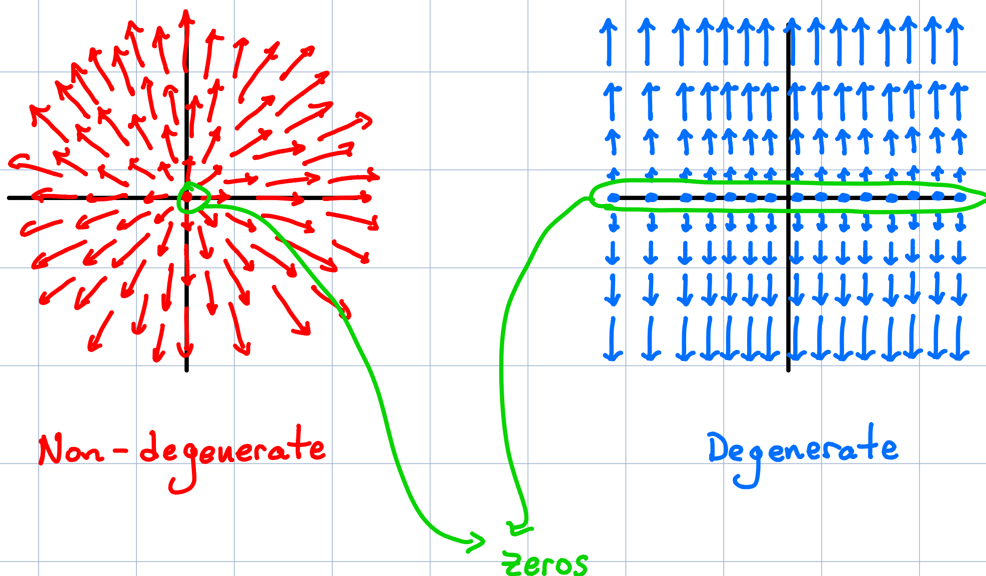
Definition: A vector field is nowhere vanishing, if the magnitudes of all vectors in the vector field are non-zero.

↪ ie, nowhere vanishing if and only if the wind is blowing everywhere.

Definition: A zero of a vector field is a point p in Σ whose associated vector has zero magnitude.

Definition: A vector field is non-degenerate if all of its zeros are isolated.

Example:



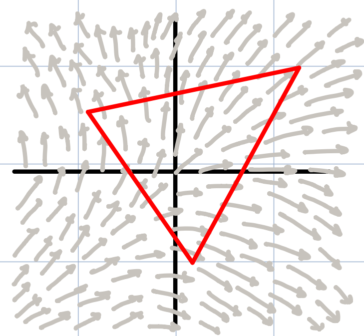
Section 3: The Hairy Ball Theorem

Proposition: If a surface Σ admits a nowhere vanishing vector field, then $\chi(\Sigma) = 0$.

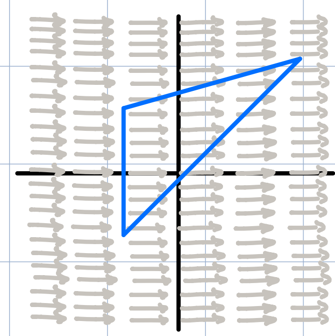
Proof:

1) Pick a triangulation of Σ such that

a) the vector field is locally constant on each triangle, ie, in a neighborhood about any triangle, the vector field looks constant



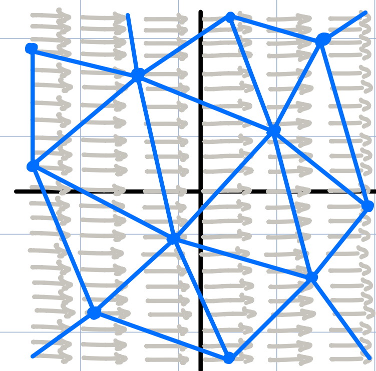
Not locally constant



Locally constant

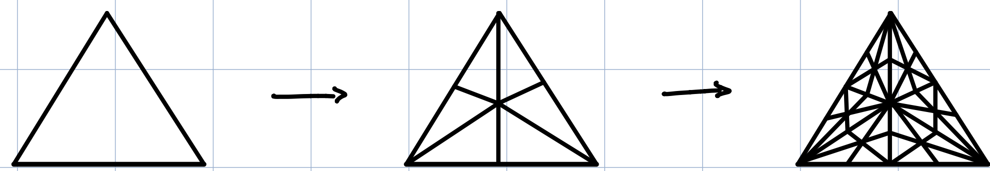
b) it is <u>perendicular</u> to the vector field V, ie, each vector in V does not lie along any edge in the triangulation

non-perpendicular



perpendicular

2) To achieve (a) above, we repeatedly divide our triangles into smaller and smaller triangles

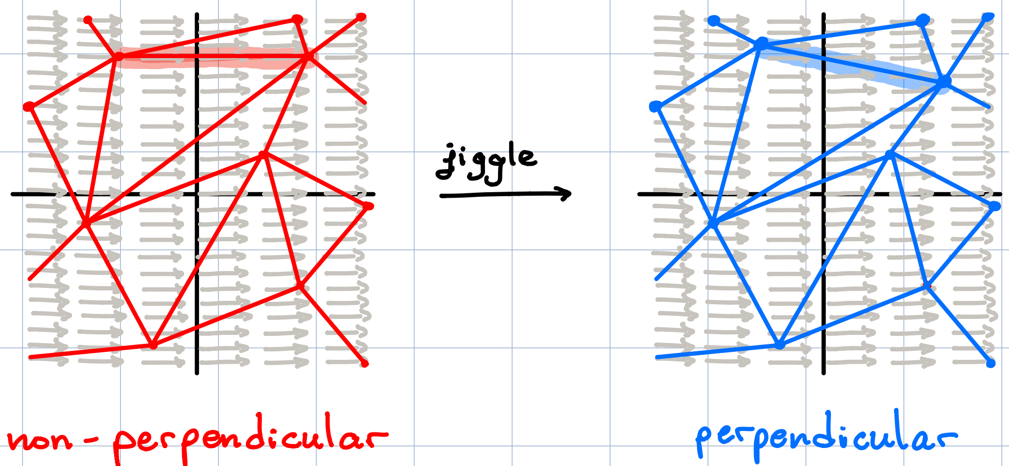


until locally our vector field looks constant

↪ In outerspace, the wind clearly bends and wraps in a non-constant manner, but to us on the surface it just appears to be in one fixed direction.

↪ Dividing Δ's $\longleftrightarrow$ getting closer to the surface.

3) To achieve (b), we start w/ the locally constant triangulation above. Then we can jiggle the vertices and edges a little so that they are perpendicular to the vector field.



4) Place a proton at each vertex and in the center of each face

5) Place an electron at the center of each edge.

6) Let the wind blow the charges.

↪ ie, push the protons and the electrons along the direction of the vector field some small amount

7) If a proton on a vertex moves into a face, then so do the electrons on the two adjacent edges.

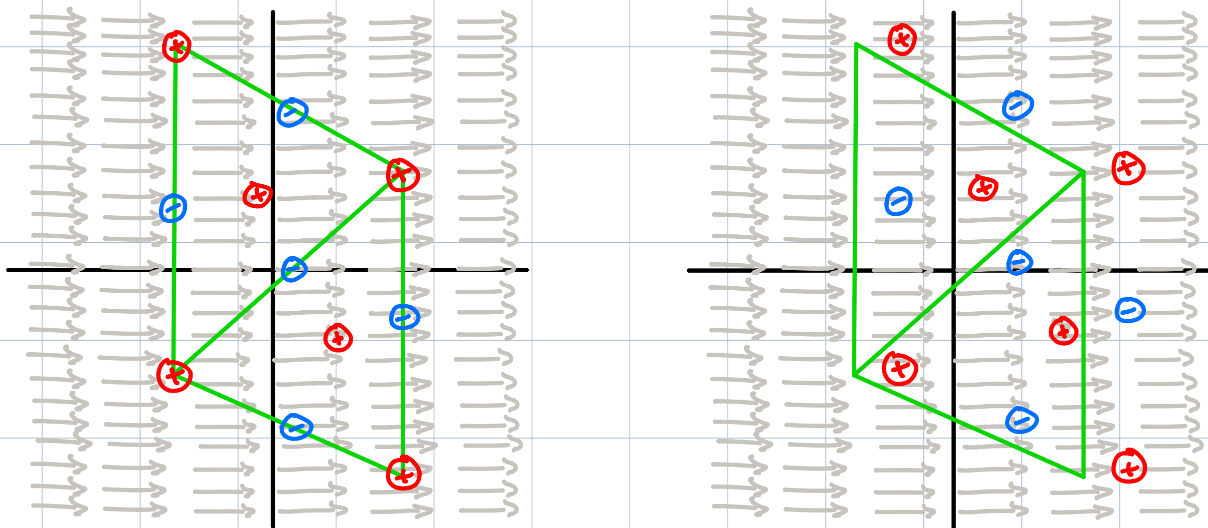
⇒ Either

i) One edge electron gets pushed in

ii) One vertex proton and two edge electrons get pushed

8) If we only push a little the face protons will not leave the faces

9) So after flowing, each region contains the same number of protons as electrons.



10) $\Rightarrow 0 = \#\text{protons} - \#\text{electrons}$

$$= V + F - E$$

$$= \chi(\Sigma)$$

$\square$

Corollary: The sphere does not admit a nowhere vanishing vector field.

Proof: If it did, then the above result $\Rightarrow \chi(S^2) = 0$.

But we know that $\chi(S^2) = 2$.

So it does not admit a nowhere vanishing vector field. $\square$

Section 4: Zeros and their indices

Construction:

1) Let $V =$ non-degenerate vector field.
   - ↪ ie, V has isolated zeros.

2) Let p be a point in Σ.

3) Construct a polygon about p whose edges are perendicular to the vector field V, ie, each vector in V does not lie along any edge in the polygon.
   - ↪ One constructs such a polygon using similar "dividing and locally constant" arguments as before.

4) Place a proton on each vertex of the polygon, an electron on each edge of the polygon, a proton in the polygon.

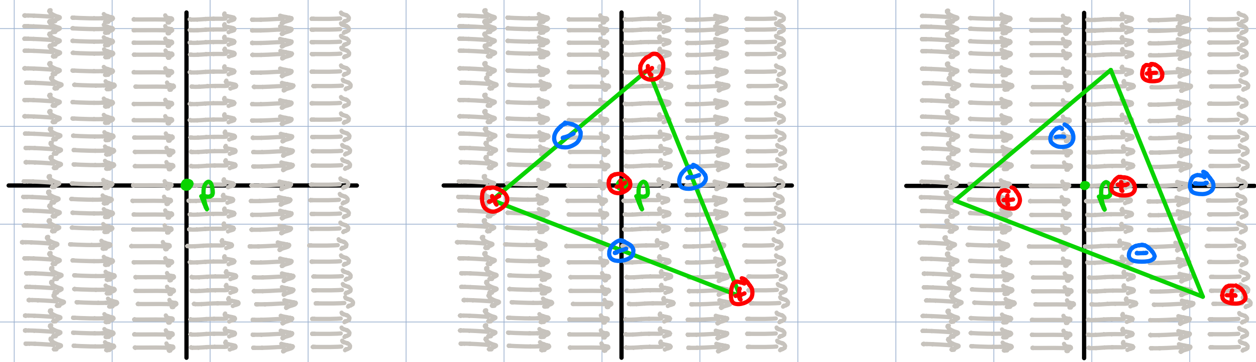
5) Push the charges along the vector field.

Definition: The index of V at p is the number

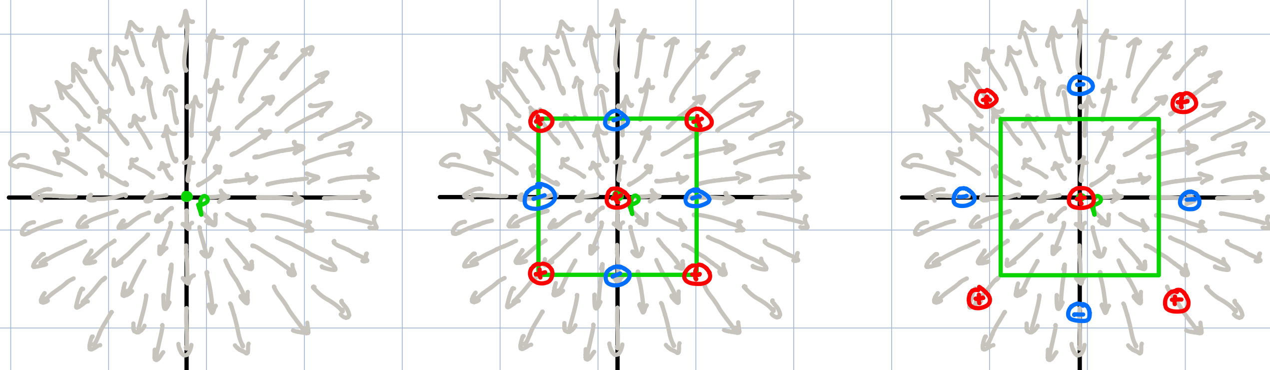
$$\mathrm{ind}(V, p) = \#\text{protons} - \#\text{electron}$$

in polygon after pushing.

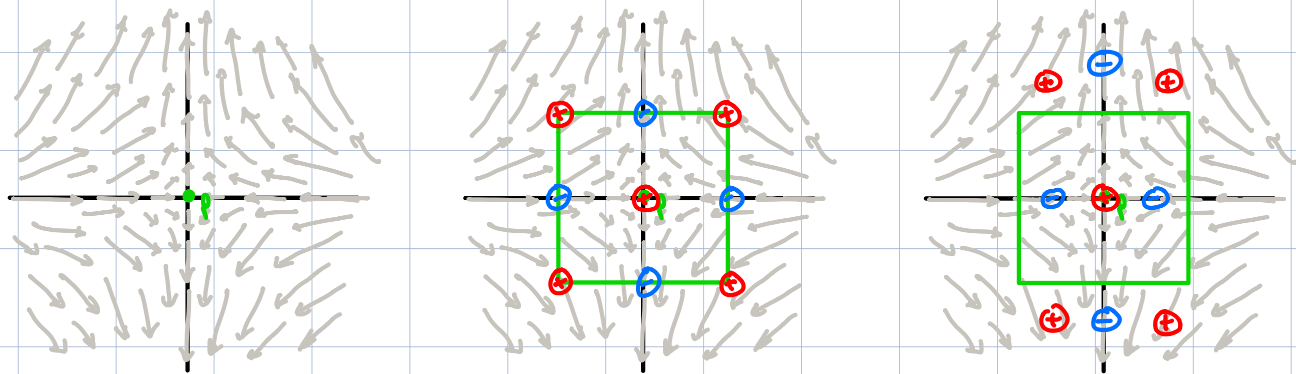
Example :



$\Rightarrow \quad \mathrm{ind}(V, p) = 0$



$\Rightarrow \quad \mathrm{ind}(V, p) = 1$



$\Rightarrow \operatorname{ind}(V, p) = -1$

Lemma: $\operatorname{ind}(V,p)$ does not depend on the choice of polygon that surrounds p.

Lemma: Given any polygonal cpx X for the disk, $\chi(X) = 1$

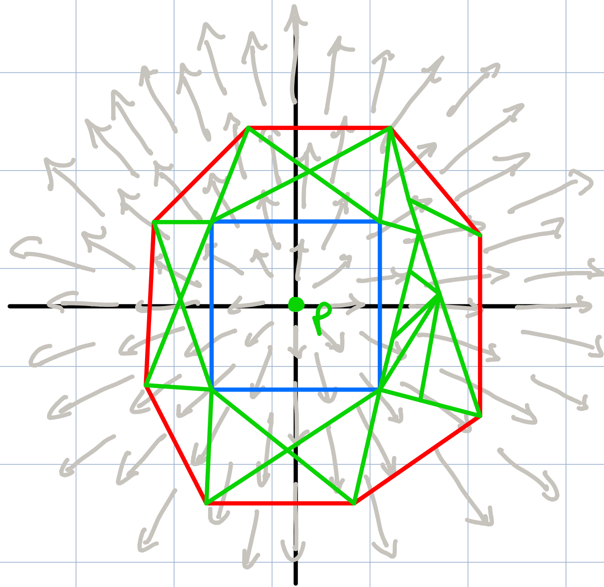
↳ Similar to proof of invar. of χ for surfaces.

Proof :

1) Notice that given any two polygons surrounding p, we may find a smaller polygon that contains p and is contained in the other two polygons.

2) $\Rightarrow$ to prove lemma, it suffices to show that if one polygon is contain in another, then they give the same index.

3) Spse we have larger polygon $\supset$ smaller polygon. Using the edges of the polygons, add more edges in between them to triangulate the region between the two polygons.



4) As before, we can arrange for this triangulation to:

i) have V be locally constant on each triangle

ii) be perendicular to the vector field V

5) Place a proton in each face and on each vertex

Place an electron on each edge.

6) By the lemma,

$$1 = \chi(\text{polygonal cpx for disk given by our triangulation})$$

$$= \#\text{protons} - \#\text{electrons}$$

7) Push the particles along the vector field.

8) Like above, the charge left on each triangle is zero.

9) $1 =$ total charge

$=$ charge in smaller polygon

$+$ charge in triangles

$+$ charge that exited larger polygon

$$= \operatorname{ind}_{\text{small}}(V, p) + 0 + 1 - \operatorname{ind}_{\text{large}}(V, p)$$

10) $\Rightarrow$ $\mathrm{ind}(V,p)$ wrt larger polygon equals $\mathrm{ind}(V,p)$ wrt smaller polygon. $\square$

Theorem: Let V be a non-degenerate vector field on Σ.

$$\chi(\Sigma) = \sum_{\substack{p \text{ zero} \\ \text{of } V}} \mathrm{ind}(V,p)$$

Proof:

1) Fix polygons about each zero whose edges are perendicular to the vector field V

2) Triangulate the remainder of Σ so that all edges are perendicular to the vector field V and so V is locally constant on each triangle.

3) Place a proton in each face and on each vertex. Place an electron on each edge.

4) Push the particles along the vector field.

5) As before, charge in each triangle is zero.

6) By definition, charge in each polygon is the index

7) $\Rightarrow \chi(\Sigma) = \text{total charge} = \sum_{\substack{p \text{ zero} \\ \text{of } V}} \operatorname{ind}(V, p)$ $\square$