Title: Zoom Lecture 4 Notes ①

Date: May 9, 2020

Outline:

1) Review of last time
2) Crossing and Unknotting Numbers
3) The Jones Polynomial

②

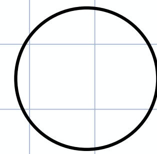
Section: Review of Last Time

Definition: (Knot) A knot is an embedded circle in S^3, ie, $K: S^1 \hookrightarrow S^3$. Two knots K_1 and K_2 are equivalent if we can push/wiggle/deform K_1 to K_2 w/out having the circles cross themselves

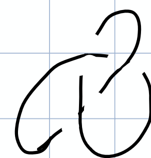
Claim: Every knot is equivalent to a copy of S^1 in $\mathbb{R}^3$, ie,

$$K: S^1 \hookrightarrow \mathbb{R}^3$$

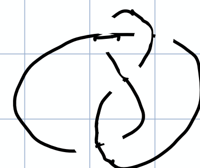
Example: 1) Unknot



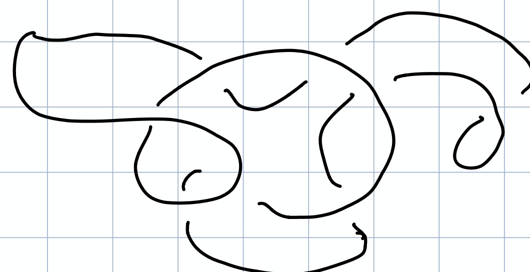
2) Trefoil



3) Figure 8



4) Unknot (complicated)



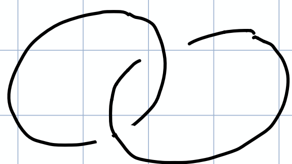
Definition: (Link) A Link is an embedding of disjoint circles into S^3, ie,

$$L : S^1 \sqcup \dots \sqcup S^1 \hookrightarrow S^3$$

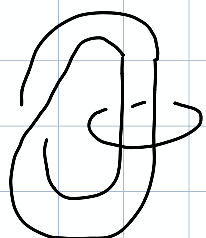
Two links L_1 and L_2 are equivalent if we can push/wiggle/deform L_1 to L_2 w/out having the circles cross themselves

Examples: (Links)

2) Hopf link

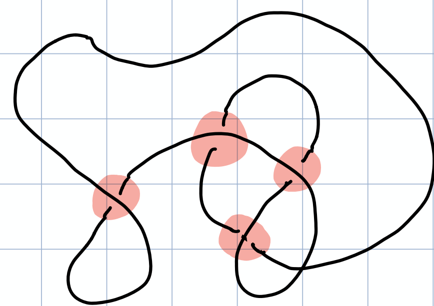


3) Whitehead Link



Definition: A Knot diagram for a knot $K \hookrightarrow \mathbb{R}^3$ is a projection/laying of K onto $\mathbb{R}^2$, given in terms arcs in the plane that meet at under/over crossings

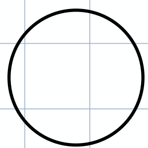
Example:



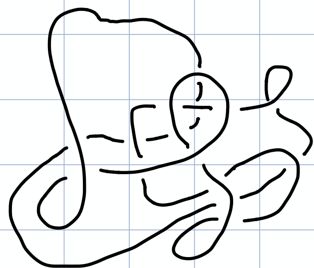
 = crossings

Remark: Same knot can have multiple knot diagrams.
Any two such dgms are related by Reidmeister moves

Example:



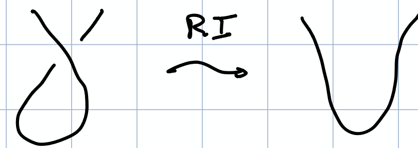
or



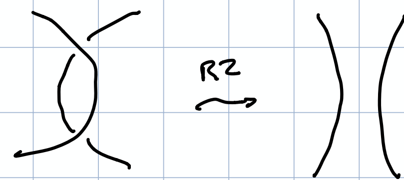
Exercise: Unknot the above picture via an explicit sequence of Reidemeister moves.

Definition: A Reidemeister Move is an alteration of a knot dgm of the following form:

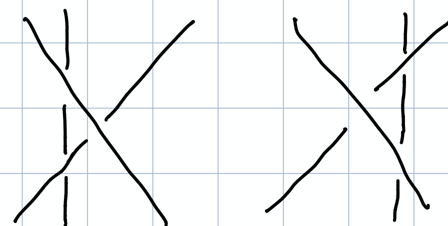
Type I:



Type II:



Type III:

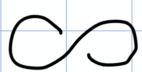


Section: Crossing and Unknotting numbers

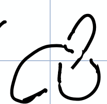
Definition: The crossing number of a knot dgm D is the # of crossings in D. Denote it be $c(D)$.

Example: 1) Unknot

$c\Big($  $\Big) = 0$

$c\Big($  $\Big) = 1$

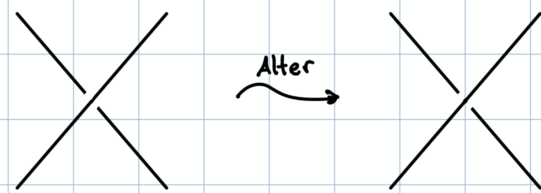
2) Trefoil

$c\Big($  $\Big) = 3$

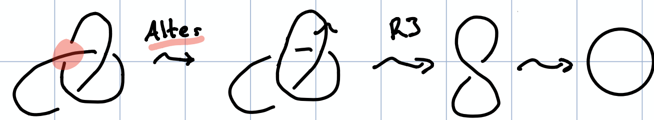
$c\Big($  $\Big) = 4$

Definition: A crossing change in a knot dgm is an alteration of an over/under crossing to an under/over crossing.

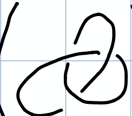
Example:



Example:



Definition: The unknotting number of a knot diagram is the minimum number of crossing changes needed to produce an unknot. Denote it by $u(D)$.

Example: $u\Big($  $\Big) = 1$

Lemma: Let D = Knot dgm for K and p be a point in D. If transversing D by starting at p, one meets every crossing first as an undercrossing (resp overcrossing), then K is the unknot

(11)

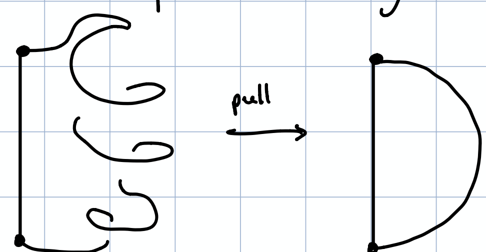
Proof: We do the undercrossing case.

A Knot dgm is a map $K : I \to \mathbb{R}^2$.

Define $\tilde{K} : I \to \mathbb{R}^3$ by $\tilde{K}(t) = (K(t), t)$

That is, lift K up. Under crossing cond., says this lift is well-defined (Why?)

Now tie the rope up in $\mathbb{R}^3$ via dropping a line and "pull the Knot tight"

$\tilde{K}$ = 

$\square$

Proposition: $u(D) \leq \frac{c(D)}{2}$

(12)

Proof: See Prop. 4.2.26 in typeset notes.

↪ Or try as exercise! $\square$

Definition: The crossing number of a knot K is (13)

$$c(K) = \min\{c(D) \mid D = \text{knot dgm for } K\}$$

The unknotting number of a knot K is

$$u(K) = \min\{u(D) \mid D = \text{knot dgm for } K\}$$

Corollary: $u(K) \leq \frac{c(K)}{2}$

Proof: Exercise! $\square$

Question: $c(K_1 \# K_2) \overset{?}{=} c(K_1) + c(K_2)$ (14)

Remark:
1) For knot diagrams this is true
2) $c(K_1 \# K_2) \leq c(K_1) + c(K_2)$ is not hard to show.
3) $c(K_1 \# K_2) \geq c(K_1) + c(K_2)$ is much harder. In fact, no one knows the answer! The relation is conjectural.

Section: The Jones Polynomial

Definition: The Kauffman bracket is the function

$$\langle \cdot \rangle : \{\text{Link Diagrams}\} \longrightarrow \left\{ \begin{array}{l} \text{Laurant} \\ \quad \text{Polynomials} \end{array} \right\}$$

↳ Poly. w/ possibly negative exponents.

determined by

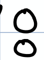
i) $\langle O \rangle = 1$ — unknot

ii) $\langle D \sqcup O \rangle = (-T^{-2} - T^{2}) \cdot \langle D \rangle$

iii) $\langle \times \rangle = T \langle)(\rangle + T^{-1} \langle \asymp \rangle$

where D = link diagram.

Example: 1) Dumb unknot

$\langle$  $\rangle = T \langle$  $\rangle + T^{-1} \langle$  $\rangle$

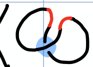
$= T + T^{-1}(-T^{2} - T^{-2})$

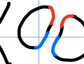
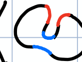
$= T - T^{1} - T^{-3}$

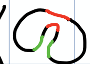
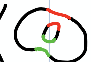
$= -T^{-3}$

2) Hopf Link

$\langle$  $\rangle$

$= T \langle$  $\rangle + T^{-1} \langle$  $\rangle$

$= T \big(T \langle$  $\rangle + T^{-1} \langle$  $\rangle \big)$

$\quad + T^{-1} \big(T \langle$  $\rangle + T^{-1} \langle$  $\rangle \big)$

$$= T\big(T(-T^{2} - T^{-2}) + T^{-1}\big) + T^{-1}\big(T + T^{-1}(-T^{2} - T^{-2})\big)$$

$$= T(-T^{3} - T^{-1} + T^{-1}) + T^{-1}(T - T - T^{-3})$$

$$= -T^{4} - T^{-4}$$

3) $\langle \text{Trefoil} \rangle = -T^{5} - T^{-3} + T^{-7}$

↳ Exercise. Use the above computations to simplify your life.

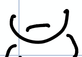
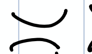
Warning: $\langle D \rangle$ is not a link invariant.
That is, it very much so depends on the link dgm chosen.

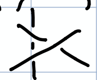
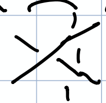
Question: How do RM change $\langle D \rangle$?

Answer:

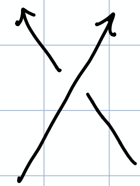
i) $\langle$  $\rangle = -T^{3} \langle$  $\rangle$

ii) $\langle$  $\rangle = -T^{-3} \langle$  $\rangle$

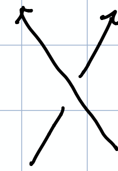
iii) $\langle$  $\rangle = \langle$  $\rangle$

iv) $\langle$  $\rangle = \langle$  $\rangle$

Definition: If L is oriented, then each crossing may be assigned a sign:



positive

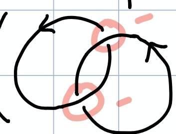


negative

Definition: The writhe of an oriented link dgm is

$$w(D) = \#\{\text{pos. crossings}\} - \#\{\text{neg. crossings}\}$$

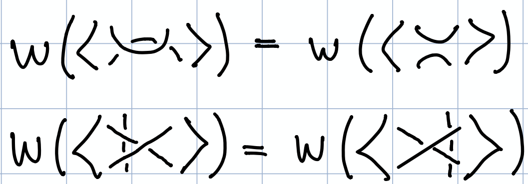
Example: i) (Writhe of Hopf Link)

$w\Big($  $\Big) = -2$

ii) (Writhe of Trefoil)

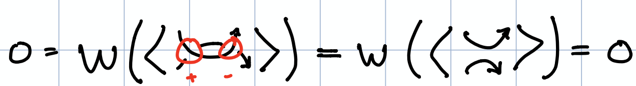
$w\Big($  $\Big) = 3$

Claim: $W(D)$ is invariant under RMs of the second and third kind, that is,



Proof: Brute force comp/observation.

E.g.:



Proposition: Let D = Link dgm for L. The polynomial:

$$K(L) = -T^{-3W(D)}\langle D\rangle$$

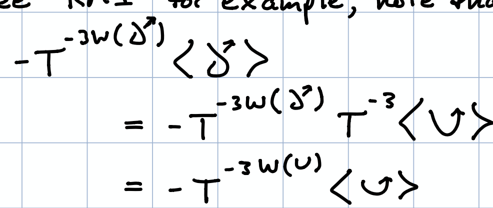
is independent of D. That is, if D, D' are two link dgms for L, then

$$-T^{-3W(D)}\langle D\rangle = -T^{-3W(D')}\langle D'\rangle$$

Proof: We just need to show that it is inv. under RMs.

We already know $\langle\,\rangle$ and $W(\,)$ are inv. under RM2 & RM3.

To see RM1 for example, note that



others are similar. $\square$

Definition: The Kauffman polynomial of an oriented link is $K(L)$.

The Jones polynomial of an oriented link is obtained from $K(L)$ by substituting $t^{1/2}$ for T^{-2}

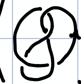
Example: $K(\text{trefoil}) = -(-T^{-4} - T^{-12} + T^{-16})$

Example: $K(\text{Hopf Link}) = -T^{-2} - T^{-10}$

Exercise: Why is the trefoil not the unknot?

Exercise: $K(K_1 \# K_2) = K(K_1) \cdot K(K_2)$

Exercise: Show:

$K\Big($  $\Big) = T^{14} + T^{6} - T^{8}$

Remark: Further readings:

When looking for readings be aware that there are multiple types of topology.

Namely, there is

↪ point-set topology

↪ geometric topology

↪ algebraic topology

We covered some basic material from the latter two.

The first is the technical/logical foundations needed to do the last two precisely.

In our class, I got around covering these technicalities by doing some hand-waving.

If you major in math, it is something that you will eventually learn, but I suggest ignoring it when getting your feet wet.

With that being said, I would maybe suggest the following texts:

For topology in general:

1) A Combinatorial Introduction to Topology by Michael Henle.

↪ He tries to avoid pointset stuff/presents it in a slightly more confined setting. It is probably the most elementary text available.

For Knot theory:

1) The Knot Book by Colin Adams

↪ It is suppose to be pretty elementary. But I've only glanced at it.

2) An Introduction to Knot Theory by Raymond Lickorish

↪ Not elementary, but it is the standard text. It is what I used to learn some knot theory.