Lecture # 3

Outline:

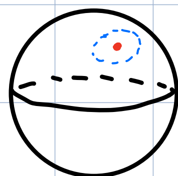
1) Review from last time
2) More on Planar diagrams
3) The Euler characteristic
4) Planarity of graphs
5) How to make new surfaces (if time permits)

Section 1: Review

Definition: A <u>surface</u> is space that locally looks like $\mathbb{R}^2$

↪ ie, zoom in close it just looks like a "piece of paper."

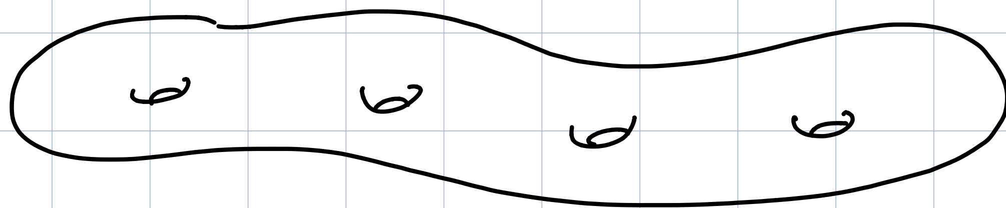
Examples: ① Sphere $= S^2$



② Torus $= T^2$

③ Klein bottle

④ Intertubes w/ multiple holes.



Remark: • We want a way of viewing surfaces as combinatorial objects.

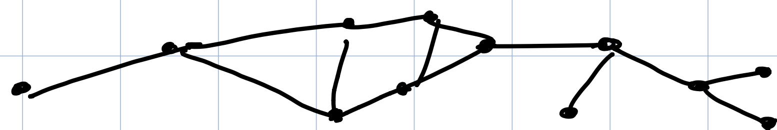
$$\begin{tikzcd} \text{Spaces} \arrow[r] & \text{Combinatorics} \arrow[l, dashed, bend right, "?"'] \end{tikzcd}$$

$$\text{Surfaces} \longrightarrow \text{Poly. cpxes} \xrightarrow{\text{Euler char.}} \text{numbers}$$

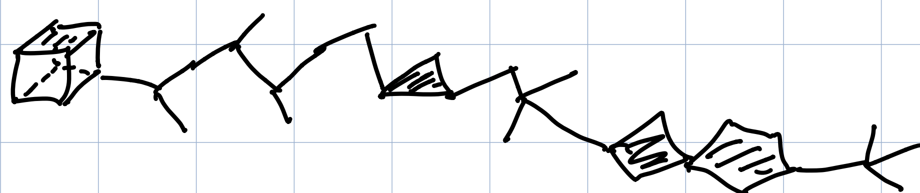
Definition: A polygonal complex is a space obtained by gluing together polygons, edges, and vertices, where by glue we mean that we identify edges w/ edges and vertices w/ vertices (could glue polygon to self)

Example:

① Graph

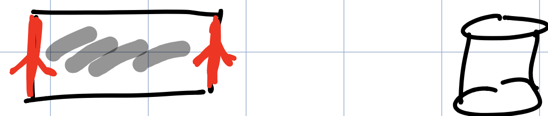


② Something Wild



④ Möbius Band

⑤ Cylinder



Remark: 1) We can always break surfaces up into polygonal cpxes

↳ A surface is homeomorphic to this associated polygonal complex

2) There are an infinite # of ways we could break it up

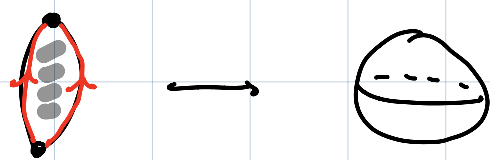
3) There are strictly infinitely many more polygonal cpxes than surfaces.

Section: More on Planar Diagrams

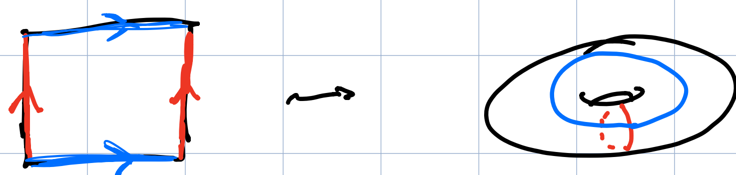
Definition: A planer diagram is a polygonal complex obtained by gluing together all pairs of edges of a single $2n$-polygon.

Examples:

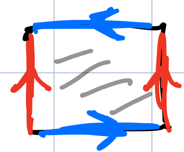
1) Sphere



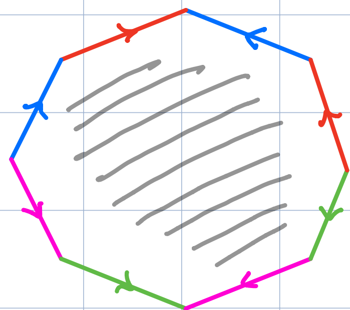
2) Torus



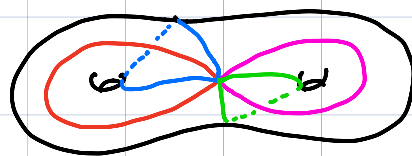
3) Klein bottle



4) Genus 2 surface



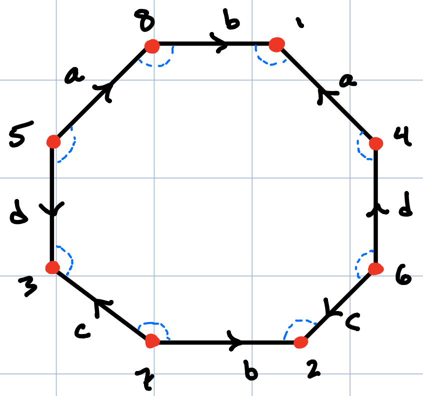
$\rightsquigarrow$

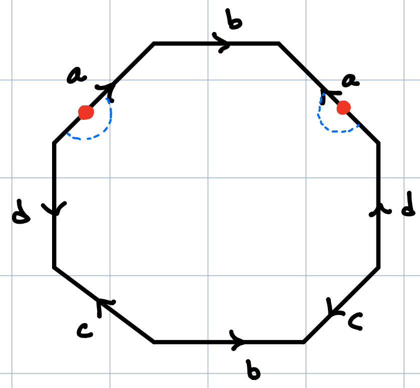


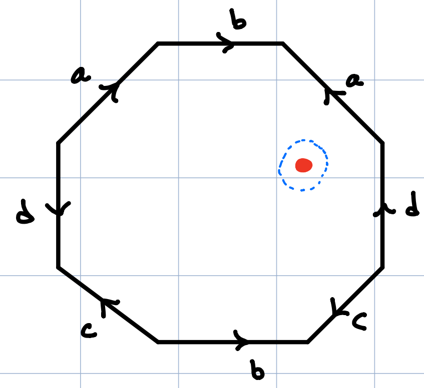
Proposition: Every planar diagram is homeomorphic to a surface.

Proof:

- Need to show that locally about every point in the planar diagram the space looks like $\mathbb{R}^2$.
- We have 3 possibilities
  1) the point is a vertex
  2) the point is in an edge
  3) the point is in the polygon.
- We think about each case

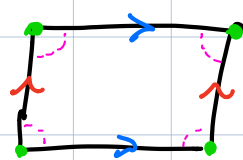


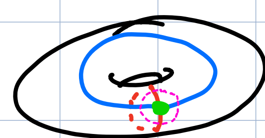




- Punchline: the gluing required for a planar diagram forces us to patch together regions that don't look like $\mathbb{R}^2$ in the "preglued" polygon into new regions that do look like $\mathbb{R}^2$ in the "glued" polygon.

- Torus case:





Section: Euler Characteristic

Definition: Let X = polygonal complex w/

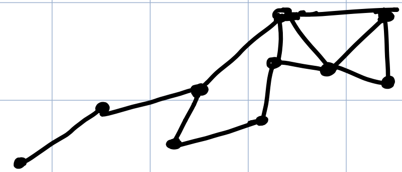
- $V(X)$ = # of vertices
- $E(X)$ = # of edges
- $F(X)$ = # of faces

The Euler characteristic of X is

$$\chi(X) = V(X) - E(X) + F(X)$$

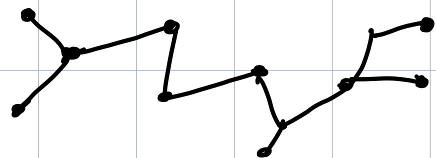
Examples:

1) Graph



: $\chi = 10 - 13 = -3$.

2) Tree

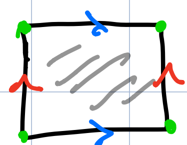


: $\chi = \underline{1} = 12 - 11 = \underline{1}$

3) Sphere

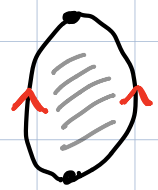
: $\chi = 8 - 12 + 6 = 2$

4) Torus



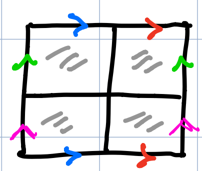
: $\chi = 1 - 2 + 1 = 0$

5) Sphere 2



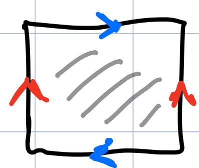
: $\chi = 2 - 1 + \underline{1} = 2$

6) Torus 2



: $\chi = 4 - 8 + 4 = 0$

7) Klein bottle



: $\chi = 1 + -2 + 1 = 0$

Remark: It appears that the Euler characteristics of polygonal complexes that are homeomorphic to the same surface always agree

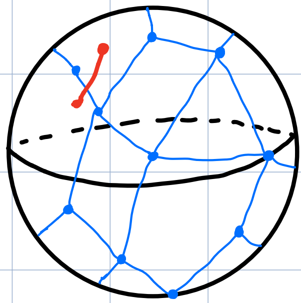
↳ Think: Given two different maps/ways of breaking up a surface into regions, they will have the same Euler characteristic.

Proposition: Let X and Y be polygonal complexes that are homeomorphic to the same surface. Then their Euler characteristics agree.

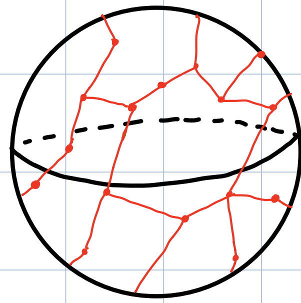
$$\chi(X) = \chi(Y)$$

Proof:

- X and Y give two different ways of breaking our surface up into polygon-like regions
- We can "overlap" X and Y on our surface, adding vertices where the edges of X intersect the edges of Y, to produce a new polygonal cpx for the surface. Call it Z.

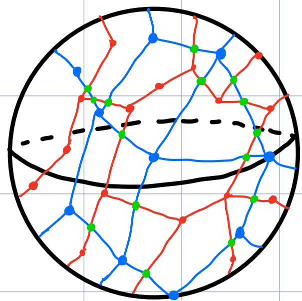


X

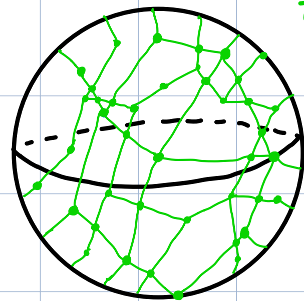


Y

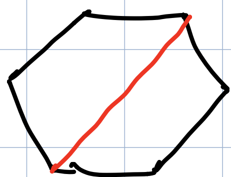
overlap



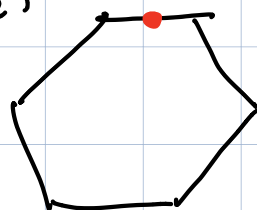
Z



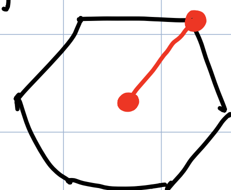
1)



2)



3)



- Note that one can obtain Z from X (similarly from Y) by
  1) Adding edge between two vertices in a polygon
  2) Adding vertex to interior of an edge
  3) Adding vertex to the interior of a polygon and connecting it to an existing vertex via an edge.
- If these don't change the Euler characteristic, then repeatedly applying them to X to get Z will give

$$\chi(X) = \chi(Z).$$

Similarly for Y, $\chi(Y) = \chi(Z)$.

- Type 1 => 1 new edge, 1 face divided into 2

$$\chi = V - (E + \cancel{1}) + (F + \cancel{1}) = V - E + F$$

- Type 2 => 1 new vertex, 1 edge divided into 2

$$\chi = (V + \cancel{1}) - (E + \cancel{1}) + F = V - E + F.$$

- Type 3 => 1 new vertex, 1 new edge.

$$\chi = (V + \cancel{1}) - (E + \cancel{1}) + F = V - E + F.$$

- $\Rightarrow \quad \chi(X) = \chi(Z) = \chi(Y).$

Definition: The Euler characteristic of a surface Σ is the Euler characteristic of any polyogonal cpx that is homeomorphic to Σ.

Remark:
- To compute $\chi(\Sigma)$, break Σ up into regions and count the # of vertices, edges, and faces.
- This allows us to prove that we are "logical beings".

Examples:

1) $\chi(S^2) = 2$
2) $\chi(T^2) = 0$
3) $\chi(\text{klein bottle}) = 0$
4) $\chi(\text{genus 2 surface}) = 1 - 4 + 1 = -2$.

Definition:

- A graph is a polygonal complex composed of edges.
- A graph is a tree if every pair of vertices is connected via a unique sequence of edges.

Definition: A graph is planar if it is given by the edges of a polygonal complex for S^2.

Fact: A graph is planar if it may be drawn in $\mathbb{R}^2$ w/out having edges intersecting / laying over each other

Proof: Remove a face from sphere and lay the remainder flat on the plane

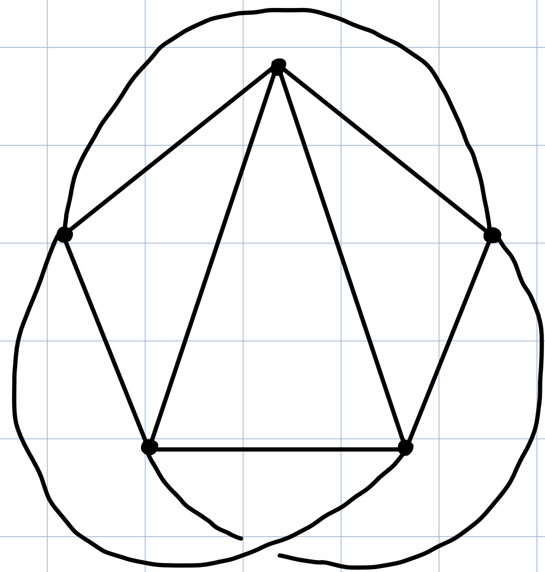
Question: Is every graph planar?

Answer: No

Reason: The Euler characteristic of the sphere puts restrictions on how edges can come together.

Notn: Let K_5 = graph w/ 5 vertices and 10 edges st every pair of vertices is connected by a unique edge.

Claim: K_5 is not a planar graph.

$K_5 =$ 

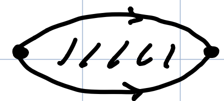
Proof:

- We use proof by contradiction. So we assume K_5 is planar and derive a contradiction. Thus our assumption will be wrong and K_5 must be non-planar.
- If K_5 is planar $\Rightarrow$ determines poly. cpx for S^2, say X.
- By the Euler characteristic proposition from today,

$$2 = \chi(S^2)$$
$$= V(X) - E(X) + F(X)$$
$$= V(K_5) - E(K_5) + F(X)$$
$$= 5 - 10 + F(X)$$
$$\Rightarrow F(X) = 7$$

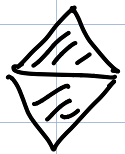
- * Note every face of X has at least 3 unique edges.

If not, then the two vertices on the face are connected via 2 different edges



But this can't happen for K_5

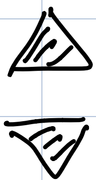
- We claim that $3F \leq 2E$



$= X$

Let $\tilde{X}$ = denote the "preglued" collection of polygons that we glue together to form X.

Note $\tilde{X}$ is itself a polygonal complex.



$= \tilde{X}$

Note

$$2E(X) = E(\tilde{X}) \geq 3F(\tilde{X}) = 3F(X)$$

↳ Used ※.

- $21 = 7 \cdot 3 = 3 \cdot F(X) \leq 2E(X) = 2E(K_5) = 20$

$\Rightarrow$ contradiction

Nexttime: 1) Colorings of Maps Theorem

2) Preliminaries for the Classification of surfaces.