Lecture # 7

Outline:

1) Review from last time

2) Brouwer's Fixed point theorem

3) Borsuk - Ulam Theorem

Section 1: Review

Definition: A closed curve in S^1 = circle is a ②continuous ①map $\gamma : S^1 \longrightarrow S^1$.

① We send every pt in S^1 to a point in S^1.

② "Continuous" = we send points infintesimally close together in S^1 to points infintesimally close together in S^1.

↪ We map S^1 into S^1 w/ out ripping or cutting it

Remark: Equivalently, a map $\gamma : S^1 \longrightarrow S^1$ may be viewed as a continuous map

$$\gamma : [0, 2\pi] \longrightarrow S^1$$

w/

$$\gamma(0) = \gamma(2\pi)$$

↳ i.e., a map of a circle is just a map of an interval that connects up at its end points.

↳ Intuitively, $\gamma : S^1 \to S^1$ gives a way of wrapping/laying a string onto a circle such that you can tie together its ends.

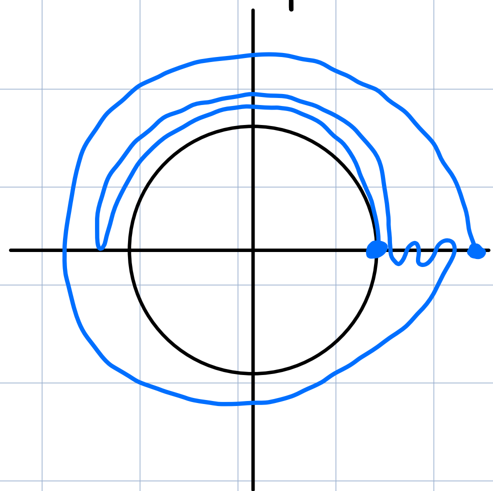
Example: 1) $\gamma_n : [0, 2\pi] \to S^1$ given by

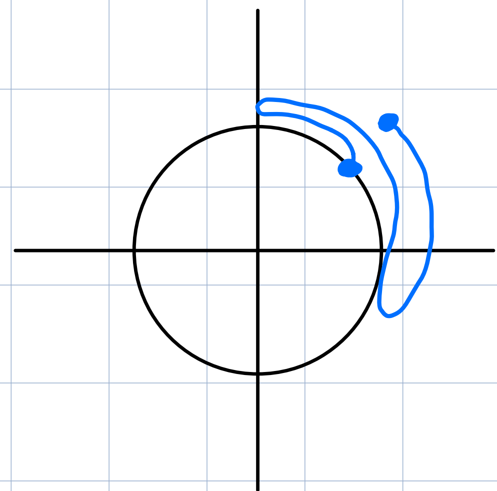
$$\gamma(t) = (\cos(nt), \sin(nt))$$

$\hookrightarrow$ γ_n wraps n-times around the circle.

2 Crazier examples

*Squish blue onto circle to get actual curves





Lemma: (Curve Lifting) Given a closed curve $\gamma : S^1 \to S^1$, there exists a function $f : [0, 2\pi] \to \mathbb{R}$ st

1) $f(0) = f(2\pi) + 2\pi \cdot n$ for some integer n
2) $\gamma(t) = (\cos(f(t)), \sin(f(t)))$

↳ f is called a lift of γ to $\mathbb{R}$.

Idea: $f(t)$ = Accumulated angle of rotation of $\gamma(t)$ measured w/ respect to $(1,0)$

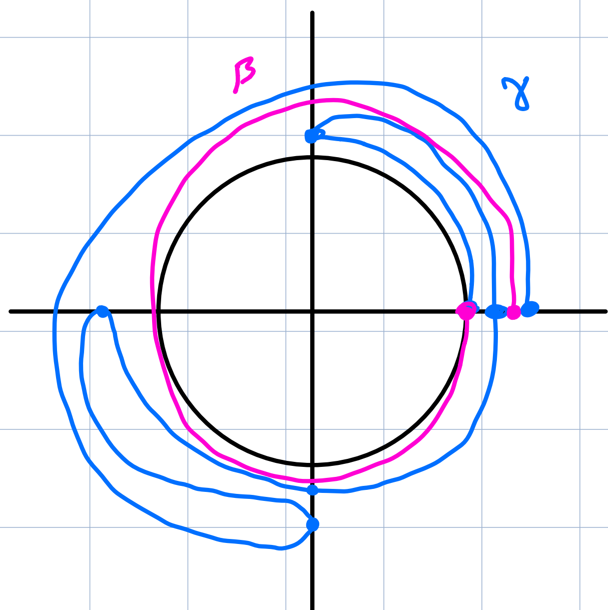
↳ rotate clockwise angle decreases

↳ rotate counter clockwise angle increases

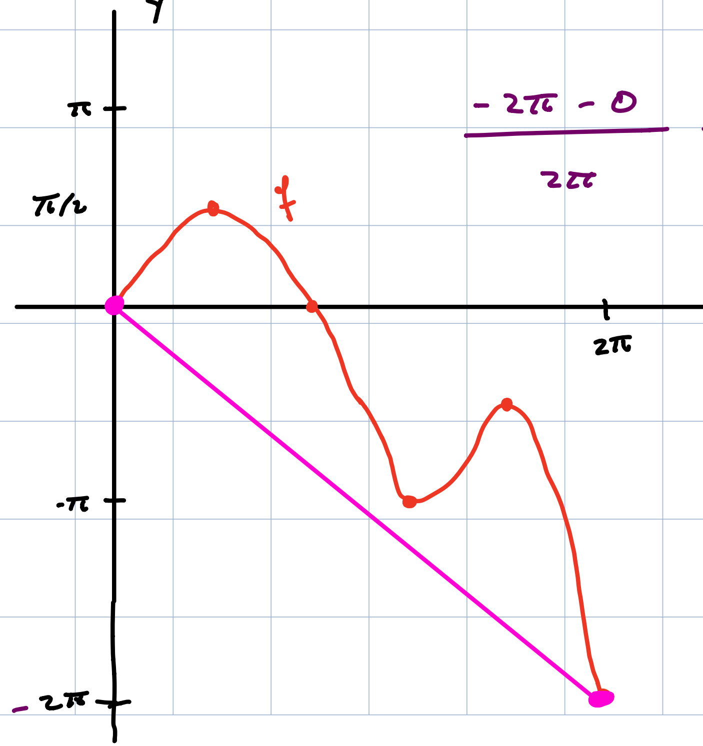
Example: 1) A lift of γ_n is given by $f(t) = n \cdot t$

$$\deg(\gamma_n) = \frac{2\pi n - 0}{2\pi} = n.$$

2)



y-axis = accumulated angle.



$$\frac{-2\pi - 0}{2\pi} = -1$$

Definition: The degree of a closed curve $\gamma : S^1 \to S^1$ is

$$\deg(\gamma) = (\tilde{f}(2\pi) - \tilde{f}(0)) / 2\pi$$

where $\tilde{f}$ is <u>any</u> lift of γ to $\mathbb{R}$.

↳ This did not depend on the choice of lift

Remark: 1) Intuitively, $\deg(\gamma)$ = signed # of times γ completely wraps around the circle

↳ signed: wraps clockwise = negative wrap

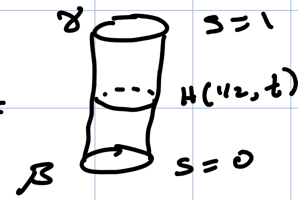
wraps counter clockwise = positive wrap

Example: 1) $\deg(\gamma_n) = n$

2) See above examples.

Definition: Two closed curves $\beta : S^1 \to S^1$ and $\gamma : S^1 \to S^1$ are **homotopic** if there is a continuous map $H : [0,1] \times S^1 \to S^1$ satisfying

1) $H(0,t) = \beta(t)$
2) $H(1,t) = \gamma(t)$

$H =$ 

Remark: Equivalently, $H : [0,1] \times [0, 2\pi] \to S^1$ w/

1) $H(0,t) = \beta(t)$
2) $H(1,t) = \gamma(t)$
3) $H(s,0) = H(s,2\pi)$

Remark:

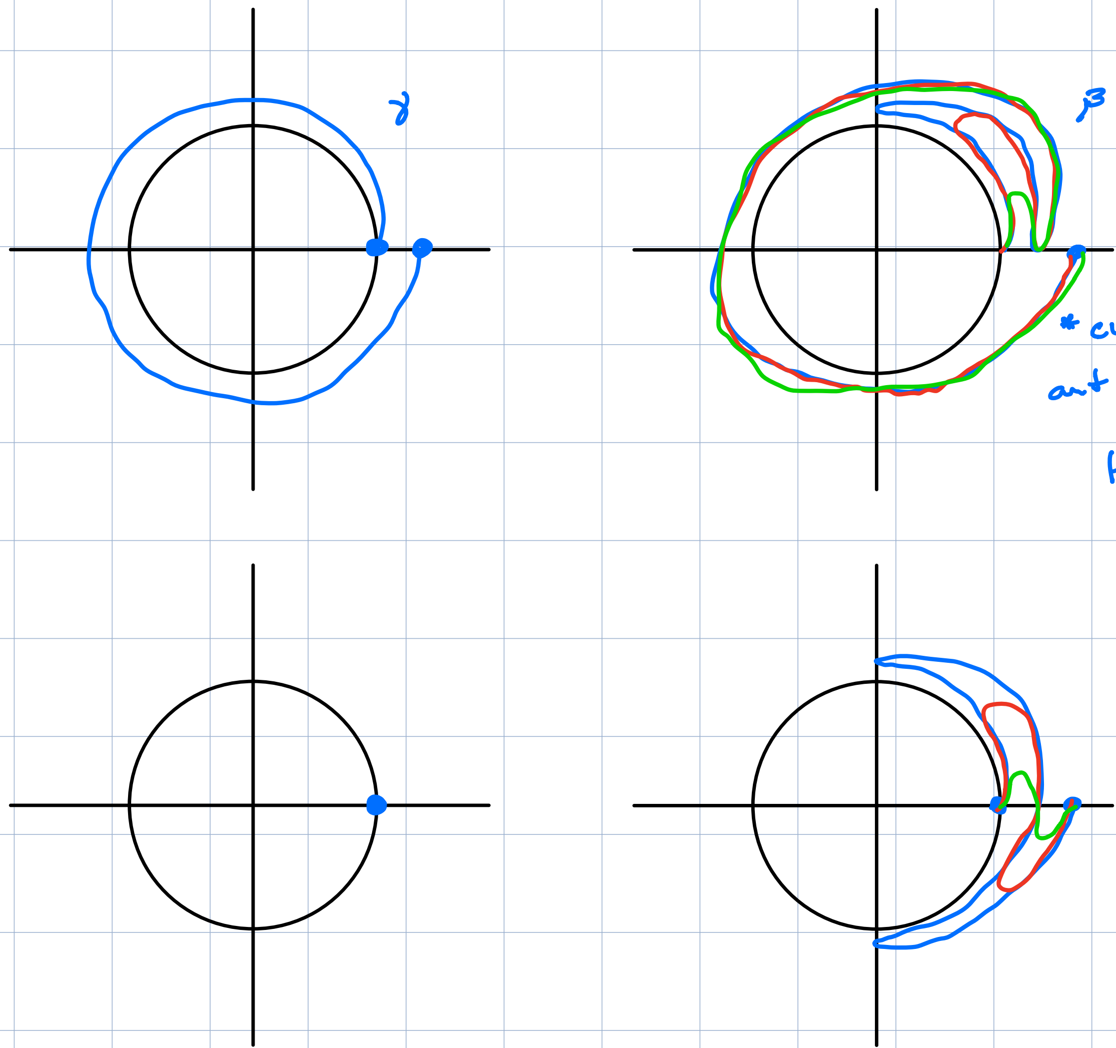
1) For each s_0 in $[0,1]$, $H(s_0, t)$ defines a closed curve in S^1.

2) H parameterizes a family of curves that interpolates between β and γ.

3) Intuitively, H parameterizes how we can push, compress, deform the image of β in S^1 to the image of γ in S^1.

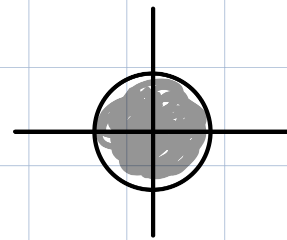
Theorem: Two closed curves $\beta : S^1 \to S^1$ and $\gamma : S^1 \to S^1$ are homotopic if and only if $\deg(\beta) = \deg(\gamma)$

Example:



* curve when ant pulls in by s_0 is $H(s_0, t)$

Section 2: Brouwer's Fixed Point Theorem



Definition: $\mathbb{D} = \{ (x, y) \text{ in } \mathbb{R}^2 \mid x^2 + y^2 \leq 1 \}$

$= $ unit disk in the plane

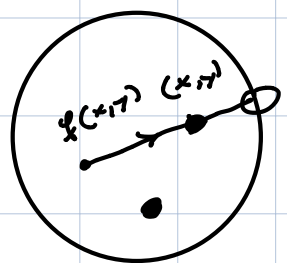
Theorem: Given a continuous map $f : \mathbb{D} \longrightarrow \mathbb{D}$, there exists a point (x, y) in $\mathbb{D}$ such that $f(x, y) = (x, y)$

↳ ie, f has a fixed point.

Proof:

1) Suppose by way of contradiction that f does <u>not</u> have any fixed points.

2) Define a map $r : \mathbb{D} \longrightarrow S^1$ as follows:



a) Consider the ray from $f(x,y)$ to (x,y)

b) follow the ray until you hit the boundary of the disk, which is a circle

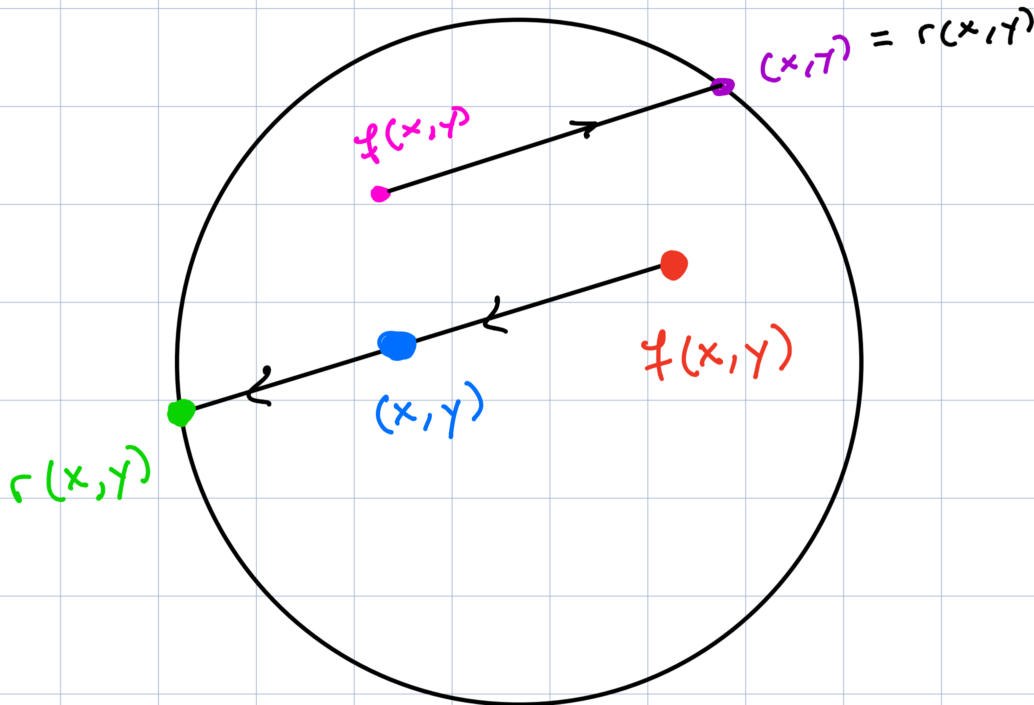
c) Set $r(x,y)$ = point where ray meets the boundary

↳ Note, that to get such a ray we needed $f(x,y) \neq (x,y)$

3) Note, r is continuous.

$$\begin{tikzcd} S^1 \arrow[d] \arrow[dr, dashed, "\gamma, \beta"] & \\ \mathbb{D} \arrow[r, "r"'] & S^1 \end{tikzcd}$$

Picture:



4) Define $\gamma : S^1 \to S^1$ as follows:

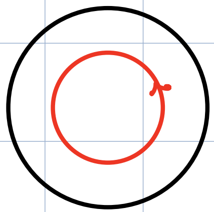
Take S^1, include it into boundary of $\mathbb{D}$, and then apply the map $r : \mathbb{D} \to S^1$.

5) Define $\beta : S^1 \to S^1$ as follows:

Take S^1, map it to $(0,0)$ in $\mathbb{D}$, and then apply the map $r : \mathbb{D} \to S^1$.

6) By construction, $\deg(\gamma) = \underline{1}$

7) β is a constant map, so $\deg(\beta) = 0$



8) Define a homotopy $H : [0,1] \times S^1 \longrightarrow S^1$ as follows:

$$H(s,t) = r\big(s \cdot \cos(t),\ s \cdot \sin(t)\big)$$

9) $H(0,t) = r(0,0) = \beta(t)$

10) $H(1,t) = \big(\cos(t), \sin(t)\big) = \gamma(t)$

11) $\Rightarrow$ γ is homotopic to β

$\Rightarrow$ $1 = \deg(\gamma) = \deg(\beta) = 0$

a contradiction

12) $\Rightarrow$ $f(x,y) = (x,y)$ for at least some (x,y) in $\mathbb{D}$. $\square$

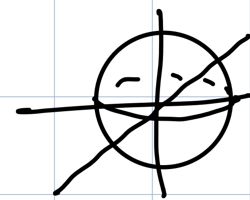
Exercise: Given a continuous function $f: [0,1] \to [0,1]$, there exists x in $[0,1]$ such that $f(x) = x$.

Hint:
1) You could use the intermediate value theorem
2) <u>or</u> try to replicate the proof of Brouwer's fixed point theorem but in one lower dimension.

Exercise: Given a map $f: S^1 \to \mathbb{R}$, there exists a point (x,y) in S^1 such that

$$f(x,y) = f(-x,-y)$$

Section 3: Borsuk-Ulam Theorem



Definition:

- $S^2 = \{(x, y, z) \text{ in } \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$

  $=$ unit sphere in $\mathbb{R}^3$

- The antipodal point of (x, y, z) in S^2 is the point $(-x, -y, -z)$

Theorem: Given a continuous map $f : S^2 \to \mathbb{R}^2$, there exists a point (x, y, z) in S^2 such that

$$f(x, y, z) = f(-x, -y, -z)$$

Example: $f: S^2 \to \mathbb{R}^2$ via a location an earth is mapped to (temperature, humidity).

So the thm $\Rightarrow$ there exists antipodal locations on the earth w/ the same temperature and humidity.

Definition: The norm of a point (x,y) in $\mathbb{R}^2$ is

$$|(x,y)| = \sqrt{x^2 + y^2}$$

$\leftrightarrow$ distance of (x,y) from the origin $(0,0)$.

Remark: If $(x,y) \neq (0,0)$, then

$$\frac{(x,y)}{|(x,y)|} = \left(\frac{x}{|(x,y)|}, \frac{y}{|(x,y)|} \right)$$

lies on $S^1 \subset \mathbb{R}^2$

$\hookrightarrow$ We've just scaled in (x,y) according to its distance from the origin so that its new distance from the origin is 1, ie, it lies on S^1.

Proof:

1) Suppose by way of contradiction that

$$f(x,y,z) \neq f(-x,-y,-z)$$

for all (x,y,z) in S^2.

2) Define $g : S^2 \to \mathbb{R}^2$ via

subtract x-coor
subtract y-coor.

$$g(x,y,z) = f(x,y,z) - f(-x,-y,-z)$$

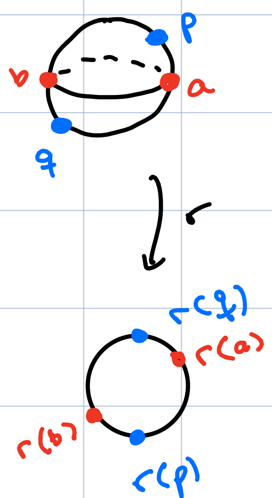
↳ By (1), $g \neq (0,0)$.

3) Define a map $r : S^2 \to S^1$ as follows:

$$r(x,y,z) = g(x,y,z) / |g(x,y,z)|$$

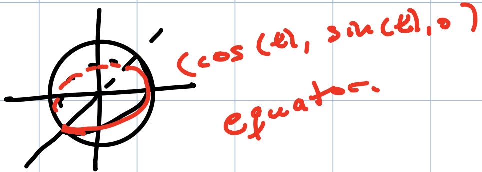
4) Note, $r(x,y,z) = -r(-x,-y,-z)$ on opposite side of circle.

↪ ie, r sends antipodal points on S^2 to antipodal points on S^1.



$$
\begin{aligned}
-r(-x,-y,-z) &= \frac{-g(-x,-y,-z)}{|g(-x,-y,-z)|} \\
&= \frac{-(f(-x,-y,-z)-f(x,y,z))}{|f(-x,-y,-z)-f(x,y,z)|} \\
&= \frac{(f(x,y,z)-f(-x,-y,-z))}{|f(x,y,z)-f(-x,-y,-z)|} \\
&= r(x,y,z)
\end{aligned}
$$

4) Define a curve $\gamma : [0, 2\pi] \to S^1$ via

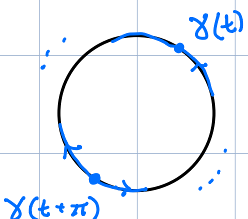


$$\gamma(t) = r\,(\cos(t), \sin(t), 0)$$

5) Notice that

$$\gamma(t+\pi) = r\,(\cos(t+\pi), \sin(t+\pi), 0)$$
$$= r\,(-\cos(t), -\sin(t), 0)$$
$$= -r\,(\cos(t), \sin(t), 0)$$
$$= -\gamma(t)$$

6) $\Rightarrow$ $\gamma(t+\pi)$ is always on the opposite side of the circle as $\gamma(t)$



7) Let f be a lift of γ

8) Let $\beta(t) := \gamma(t+\pi)$.

9) Note $f(t+\pi)$ is a lift of β.

10) (6) $\Rightarrow f(t) - f(t+\pi) =$ odd multiple of π, say $k \cdot \pi$

11) $f(0) = f(\pi) + k\pi = f(2\pi) + 2k\pi$

12) $\Rightarrow \deg(\gamma) = \dfrac{f(2\pi) - f(0)}{2\pi} = k \neq 0$

13) But γ is homotopic to a constant curve

↳ shrink equator down to southpole and apply r.

$\Rightarrow \deg(\gamma) = 0$, a contradiction

14) $\Rightarrow f(x,y,z) = f(-x,-y,-z)$ for some (x,y,z) $\square$