Title: Zoom Lecture 2 Notes (1)

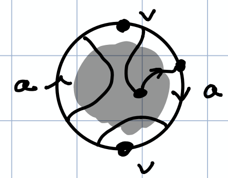
Date: April 18^{th}, 2020

Outline:
1) Introduction
2) Review of Word Groups
3) Review of Homology
4) Homology of Surfaces
5) Surface Embedding Theorem
6) Manifolds
7) Whitney's Embedding Theorem

Section: Introduction (2)

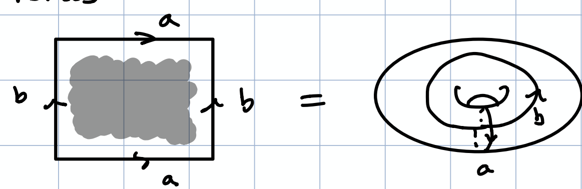
Definition: A surface is a space that locally looks like $\mathbb{R}^2$.

Example: 1) Real Projective Plane



$\rightsquigarrow$ Can't picture.

2) Torus



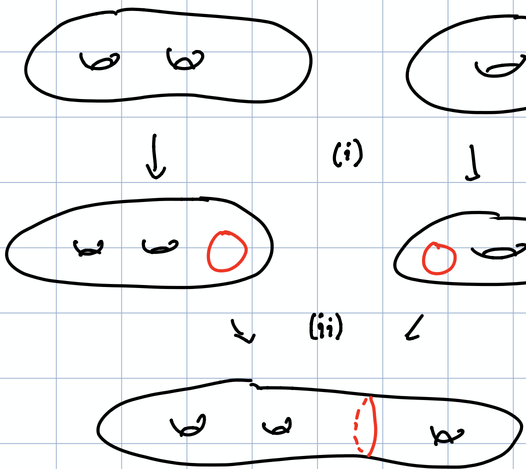
Definition: (Connect sum) The connect sum of two surfaces X_1 and X_2 is given by:

i) Remove closed disks in X_1 and X_2 to obtain two surfaces w/ boundaries, say Y_1 and Y_2.

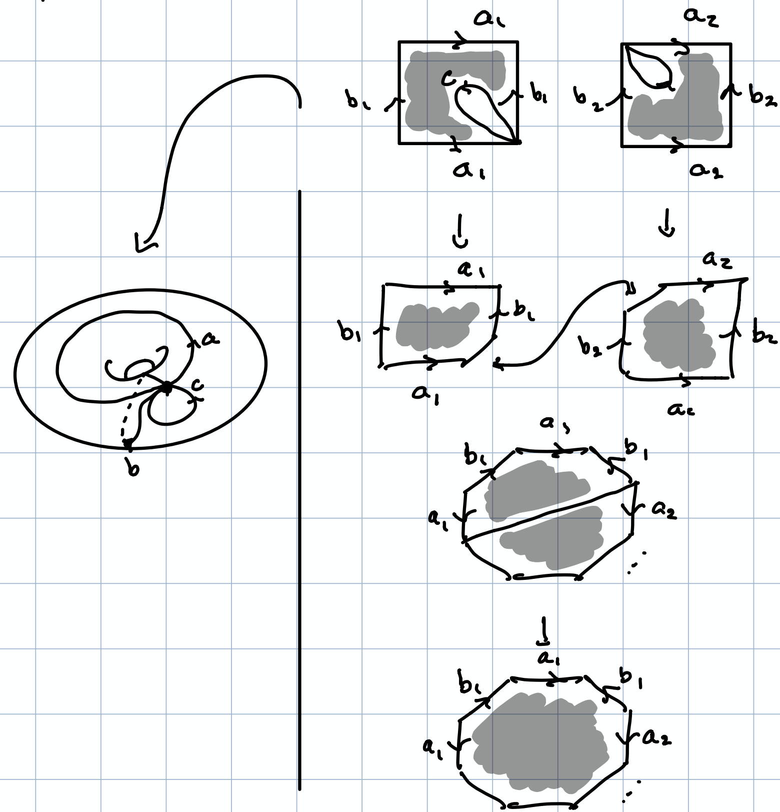
ii) Glue Y_1 to Y_2 along their boundaries

We write $X_1 \# X_2$ for the result.

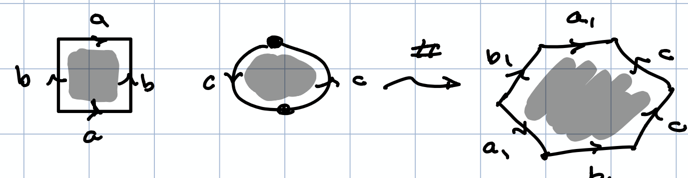
Picture:



Example: 1) $T^2 \# T^2$



2) $P^2 \# T^2$



Theorem: (Classification of surfaces) Every surface is homeomorphic to a connect sum $\underbrace{T^2 \# \dots \# T^2}_{r \text{ copies}} \# \underbrace{P^2 \# \dots \# P^2}_{s \text{ copies}} \# S^2$

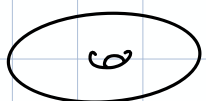
for some $r, s \geq 0$

Example: Klein bottle $= P^2 \# P^2$

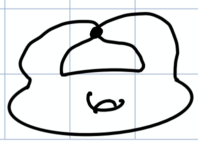
Definition: (Embedding) An embedding of a surface X is a continuous map $i : X \hookrightarrow \mathbb{R}^N$ st $i(x_0) = i(x_1) \implies x_0 = x_1$

$\hookrightarrow$ i maps each point in X to a unique point in $\mathbb{R}^N$

Example:



= Embedded



= not embedded

Theorem: The following are equivalent

1) There exists an embedding $X \hookrightarrow \mathbb{R}^3$
2) $H_2(X) \neq 0$
3) X is orientable
4) $X \simeq T^2 \# \dots \# T^2 \# S^2$

Remark: What we already know:

What we will show:

$$\begin{tikzcd}
① \arrow[rr, Rightarrow] & & ② \arrow[d, Leftrightarrow] \\
③ \arrow[rr, Leftrightarrow] & & ④ \arrow[ull, Rightarrow]
\end{tikzcd}$$

Section: Review of Word Groups

Definition: A word group consist of:

1) An alphabet $a_0, \ldots, a_\ell, a_0^{-1}, \ldots, a_\ell^{-1}$
2) List of generators $w_1, \ldots, w_m$ st each w_i is word spelled w/ alphabet
3) List of relations $r_1, \ldots, r_n$ st each r_i is word spelled w/ w_i, w_i^{-1}.

Gives a group $\langle w_1, \ldots, w_m \mid r_1, \ldots, r_n \rangle$

1) Elements are words obtained from concatenating copies of w_i or w_i^{-1}.
   Two words are equiv if
   i) Rearrange letters
   ii) Cancel a_i w/ a_i^{-1}
   iii) Remove subword that is a r_i or r_i^{-1}
2) Addition of words is by concatenation.

If $\{r_i\}$ = empty, then $\langle w_1, \ldots, w_n \rangle$ is called a free word group.

Remark: (Grp Hom of word groups)

$\varphi : \langle w_1, \ldots, w_n \rangle \longrightarrow \langle v_1, \ldots, v_\ell \rangle$

φ is a group hom when it assigns words in w_i's to words in v_j's

Remark: $\varphi : \langle w_1, \ldots, w_n \rangle \longrightarrow \langle v_1, \ldots, v_\ell \rangle$

$\mathrm{Ker}(\varphi) = \langle x_1, \ldots, x_k \rangle$ st

x_i = words in w_i's or w_i^{-1}'s st

$\varphi(x_i)$ = empty word $= 0$

$\hookrightarrow$ $\mathrm{Ker}(\varphi)$ is a word group!

Example: $\varphi : \langle e_0, e_1, e_2 \rangle \longrightarrow \langle v_0, v_1, v_2 \rangle$ 9

$$\varphi(e_0) = v_2 v_1^{-1}$$
$$\varphi(e_1) = v_2 v_0^{-1}$$
$$\varphi(e_2) = v_1 v_0^{-1}$$

What is $\mathrm{Ker}(\varphi)$?

LHS, gen. word looks like $e_0^n e_1^k e_2^\ell$

$$\varphi(e_0^n e_1^k e_2^\ell) = v_2^n v_1^{-n} v_2^k v_0^{-k} v_1^\ell v_0^{-\ell} \overset{?}{=} 0$$

Happens when: $n = -k$, $n = \ell$, $\ell = -k$

$\Rightarrow n = \ell = -k$

$\Rightarrow \mathrm{Ker}(\varphi) = \{ e_0^n e_1^{-n} e_2^n \}$
$= \langle e_0 e_1^{-1} e_2 \rangle$

Remark: If $B \subseteq \langle w_1, \dots, w_n \rangle = G$ is a collection of words obtained from w_i,

$$B = \langle b_0, \dots, b_k \rangle$$

Then we obtain new word group G/B

$$G/B = \langle w_1, \dots, w_n \mid b_0, \dots, b_k \rangle$$

Example: $\langle b \rangle \subseteq \langle a, b, c \rangle$

$\Rightarrow \langle a, b, c \mid b \rangle \cong \langle a, c \rangle$ 10

Section: Review of Homology

Construction: Let X = polygonal complex.

Pick direction for each edge

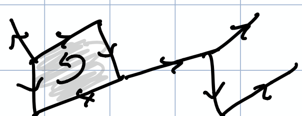
Pick orientation of each polygon

↳ way to sweep out edges.

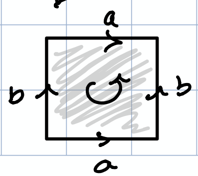
These need to be compatible w/ any gluings that we did. Recall diff between Torus and Klein bottle.

Remark: By "sweep", I mean clockwise or counter-clockwise direction to read of seq. of edges on boundary

Example: 1)



2)



Construction: (continued)

Label vertices : $v_0, \dots, v_e$

edges : $e_0, \dots, e_n$

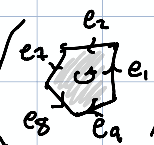
faces : $f_0, \dots, f_m$

Define

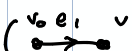
$$\begin{aligned} C_0(X) &= \langle v_0, \dots, v_e \rangle \\ C_1(X) &= \langle e_0, \dots, e_n \rangle \\ C_2(X) &= \langle f_0, \dots, f_m \rangle \end{aligned}$$

with maps $\partial_1 : C_1(X) \to C_0(X)$ and $\partial_2 : C_2(X) \to C_1(X)$.

Define

$\partial_2(\text{face}) = \partial_2\Bigg($  $\Bigg)$

$$= e_9^{-1} e_8 e_7 e_2 e_1$$

$\partial_1(\text{edge}) = \partial_1\big($  $\big)$

$$= v_1 v_0^{-1}$$

Lemma: $\partial_1 \circ \partial_2 = 0$ and consequently

$$\mathrm{Im}(\partial_2) \subseteq \mathrm{Ker}(\partial_1)$$

Definition: The homology groups of X are

$$H_0(X) = \langle v_0, \dots, v_e \mid \partial_1(e_0), \dots, \partial_1(e_n) \rangle$$

↳ Spse $\partial_1(e) = vw^{-1}$

$\Rightarrow v \simeq w$ as words

$\Rightarrow H_0 = \#$ of connected components w/ one vertex for each component

$$H_1(X) = Ker(\partial_1) / Im(\partial_2)$$

$= \langle x_0, \dots, x_k \mid \partial_2(f_0), \dots, \partial_2(f_n) \rangle$

$=$ words that form loops.

Two loops are equiv if we can push across faces.

$$H_2(X) = Ker(\partial_2)$$

$= \langle y_1, \dots, y_b \rangle$

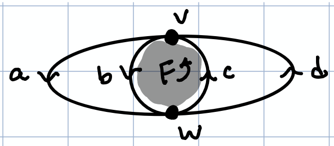
where $\partial_2(y_i) = 0$

↳ $\partial_2 = 0$ iff edges pair off

$$\partial_2(y) = e_0 e_1^{-1} e_2 e_1^{-1} e_0^{-1} e_1^{2} e_2^{-1}$$

$\Rightarrow$ form 2-dim'l voids.

Example:



$H_0 = \langle v, w \mid vw^{-1} \rangle \cong \langle v \rangle$

$\mathrm{Im}(\partial_2) = \partial_2(F) = cb.$

For $\mathrm{Ker}(\partial_1)$ we notice that

$$\partial_1(a) = wv^{-1}$$
$$\partial_1(b) = wv^{-1}$$
$$\partial_1(c) = vw^{-1}$$
$$\partial_1(d) = vw^{-1}$$

So

$$\partial_1(ad) = \cancel{wv^{-1}}\,\cancel{vw^{-1}} = 0$$

One can deduce

$\mathrm{Ker}(\partial_1) = \langle ad, ab^{-1}, dc^{-1}, bc \rangle$

Consequently,

$$H_1 = \langle ad, ab^{-1}, dc^{-1}, bc \mid cb \rangle$$
$$= \langle ad, ab^{-1}, dc^{-1} \mid cb \rangle$$

Section: Homology of Surfaces

Claim: X = surface, then $H(X)$ is independent of the choice of polygonal structure.

Question: 1 What is $H(X)$ when

$$X = T^2 \# \dots \# T^2$$

2 What is $H_2(X)$ when

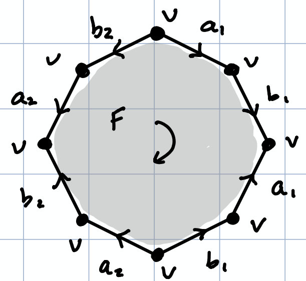
$$X = T^2 \# \dots \# T^2 \# P^2 \# \dots \# P^2$$

Answer: 1 $H_2(X) \neq 0$

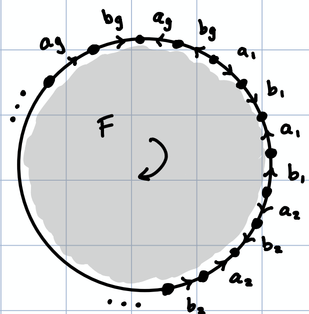
2 $H_2(X) = 0$

① **Solution:** X = surface, then to compute $H(X)$ we should pick a nice polygonal str. Using the connect sum operation w/ planar diagrams, we could create a planar dgm for a genus g surface by gluing edges of $4g$-gon to themselves in pairs as illustrated below.

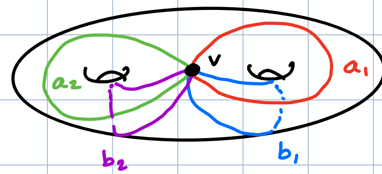
$g = 2$:



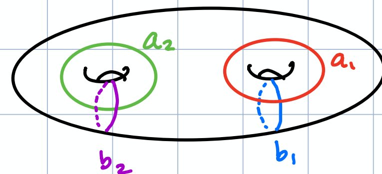
g = arbitrary:



For $g = 2$, this look like



21 pushing around curves



$H_0 = \langle v \rangle$

$H_1 = \langle a_1, b_1, a_2, b_2 \rangle$

$H_2(T^2 \# T^2) = \ker(\partial_2)$

$\partial_2(F) = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1}$

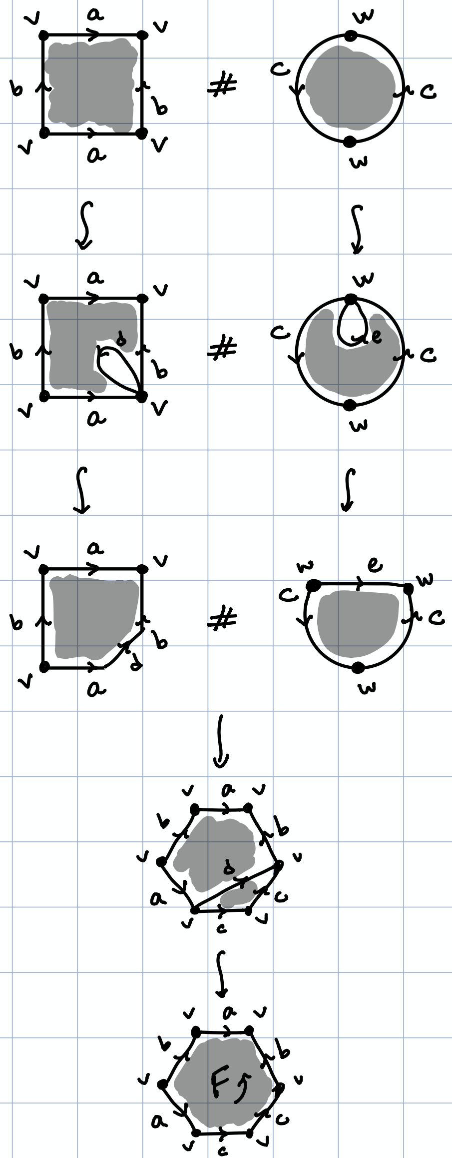
$= 0$

$\Rightarrow \ker(\partial_2) \neq 0$

$\Rightarrow H_2 \neq 0$

The argument for g = arbitrary is similar.

② Solution:



⑲

⑳

We compute

$$\partial_2(F) = b a^{-1} b^{-1} a c c = c^2 \neq 0$$

$\Rightarrow \operatorname{Ker}(\partial_2) = 0$

$\Rightarrow H_2 = 0$

The more general case is analogous.

Section: Proof of Surface Embedding Theorem

Theorem: The following are equivalent

1) There exists an embedding
$$X \hookrightarrow \mathbb{R}^3$$
2) $H_2(X) \neq 0$
3) X is orientable
4) $X \simeq T^2 \# \dots \# T^2 \# S^2$

Remark: What we already know:

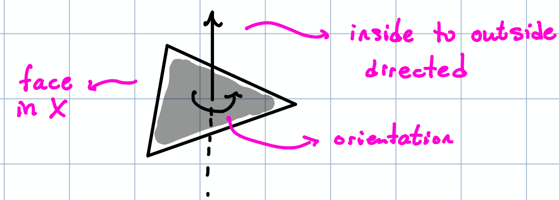
What we show today:

What we will show now:

$$\begin{tikzcd} ① \arrow[r, Rightarrow] & ② \arrow[d, Leftrightarrow] \\ ③ \arrow[r, Leftrightarrow] & ④ \arrow[ul, Rightarrow] \end{tikzcd}$$

Claim: If X can be embedded in $\mathbb{R}^3$, then $H_2(X) \neq 0$.

Idea: "Pretend" that X has an "inside and outside". We orient faces via a right hand rule:

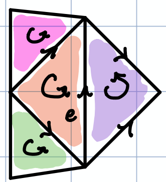


Of course, apriori we don't know what is "inside and outside". But every time a line meets X it either "enters" or "exits" which is enough to get orientations that will cancel out on edges to produce a 2-dim'l void and thus $H_2 \neq 0$.

Proof: The proof will be broken up into parts.

1) What we need to show.

If we can orient all the faces so that any two adjacent faces always satisfy:



then $\partial_2(F_1 \dots F_K) = 0$

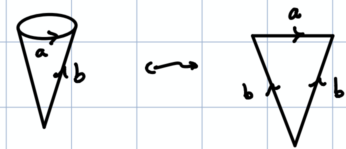
$\Rightarrow H_2(X) = \operatorname{Ker}(\partial_2) \neq 0.$

Indeed, this implies that edges in $\partial_2(F_1 \dots F_K)$ will pair off in cancelling pairs of edges.

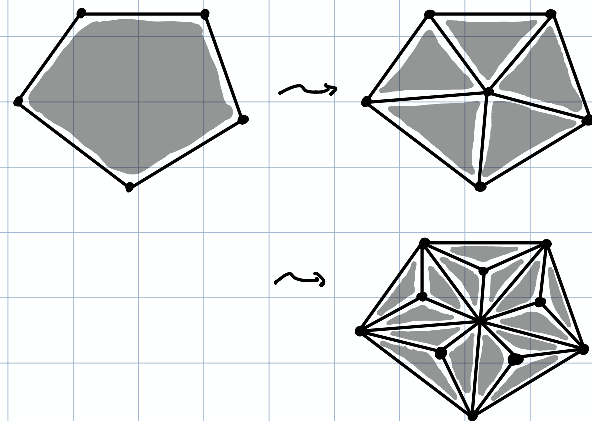
2) Simplifying the geometry.

Since $H_2(X)$ is ind. of poly. str. we pick a structure st each face is never glued to itself

↪ ie, not



To obtain this we just "refine" faces until we achieve it:



3) Simplifying the Embedding.

We have $X \hookrightarrow \mathbb{R}^3$.

Fix a direction in $\mathbb{R}^3$

After rotating/translating X and refining poly. structure, we may assume:

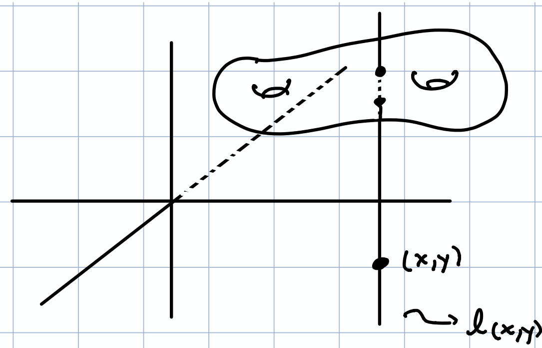
1) X is above xy-plane
2) Every line

   $\ell_{(x,y)}$ = vertical line through the point (x,y)

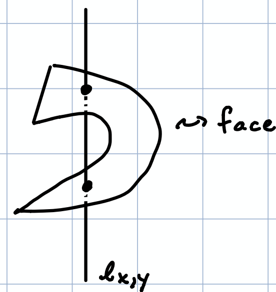
   meets each face at most once
3) $\ell_{(x,y)}$ never "runs along" an edge in X.

↪ Again, we may need to refine the polygonal structure to obtain this result.

Picture: (1)

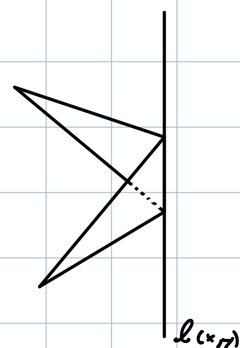


(2)



= can't happen

(3)



= can't happen

4) Assigning orientations.

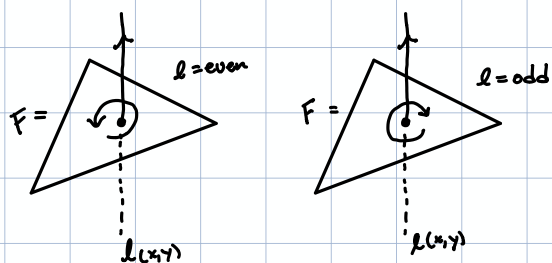
Given F = face

Pick (x,y) st $\ell_{(x,y)} \cap F \neq \emptyset$

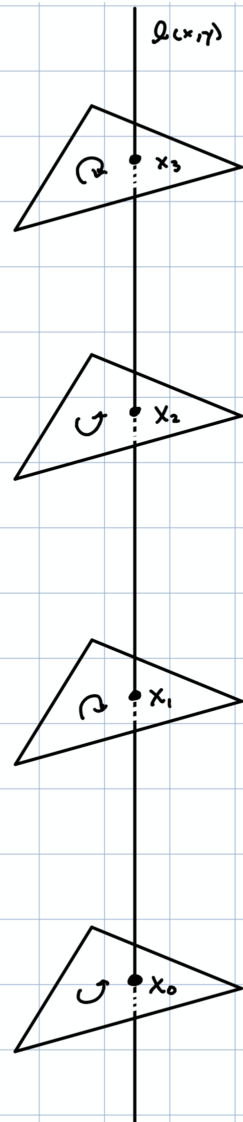
Let $x_1, \ldots, x_k$ = ordered intersection points of $\ell_{(x,y)}$ w/ X

Spse $\ell_{(x,y)} \cap F = x_\ell$

Define orientation for F by



Picture:



5) Orientations are well-defined (29)

Spse (x', y') also has $\ell_{(x',y')} \cap F \neq \emptyset$

Push $\ell_{x,y} \rightsquigarrow \ell_{x',y'}$, 2 things can happen:

1) don't cross edge

2) cross edge(s)

If (1), then we don't lose any intersections

$\Rightarrow$ same sign

If (2), then either

(a) cross one face into another

(b) push off two faces at once

$\Rightarrow$ parity of ℓ will remain the same.

$\Rightarrow$ well-defined, ie, orientation didn't depend on $\ell_{(x,y)}$.

6) Check adjacency condition. (30)

Spse F_0, F_1 share edge e

Pick (x,y) st $\ell_{(x,y)}$ lies close to e and $\ell_{(x,y)} \cap F_0 \neq \emptyset$

Push $\ell_{(x,y)}$ across e. Either

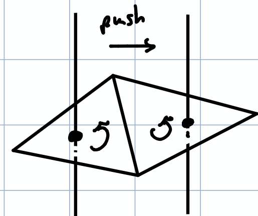
1) ℓ slides over into F_1

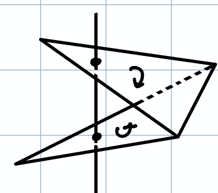
2) ℓ slides off F_0 and F_1

$\Rightarrow$ sweep opposingly across e.

Which is what we originally wanted!

Picture:

1) 

2) 

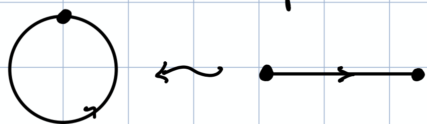
$\square$

Section: Manifolds

Definition: (Manifold) An n-manifold is a space that locally looks like $\mathbb{R}^n$.

Example: circle = 1-manifold
surface = 2-manifold

Example: $S^1 = [0,1]$ w/ end points identified



S^2 = disk w/ boundary pts identified

$S^n = (B^n = n\text{-dim'l ball})$ w/ points on boundary all collapsed to a unique point.

Example: We have another (equivalent) description of spheres

$$\text{Circle} = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$$
$$S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1\}$$
$$S^3 = \{(x,y,z,w) \in \mathbb{R}^4 \mid x^2+y^2+z^2+w^2=1\}$$
$$\vdots$$
$$S^n = \{(x_0,\dots,x_n) \in \mathbb{R}^{n+1} \mid \textstyle\sum_{i=0}^{n} x_i^2 = 1\}$$

These give rise to std embeddings:

$$S^1 \hookrightarrow \mathbb{R}^2$$
$$S^2 \hookrightarrow \mathbb{R}^3$$
$$S^3 \hookrightarrow \mathbb{R}^4$$
$$\vdots$$
$$S^n \hookrightarrow \mathbb{R}^{n+1}$$

Theorem: For every group G and $n \geq 4$, there exists an n-manifold M^n st

$$\pi_1(M^n) = G$$

Theorem: It is impossible to classify all groups.

Corollary: There is no classification of n-manifolds for $n \geq 4$

Remark: For $n = 2$, we proved the classification thm

For $n = 3$, there is a classification theorem but it is much more complicate.

↪ It is intimately related to Knots

Section: Whitney's Embedding Theorem

Theorem: (Whitney) Every n-manifold can be embedded in $\mathbb{R}^{2n}$.

Theorem: (Whitney) Every n-manifold that can be covered by a finite collection of n-dim'l balls can be embedded in $\mathbb{R}^N$ for some N.

Proof: Let $B_1, \dots, B_k$ be the balls that cover M

Associated to each B_i, we have a composition

$$\begin{tikzcd} M \arrow[r, "(1)"] \arrow[rrr, bend right, "\varphi_i"'] & M \cup S^n \arrow[r, "(2)"] & S^n \arrow[r, "(3)"] & \mathbb{R}^{n+1} \end{tikzcd}$$

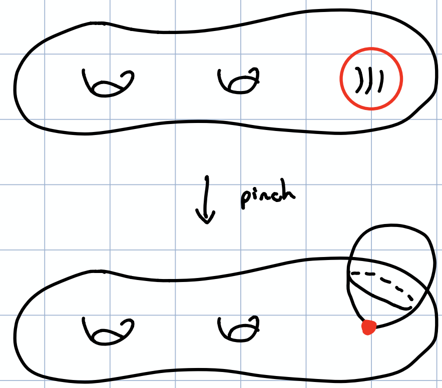
(1) Pinch the boundary of B_i to a point

(2) Collapse M part to a single point

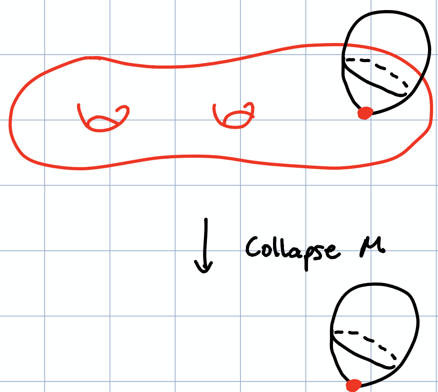
(3) Std embedding

Picture:

(1)



(2)



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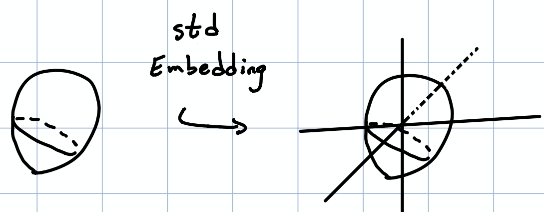

Proof: (continued)

$\varphi_i : M \to S^n \hookrightarrow \mathbb{R}^{n+1}$

Note φ_i embeds B_i in $\mathbb{R}^{n+1}$, but sends points outside of B_i to same point

Combine them to obtain embedding!

Defn

$$\Phi : M \to \mathbb{R}^{n+1} \times \dots \times \mathbb{R}^{n+1} = \mathbb{R}^{kn+k}$$

By

$$\Phi(p) = (\varphi_1(p), \dots, \varphi_k(p))$$

This is an embedding.

Indeed, if $\Phi(p) = \Phi(q)$, then $\varphi_i(p) = \varphi_i(q)$ $\forall i$.

$\Rightarrow$ p, q in same ball, say B_j

$\Rightarrow$ $\varphi_j(p) = \varphi_j(q)$

$\Rightarrow$ same since φ_j embeds points in B_j.

$\square$