Lecture # 5

Outline:

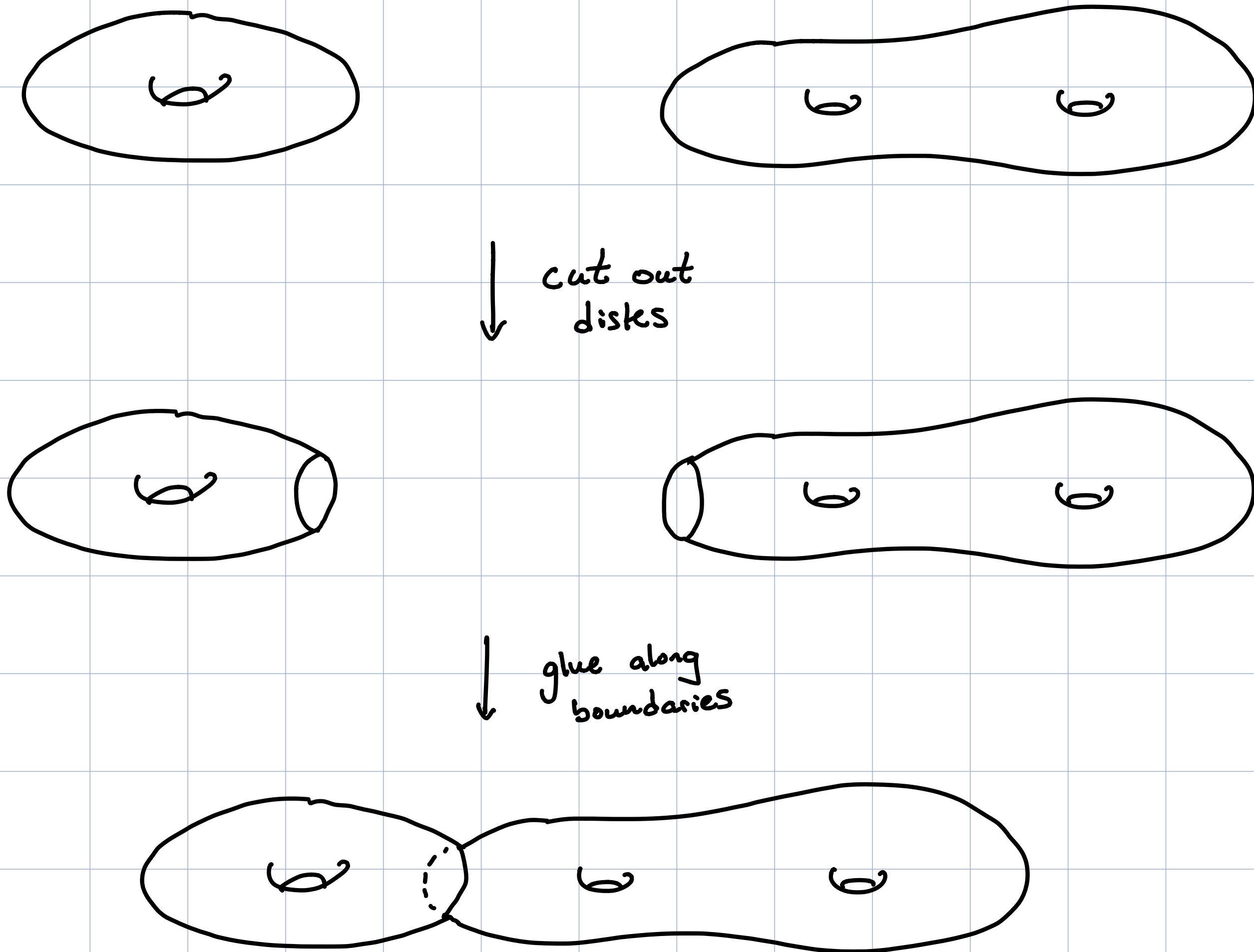
1) Review from last time
2) Curves in Surfaces and Orientability
3) Preliminaries on Graphs
4) 2 - dimension Poincare Conjecture
5) Classification of Surfaces

Section 1: Review

Definition: Given two surfaces X and Y, the connect sum of X and Y, denoted $X \# Y$, is obtained via

1) Remove an open disk from both X and Y to create two surfaces w/ "boundaries"
2) Glue the resulting boundaries together to create the new surface $X \# Y$.

Picture:



Example:

1) $T^2 \# T^2 =$ genus 2 surface

2) $S^2 \# S^2 = S^2$

3) $S^2 \# T^2 = T^2$

4) $\left.T^2 \# \dots \# T^2\right\}$ g-times = genus g surface.

Proposition: $\chi(X \# Y) = \chi(X) + \chi(Y) - 2$

$\hookrightarrow$ $Y = T^2$, $\chi(T^2) = 0$

$\chi(X \# T^2) = \chi(X) - 2$

$\chi(X) = \chi(X \# T^2) + 2$

Section 2: Curves in Surfaces and Orientability

Definition:

- A closed curve in a surface Σ is a ② continuous ① map $\gamma : S^1 = \text{circle} \longrightarrow \Sigma$.

  ① We send every pt in S^1 to a point in Σ.

  ② "Continuous" = we send points infintesimally close together in S^1 to points infintesimally close together in Σ

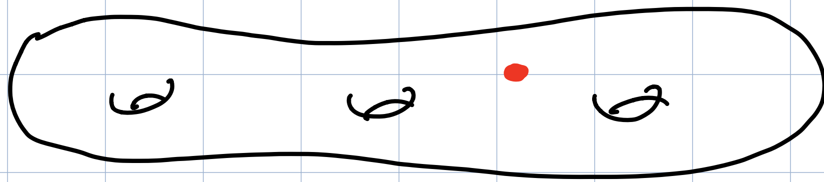
  ↪ We map S^1 into Σ w/out ripping or cutting it

- A curve is simple if the image of the curve in Σ does not cross/meet itself and the circle can be "pushed"/deformed to look like a seq. of edges

Examples:

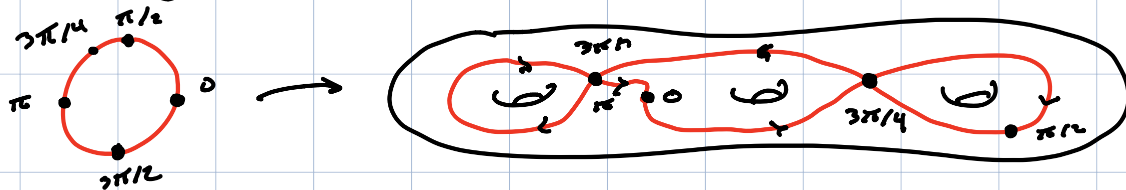
1) Constant curve

non-simple



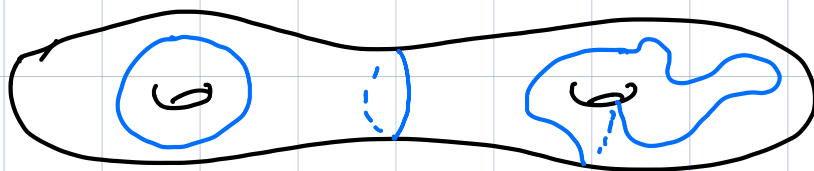
2) Crossing curve

non-simple



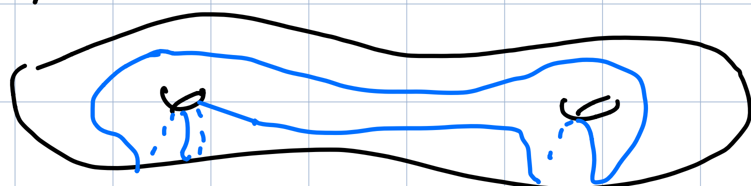
3) Simple closed curves

simple



4) Crazy curves

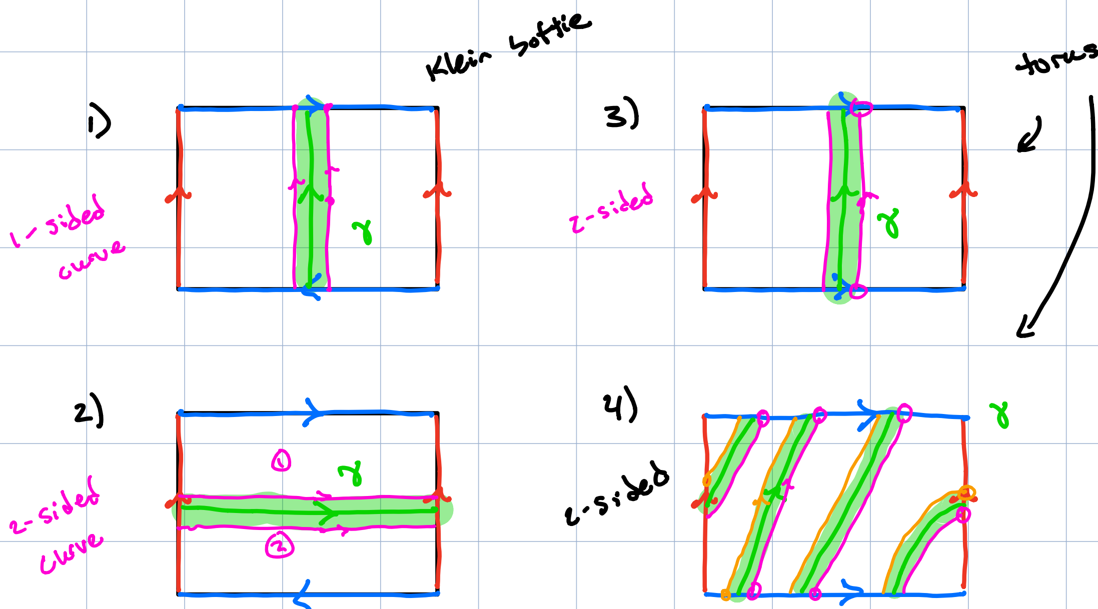
simple



Definition:

- A simple closed curve is 1-sided if a small thickening of the curve in Σ is a Möbius band.
- A simple closed curve is 2-sided if a small thickening of the curve in Σ is a cylinder

Examples:

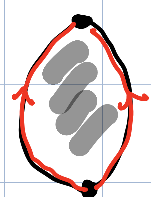


Remark: If there are 1-sided curves on Σ, then we don't know what is up/down or in/out. We have no reference outward direction.

Definition: A surface is orientable if it has <u>no</u> 1-sided curves.

Example:

1) Connect sums of tori = orientable
2) Klein bottle is non-orientable



Definition: A surface is <u>compact</u> if it admits a polygonal complex structure w/ a finite # of vertices, edges, and faces.

Theorem: Every compact orientable surface is homeomorphic to a connect sum $T^2 \# \cdots \# T^2 \# S^2$ for some # of T^2's.

Section 3: Preliminaries on Graphs

Definition:

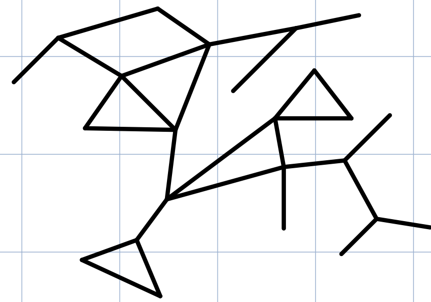
- A <u>graph</u> is a polygonal complex composed of edges.
- A graph is a <u>tree</u> if every pair of vertices is connected via a unique sequence of edges.
  - ↪ A tree is a graph w/ <u>no</u> loops

Proposition: Let Γ = connected graph. There exists a subcollection of edges of Γ that form a tree T that touches every vertex in Γ. T is called a <u>spanning tree for Γ</u>

Remark: A graph can have multiple spanning trees.

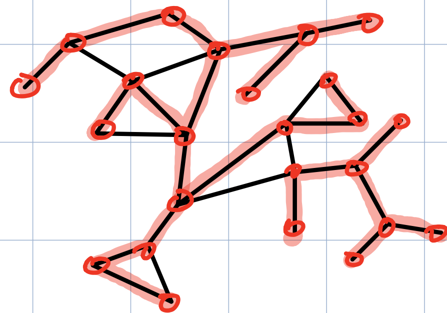
Example:

Graph



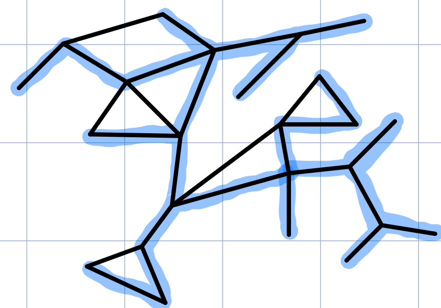
$= \Gamma$

Spanning Tree



$= T$

Another spanning tree



Proof :

1) Buildup Γ one edge at a time.

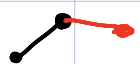
$$\Gamma_0 \xrightarrow[\text{edge}]{\text{Add}} \Gamma_1 \xrightarrow[\text{edge}]{\text{Add}} \Gamma_2 \xrightarrow[\text{edge}]{\text{Add}} \cdots \xrightarrow[\text{edge}]{\text{Add}} \Gamma_n = \Gamma$$

2) We sequentially build spanning trees T_i for Γ_i.

3) $\Gamma_0 =$ edge, $T_0 = \Gamma_0$

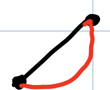
4) $\Gamma_0 \longrightarrow \Gamma_1$: either

a) A new vertex is added to Γ_0 to create Γ_1



↳ create new "step"

b) No new vertex is " " " " " "



↳ create a loop

5) If a) => Set $T_1 = T_0 \cup$ new edge

If b) => Set $T_1 = T_0$

6) $\tilde{\Gamma}_i \longrightarrow \tilde{\Gamma}_{i+1}$: either

a) A new vertex is added to $\tilde{\Gamma}_i$ to create $\tilde{\Gamma}_{i+1}$

b) No new vertex is " " " " " "

7) If a) $\Rightarrow$ Set $T_{i+1} = T_i \cup$ new edge

If b) $\Rightarrow$ Set $T_{i+1} = T_i$

8) By construction, each T_i is a tree and touches every vertex of $\tilde{\Gamma}_i$. So repeated we obtain the result. $\square$

Lemma: Let Γ = connected graph. We have

$$V(\Gamma) - E(\Gamma) = \chi(\Gamma) \leq 1$$

w/ equality iff Γ is a tree.

Proof: 1) If Γ = tree, then we claim that $\chi(\Gamma) = 1$

i) Build up Γ sequentially: $\Gamma_1, \Gamma_2, \Gamma_3, \ldots, \Gamma_n = \Gamma$.

ii) Since Γ is a tree each time we add an edge, we also add another vertex

↳ if not, we would conn. two vertices via at least 2 different seqs of edges

iii) So Γ_1 = edge $\Rightarrow \chi(\Gamma_1) = 2 - 1 = 1$

$$\Gamma_2 = V(\Gamma_1) - E(\Gamma_1) + 1 - 1 = 1$$

iv) Repeatedly, $\chi(\Gamma_{i+1}) = V(\Gamma_i) - E(\Gamma_i) + 1 - 1 = 1$

v) $\Rightarrow \chi(\Gamma = \text{tree}) = 1$

2) Spse Γ is not necessarily a tree.

Let T = spanning tree for Γ.

$$\chi(\Gamma) = V(\Gamma) - E(\Gamma)$$
$$= V(T) - E(T) - E(\text{not in } T)$$
$$= \chi(T) - E(\text{not in } T)$$
$$\leq 1$$

3) Note, if $E(\text{not in } T) = 0$, then $\Gamma = T$.

$\Rightarrow \chi(\Gamma) = 1$ if and only if Γ = tree.

Theorem: Let Σ = compact surface. Then $\chi(\Sigma) \le 2$ and $\chi(\Sigma) = 2$ if and only if Σ is homeomorphic to S^2.

Proof:

1) Fix a polygonal cpx X that gives Σ.

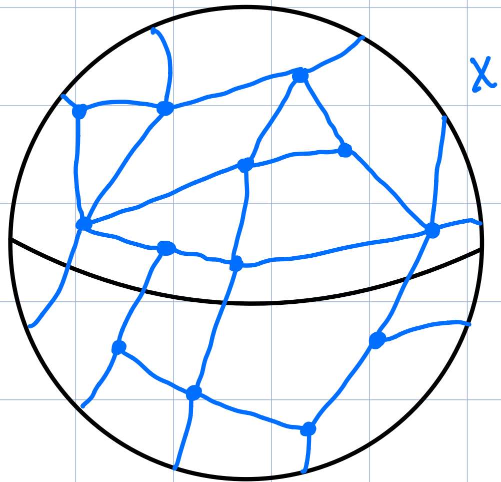
2) Let T = spanning tree for the graph that is made up of the edges of X.

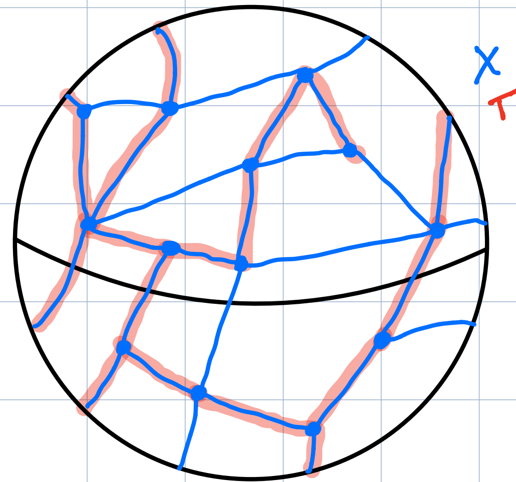
3) Define a graph Γ (that can be drawn on X) via:

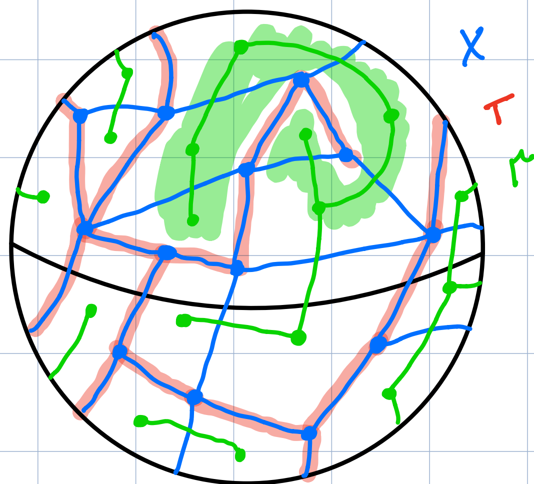
   a) place a vertex in the center of each face of X.

   b) Connect two vertices via an edge for each edge in X that is not in T that their faces share

Picture :







4) $$\begin{aligned} \chi(\Sigma) &= \chi(X) \\ &= V(X) - E(X) + F(X) \\ &= V(T) - E(T) - E(\Gamma) + V(\Gamma) \\ &= \chi(T) + \chi(\Gamma) \\ &\leq 2 \end{aligned}$$

prev. lemma χ of graphs.

↪ This gives the first claim

5) Spse $\chi(\Sigma) = 2$, then $\chi(\Gamma) = 1$

6) $\Rightarrow$ Γ is a tree

7) Thicken T and Γ into weird looking disks, which are trees, until they fill out Σ.

8) $\Rightarrow$ Σ is gluing of two disks along their boundaries

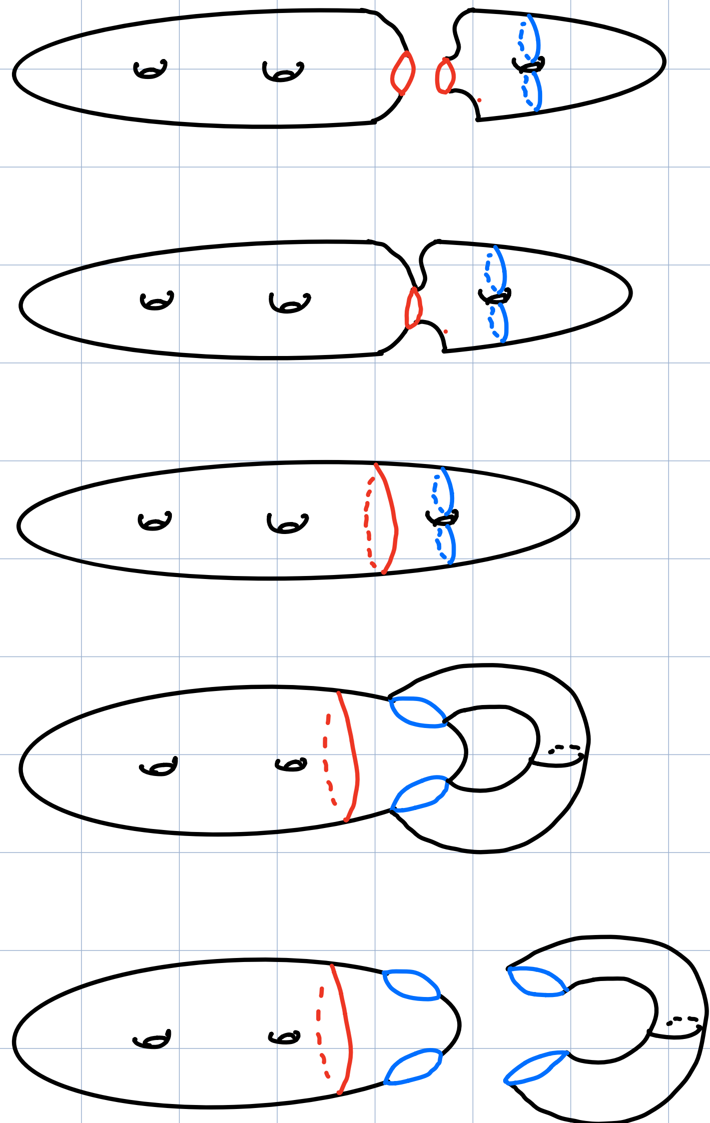
9) $\Rightarrow$ Σ is homeomorphic to S^2. $\square$

Lemma: If a surface Σ has a 2-sided curve that does not separate Σ into two pieces, then Σ is homeomorphic to $\Sigma' \# T^2$ for some surface Σ'.

Proof:

1) Let γ = 2-sided curve in Σ.
2) Thicken γ to cylinder in Σ.
3) Note, $\Sigma' \# T^2$ can also be obtained via:
   i) Remove two disjoint disks from Σ'.
   ii) Connect these boundaries via gluing in a cylinder.
4) So removing γ from Σ and capping off the boundaries w/ disks undoes a connect sum.
5) Upshot, γ let's us realize Σ as connect sum w/ T^2.

Picture :



Proof:

1) Let X = poly. cpx for Σ
2) Let T and Γ be defined as before.
3) If Γ = tree, then as argued before $\Sigma = S^2$.
4) So we assume Γ is not a tree.

   $\Rightarrow$ Γ has a loop γ = 2-sided curve
5) We claim that γ does not separate Σ.

   ↪ If not, $\Sigma - \gamma = \Sigma_0 \cup \Sigma_1$ two separate pieces

   ↪ If we remove the faces and edges that γ touches in X, then this divides X into poly. cpxes X_0 and X_1 for Σ_0 and Σ_1

↪ γ doesn't meet T

⇒ T is completely contained in, say, X_0

↪ But T contains all the vertices of X.

⇒ X_1 has no vertices and thus no polygons

⇒ X_1 is empty, a contradiction.

6) By previous lemma, $\Sigma = \Sigma' \# T^2$ for some surface Σ'.

7) $\chi(\Sigma') = \chi(\Sigma) + 2$

8) ⇒ Repeating this setup w/ Σ replaced by Σ' realizes $\Sigma' = \Sigma'' \# T^2$. $(\Sigma = \Sigma'' \# T^2 \# T^2)$

9) Eventually, this will terminate as $\chi(\Sigma'') = \chi(\Sigma') + 2$, ie, eventually $\chi(\Sigma'') = 2$ and thus $\Sigma'' = S^2$. $\square$

Nextime:

- ?????