Title: Zoom Lecture 3 Notes ①

Date: May 2, 2020

Outline:
1) 3-Manifolds
2) Knot Diagrams
3) Connect sums of Knots
5) Seifert Genus
6) Prime Knots

②

Section: 3-Manifolds

Definition: (3-manifold) A 3-manifold is a space that locally looks like a 3-dim'l ball.

Example: (3-sphere 3 ways)

1)

$$S^3 = \{(x,y,z,w) \in \mathbb{R}^4 \mid \overbrace{x^2+y^2+z^2+w^2=1}^{\text{Points that are distance 1 from } (0,0,0,0)}\}$$

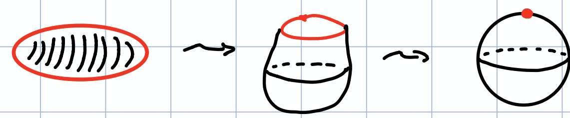
2) $S^3 = B^3$ w/ boundary points all identified.

3) $S^3 = 1$ point compactification of $\mathbb{R}^3$

Remark: Helpful to think of analogues of above for S^2 and B^2 and $\mathbb{R}^2$.

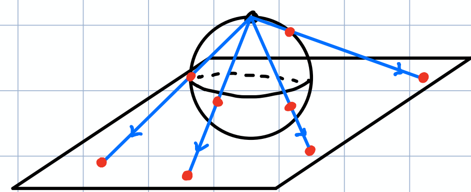
1) $S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1\}$

2)



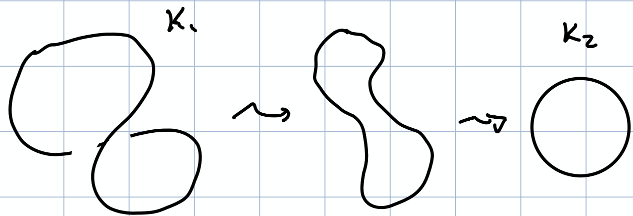
3) $S^2 = 1$ point compactification of $\mathbb{R}^2$

↳ After removing the north pole, we can lay the rest of the sphere onto the plane via a stereographic projection



Definition: (Knot) A knot is an embedded circle in S^3, ie, $K: S^1 \hookrightarrow S^3$.
Two knots K_1 and K_2 are equivalent if we can push/wiggle/deform K_1 to K_2 w/out having the circles cross themselves

Picture:



Claim: Every knot is equivalent to a copy of S^1 in $\mathbb{R}^3$, ie,

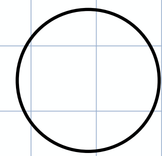
$$K: S^1 \hookrightarrow \mathbb{R}^3$$

Proof: $S^3 = \mathbb{R}^3 \cup \{\infty\}$

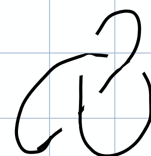
Wiggle K to miss $\{\infty\}$

$\Rightarrow K \subseteq \mathbb{R}^3 \subseteq \mathbb{R}^3 \cup \{\infty\} = S^3$ $\square$

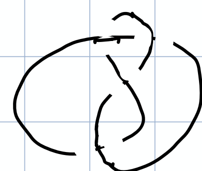
Example: 1) Unknot



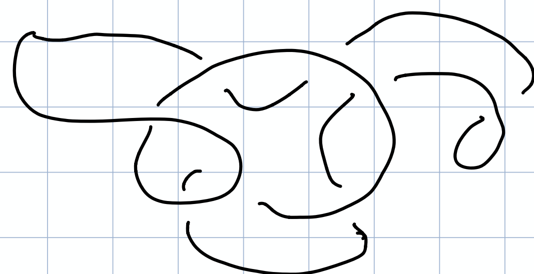
2) Trefoil



3) Figure 8



4) Unknot (complicated)



Remark: (Surgery along a Knot)

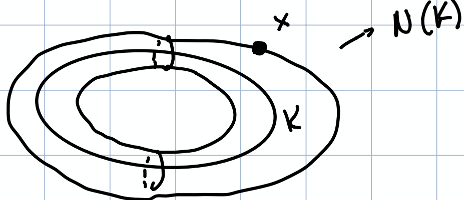
Given a Knot $K \hookrightarrow S^3$, we can use it to alter S^3 to a different 3-manifold.

We outline this in steps

Step 1: Given a Knot $K : S^1 \hookrightarrow S^3$, we may "thicken" it to obtain an embedded donut (solid torus) in S^3.

Denote it by $i : N(K) \hookrightarrow S^3$

Picture:



Step 2: Remove $N(K)$ from S^3.

Step 3: Glue in another donut in a "different" manner.

Namely, pick a homeomorphism of the torus, say $\varphi : T^2 \to T^2$.

Let X = donut w/ boundary T^2

Glue $x \in T^2 \cong \partial N(K) \subseteq S^3 \setminus N(K)$ to $\varphi(x) \in T^2 \cong \partial X \subseteq X$.

$\Rightarrow$ Glue $S^3 \setminus N(K)$ and X to obtain a new 3-manifold!

Step 4: Denote this new space by $M(K, \varphi)$.

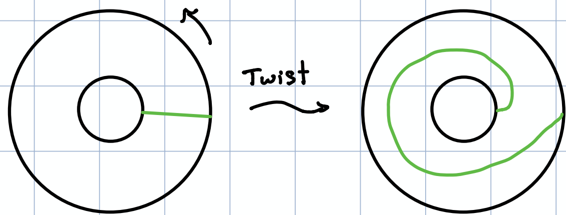
Remark: Why different?

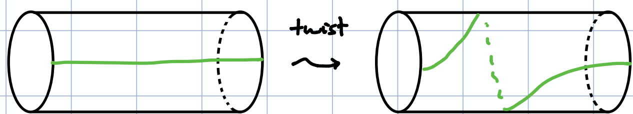
↪ Think about how different gluings of cylinder can produce T^2 vs Klein bottle.

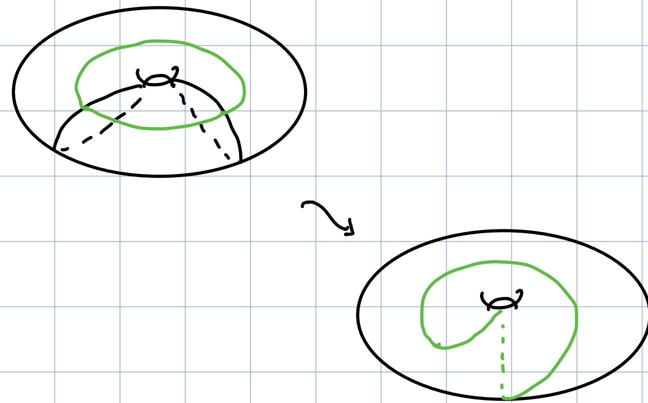
↪ This is 3-dim'l analogue.

Fact: If $K_1 \sim K_2$, then $M(K_1, \varphi)$ is homeomorphic to $M(K_2, \varphi)$. □

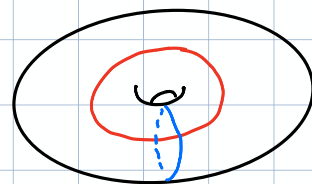
Example: (Dehn Twist)







Theorem: "Every" homeo of T^2 can be produced from composing a combination of Dehn twists about either of the curves below:



□

Corollary: $\exists$ a listable number of homeomorphisms of the torus. □

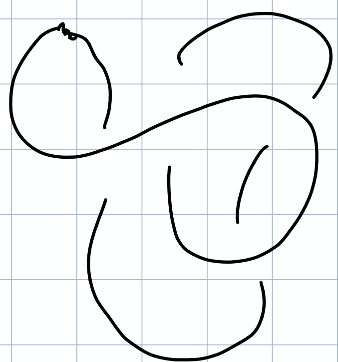
Definition: (Link) A Link is an embedding of disjoint circles into S^3, ie,

$$L : S^1 \sqcup \dots \sqcup S^1 \hookrightarrow S^3$$

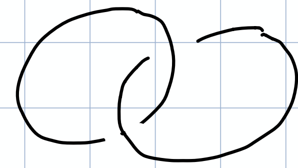
Two links L_1 and L_2 are equivalent if we can push/wiggle/deform L_1 to L_2 w/out having the circles cross themselves

Examples: (Links)

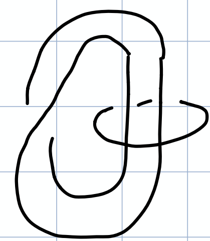
1) Any Knot



2) Hopf link



3) Whitehead Link



4) Olympics Logo

Remark: (Surgery along a link) (13)

Same procedure as for knot, but now do it simultaneously for each copy of S^1 in the link.

So if

$$L = \underbrace{S^1 \cup \dots \cup S^1}_{K\text{-times}}$$

then we pick K homeos $\varphi_K : T^2 \to T^2$ and now obtain $M(L, \varphi_1, \dots, \varphi_K)$.

Theorem: (Dehn) Every 3-manifold can be obtain by performing surgery along a link in S^3.

Proof: The idea is to reverse engineer the above process. $\square$

Theorem: There is a countably infinite number of 3-manifolds. That is, we can describe/list/classify all 3-manifolds. (14)

Proof: Dehn's thm $\Rightarrow$ every 3-manifold is some

$$M(L, \varphi_1, \dots, \varphi_K)$$

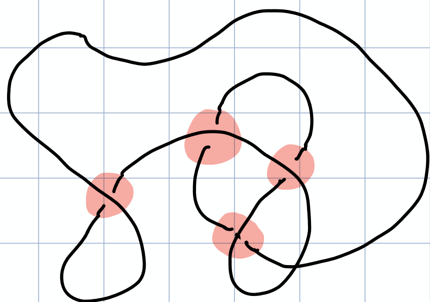
We show below that there is a countable # of links. The possible φ_i are countably as we saw above.

$\Rightarrow$ List all $M(L, \varphi_1, \dots, \varphi_K)$ $\square$

Section: Knot Diagrams

Definition: A knot diagram for a knot $K \hookrightarrow \mathbb{R}^3$ is a projection/laying of K onto $\mathbb{R}^2$, given in terms arcs in the plane that meet at under/over crossings

Example:

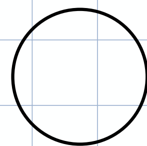


 = crossings

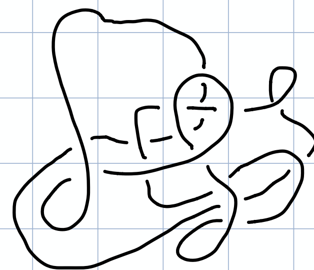
Remark: Same knot can have multiple knot diagrams

$\Rightarrow$ Need notion of equivalence for knot diagrams.

Example:

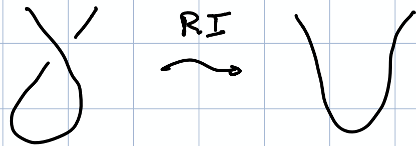


or

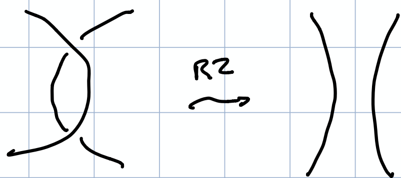


Definition: A Reidemeister Move is an alteration of a Knot dgm of the following form:

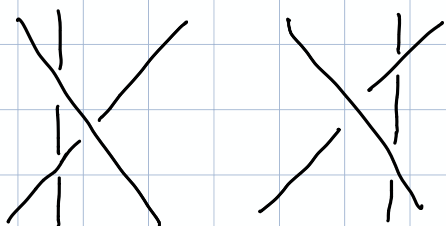
Type I:



Type II:



Type III:



Definition: Two Knot diagrams are equivalent iff they are related via a seq. of Reidemeister moves.

Theorem: Two Knots are equivalent iff they have equivalent Knot diagrams.

Proof: Project the wiggling upstair onto the table and observe that as we wiggle we are just performing Reidemeister moves. $\square$

Corollary: There is a listable number of links

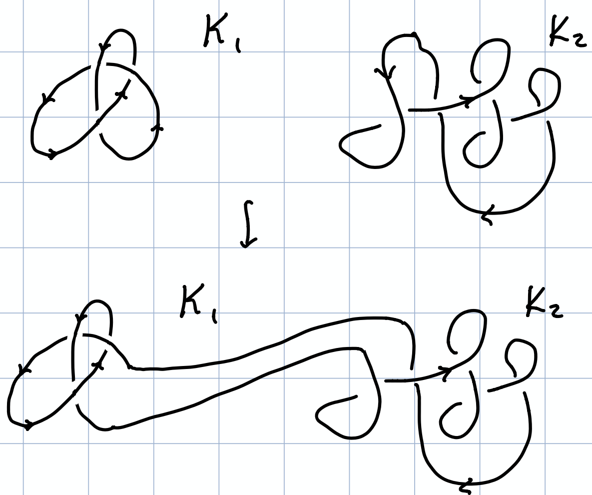
Proof: Links $\longleftrightarrow$ Link dgms $\longleftrightarrow$

$\longleftrightarrow$ Lists of crossing info. $\square$

Section: Connect Sums of Knots

Definition: An oriented knot is a knot w/ a choice of direction:

Definition: The connect sum of two oriented knots K_1 and K_2 is the knot $K_1 \# K_2$ obtained as follows:



Remark: $K_1 \# K_2$; think as you further knot K_1 according to K_2.

Theorem: If K is not the unknot, then $K \# K_1$ is not the unknot for all knots K_1.

Idea:
$$0 = (1 + -1) + (1 + -1) + (1 + -1) + \dots$$
$$= 1 + (-1 + 1) + (-1 + 1) + (-1 + 1) \dots$$
$$= 1$$

↳ One learns in Calculus that such an infinite sum is not well-defined and thus $0 \neq 1$.

↳ In knot theory / topology, we can make sense of these "infinite sums"

Proof: Spse $\exists K_1$ st $K \# K_1 = U$

Consider $K \# K_1 \# K \# K_1 \# K \# \cdots$

$$\overbrace{(K \# K_1)}^{U} \# \overbrace{(K \# K_1)}^{U} \# \overbrace{(K \# \cdots)}^{U} = U$$

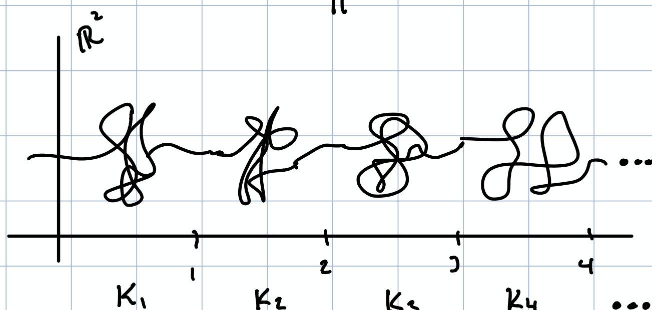
$$\overbrace{K}^{K} \# \overbrace{(K_1 \# K)}^{U} \# \overbrace{(K_1 \# K)}^{U} \overbrace{\# \cdots}^{U} = K$$

$\Rightarrow K = U$, that is, K was already the unknot, a contradiction. $\square$

Remark: Technically, infinite $\#$ is not a knot; its a wild knot, which we make sense of in the following way:

$$K_1 \# K_2 \# K_3 \# K_4 \# \cdots$$

$$\|$$

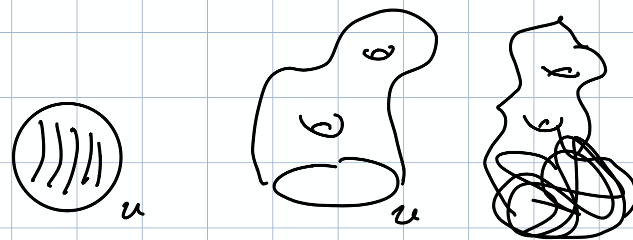


Section: Seifert Genus

Question: Does there exist a surface in S^3 w/ a single boundary component whose boundary is a knot K?

Such a surface is called a Seifert surface for K.

Picture:



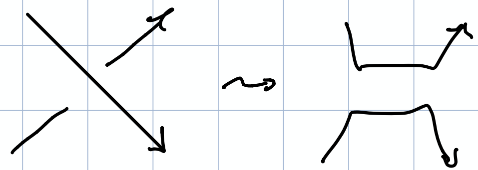
Exercise: Trefoil can be bounded by a torus w/ boundary.

Proof: Use the below algorithm to try to create it $\square$

Proposition: Every Knot admits a Seifert surface.

Proof: 1) Orient the Knot

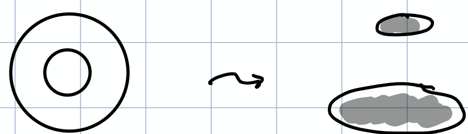
2) Resolve crossings



3) Produces collection of circles in plane.

So they all bound disks.

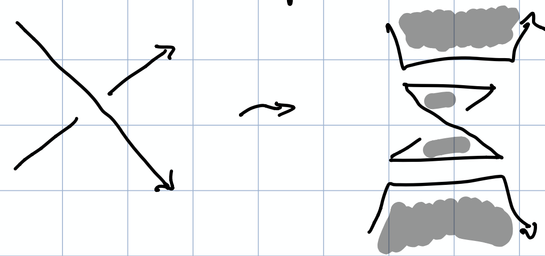
4) If



then lift disks off of each other w/ smallest to biggest going to highest to lowest.

5) Add in all the disks.

6) Add twist strips



7) Observe that you have produced the desired surface.

Remark: W/ care, one can ensure that the resulting surface is a connect sum of tori.

Example:

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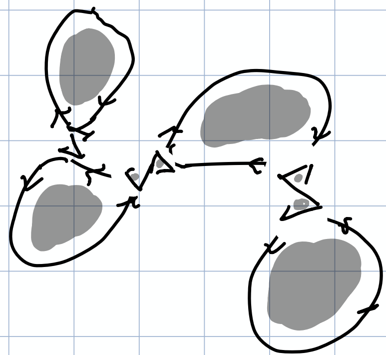

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Definition: The Seifert genus of K is the minimum genus among all Seifert surfaces for K.

Denote it by $g(K)$.

Proposition: K = unknot iff $g(K) = 0$

Proof: $(\Longrightarrow)$ Easy

$(\Longleftarrow)$ Very Hard $\square$

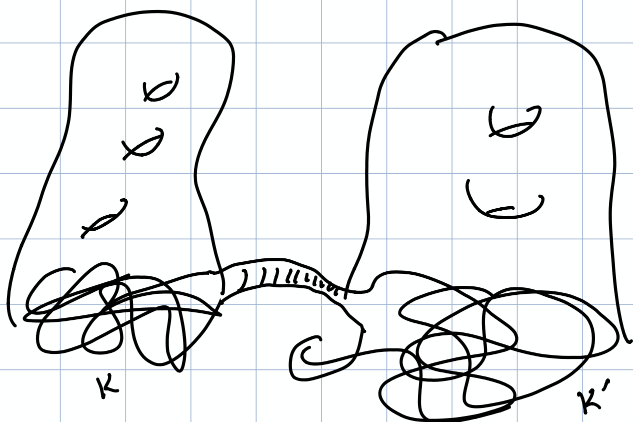
Proposition: $g(K \# K') = g(K) + g(K')$

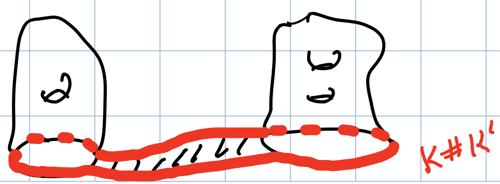
Proof: First, we show

$$g(K \# K') \leq g(K) + g(K')$$

We surger together the Seifert surfaces for K and K' to obtain one for $K \# K'$

Picture:





Second, we show

$$g(K \# K') \geq g(K) + g(K')$$

Let X = Seifert surf for $K \# K'$ w/ minimal genus.

Split $\mathbb{R}^3$ by a plane that divides K from K', call it P

After wiggling X, we may assume $X \cap P$ is collection of circles or arcs.

After taking care, one can push X off of P st $X \cap P$ is single arc.

This now splits X into seifert surfaces for K and K'.

$\Rightarrow$ Gives result.

Section: Prime Knots

Definition: (Prime Knot) A knot K is prime if $K = K_1 \# K_2 \Rightarrow K_1$ or K_2 is the unknot.

Lemma: If $g(K) = 1$, then K is prime.

Proof: If $K = K_1 \# K_2$

$$\Rightarrow 1 = g(K) = g(K_1 \# K_2)$$
$$= g(K_1) + g(K_2)$$
$$\Rightarrow g(K_1) = 0$$
$$\Rightarrow K_1 = \text{unknot}$$

$\square$

Proposition: Every knot can be written as a connect sum of prime knots.

Proof: See Lemma 4.2.21 in typeset notes $\square$

↳ Or try using the above result to prove it yourself

Theorem: Every knot can be written as a "unique" connect sum of prime knots.

Proof: It remains to prove uniqueness. This is complicated! See Section 4.2.5 in typeset notes.

Exercise: Show that there exists infinitely many knots. (You may assume that there exists a non-trivial knot) (31)

Solution: Let K = non-trivial knot.

Define
$$\begin{aligned} K_1 &= K \\ K_2 &= K \# K = K \# K_1 \\ K_3 &= K \# K \# K = K \# K_2 \\ &\vdots \\ K_n &= K \# K_{n-1} \end{aligned}$$

We claim that $g(K_n) \neq g(K_m)$ for all $m \neq n$. Consequently, $K_n \neq K_m$.

$$\begin{aligned} g(K_n) &= g(K \# K_{n-1}) \\ &= g(K) + g(K_{n-1}) \\ &\vdots \\ &= \sum_{i=1}^{n} g(K) \\ &= n \cdot g(K) \end{aligned}$$

$\Rightarrow g(K_n) = g(K_m)$ iff $n = m$. $\square$

Exercise: Use Seifert genus to show that K not the unknot implies $K \# K'$ is not the unknot. (32)

Proof:
$$\begin{aligned} g(K \# K') &= g(K) + g(K') \\ &\geq g(K) \\ &> 0 \quad \} \text{ K not unknot} \end{aligned}$$

$\Rightarrow K \# K'$ not the unknot. $\square$