Title: Zoom Lecture 1 Notes ①

Date: April 4^{th}, 2020

Section: Administrative Details

Remark: Zoom Etiquette

1) Keep microphone muted when not speaking
2) When microphone is muted, holding spacebar will temporarily unmute you
   ↳ good for quick questions
3) Using the chat feature you can type questions and raise hand during lecture.
4) Initially, let's say that you can use the spacebar feature to interupt me or ask questions
5) Enabling video is appreciated so I won't be talking to myself

②

Remark: Class Format

1) 1.5 – 2 hr of lecture from iPad w/ breaks
2) During lecture, I'll periodically give mini-exercises that you'll have a a few minutes to think about.
   ↳ My attempt to keep you engaged
3) Conclude w/ exercises, hints, and QA.

Remark: Notes

1) I'll post "non-filled in" handwritten note before class on class webpage.
2) Post "filled in" notes after class.
3) Notes for material that was skipped due to various constraints has been posted to class webpage.

Remark: Outline for remaining classes (3)

1) Homology
   ↪ Algebraic description/detection of voids in a space
   ↪ Prove that non-orientable surfaces can not be realized in $\mathbb{R}^3$
      $\Rightarrow$ Can't actually visualize.

2) Differential Topology
   ↪ Vector fields
   ↪ Euler characteristic via calculus
   ↪ Hairy Ball Theorem

3) Wild World of Topology
   ↪ Examples of crazy things that can happen in topology
   ↪ Stable Homotopy Theory
   ↪ What I do

(4)

Before: Topology $\longrightarrow$ Groups

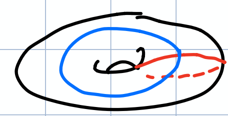
$X \longrightarrow \pi_1(X)$

↪ Encodes loops

Now: $X \longrightarrow H_\bullet(X)$

↪ Encodes voids

Example: Torus

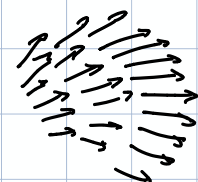


↪ 0-dim'l void is conn component

↪ 1-dim'l void is loops so ◯ ◯

↪ 2-dim'l void is inside versus outside.

↪ Vector field = arrow at every point that "continuously vary"

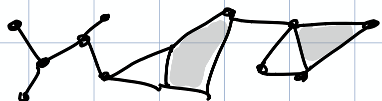


Hairy ball = comb a ball/surface w/ hair, when must it have a cowlick?

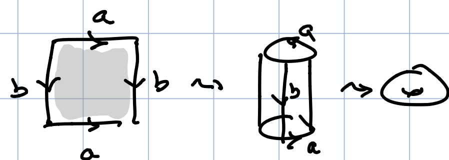
Section: Motivation for Homology

Definition: A __polygonal complex__ is a gluing of vertices, edges, and polygons. Gluing means edges are matched along vertices, polygons are matched along edges.

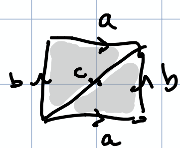
Example: 1) Random complex



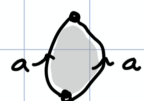
2) Torus



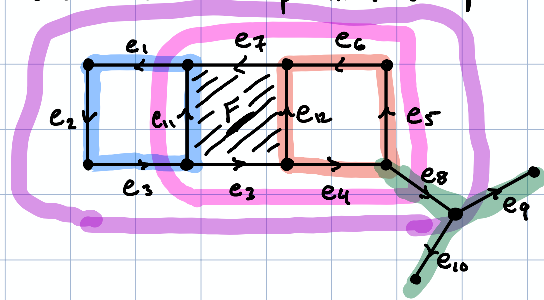
3) Torus (2)



4) Sphere



Example: What are the loops in this cpx?



$e_{11}e_3e_2e_1$

$e_{12}^{-1}e_6e_5e_4$

equiv since push across F

$e_{11}^{-1}e_7e_6e_5e_4e_3$

Purple = Blue + pink

Green isn't really loop. So we shouldn't care about it.

So in terms of loops homology should encode:

1) Ignore non-loops and identify actual loops
2) Have notion of adding loops
3) Face give ways to deform loops to equivalent loops.

Section: Group Theory and Word Groups ⑦

Definition: A commutative group is a set G and an operation $\star$ that eats two elements of G and spits out a new one st:

1) (Unital) $\exists\, e_G \in G$ st for all $g \in G$, $e_G \star g = g$.

   e_G is called the identity/unit.

2) (Inverses) for all $g \in G$, $\exists\, g^{-1} \in G$ st $g \star g^{-1} = e_G$

3) (Associative)

$$(g_1 \star g_2) \star g_3 = g_1 \star (g_2 \star g_3)$$

4) (Commutative)

$$g_1 \star g_2 = g_2 \star g_1$$

Example: ⑧

1) $(\mathbb{Z}, +)$; So $G = \mathbb{Z}$, $\star = +$

   So we add numbers to get new one.

   $e_G = 0$

2) $(\mathbb{R}_{>0}, \cdot)$; So $G = \mathbb{R}_{>0}$

   $\star =$ multiplication of positive real #s.

Definition: A word group consist of:

1) An alphabet, ie, list of symbols
   $a_0, \ldots, a_\ell, a_0^{-1}, \ldots, a_\ell^{-1}$
2) List of generators $w_1, \ldots, w_m$ st each w_i is word spelled w/ alphabet
3) List of relations $r_1, \ldots, r_n$ st each r_i is word spelled w/ w_i, w_i^{-1}.
   where w_i^{-1} is same word as w_i but w/ $a_i \mapsto a_i^{-1}$ or $a_i^{-1} \mapsto a_i$

Eg. $w = abc$
$w^{-1} = a^{-1}b^{-1}c^{-1}$

Gives a group $\langle w_1, \ldots, w_m \mid r_1, \ldots, r_n \rangle$

1) Elements are words obtained from concatenating copies of w_i or w_i^{-1}.
   Two words are equiv if
   i) Rearrange letters
   ii) Cancel a_i w/ a_i^{-1}
   iii) Remove subword that is a r_i or r_i^{-1}
2) Addition of words is by concatenation.

If $\{r_i\}$ = empty, then $\langle w_1, \ldots, w_m \rangle$ is called a free word group.

Example: 1) $\langle a, b \rangle$

For example, things look like
$aab \sim aba$, $abb \sim bab \sim bba$
We write $aa = a^2$, etc.
So $aaabbab \sim a^4b^3 \sim$ a 4x then b 3x.

2) $\langle a, b, c \mid bc^{-1} \rangle$ → equiv to saying $b = c$.

The relation $bc^{-1} \Rightarrow b = c$. Indeed,

$$b \overset{(ii)}{\sim} bcc^{-1} \overset{(i)}{\sim} cbc^{-1} \overset{(iii)}{\sim} c$$

So any word we get w/ a, b, c is equiv to a word we get w/ just a, b
$\Rightarrow \langle a, b, c \mid bc^{-1} \rangle \simeq \langle a, b \rangle$
where $\simeq$ means they abstractly give the same "language".

3) $\langle a, b, c \mid c^2, b^2, ab \rangle$.

As in (2), ab a rel. $\Rightarrow a \sim b^{-1}$

$\Rightarrow a^2 = (b^{-1})^2 = (b^2)^{-1} = 0$

So abstractly

$\langle a, b, c \mid c^2, b^2, ab \rangle \simeq \langle a, c \mid c^2, a^2 \rangle$

4) $(*) = \langle a, b, c, d \mid ac, db, a^2, b^2, c^2, d^2, ad \rangle$

Let's simplify.

As above, $a \sim c^{-1}$, $a \sim d^{-1}$, $b \sim d^{-1}$

So $(*) \simeq \langle a \mid a^2 \rangle$

$\hookrightarrow$ we can replace b, c, d or inverse w/ an a or a^{-1}.

(13)

Remark: $\langle w_1, \ldots, w_m \mid r_1, \ldots, r_n \rangle$ gives a presentation of a word group. Different pres. can give same group as we saw above.

Claim: $\langle w_1, \ldots, w_m \mid r_1, \ldots, r_n \rangle$ is a commutative group.

Proof: (Unital)

Identity = empty word.

(Inverses)

Switch a_i w/ a_i^{-1} or a_i^{-1} w/ a_i to get inverse. Eg. $(a_0 a_1^{-1} a_2)^{-1} = a_0^{-1} a_1 a_2^{-1}$.

Indeed,

$$(a_0 a_1^{-1} a_2)(a_0^{-1} a_1 a_2^{-1}) \overset{(i)}{=} a_0 a_0^{-1} a_1 a_1^{-1} a_2 a_2^{-1} \overset{(ii)}{=} 0$$

(Associative + comm.)

Follows from rearranging letters. $\square$

(14)

Definition: A group homomorphism between groups G and H is a map $\varphi: G \to H$ st

$$\varphi(g_1 \cdot g_2) = \varphi(g_1) \cdot \varphi(g_2)$$

↳ It respects the additive/grp structure

Example: 1) $\varphi: \mathbb{Z} \to \mathbb{Z}$ by $\varphi(n) = 77 \cdot n$

$$\begin{aligned}\varphi(n+m) &= 77 \cdot (n+m) \\ &= 77n + 77m \\ &= \varphi(n) + \varphi(m)\end{aligned}$$

$\Rightarrow \varphi$ is grp hom

2) $\varphi: (\mathbb{R}, +) \to (\mathbb{R}_{>0}, \cdot)$, $\varphi(x) = e^x$

$$\begin{aligned}\varphi(x+y) &= e^{x+y} \\ &= e^x e^y \\ &= \varphi(x) \cdot \varphi(y)\end{aligned}$$

$\Rightarrow \varphi$ is grp hom.

Remark: A grp homomorphism between free word groups is assignment of words in one to words in another

- $\hookrightarrow$ Suffices to specify what happens to generators
- $\hookrightarrow$ $\varphi : \langle w_1, \dots, w_n \rangle \to \langle v_1, \dots, v_\ell \rangle$

  $\varphi(w_i) = v_1^{k_1} \cdots v_\ell^{k_\ell}$ where $k_j \in \mathbb{Z}$.

Example: 1) $\langle e_0, e_1, e_2 \rangle \xrightarrow{\partial} \langle v_0, v_1, v_2 \rangle$

by
$$\partial(e_0) = v_2 v_1^{-1}$$
$$\partial(e_1) = v_2 v_0^{-1}$$
$$\partial(e_2) = v_1 v_0^{-1}$$

Not hard to $\checkmark$ that its a grp hom

Definition: The Kernel of a group homomorphism $\varphi : G \to H$ is

$$\mathrm{Ker}(\varphi) = \{ g \in G \mid \varphi(g) = e_H \}$$

Question: What is Kernel for homomorphism of free word groups?

Answer: Words that don't have translations

$\hookrightarrow$ Words sent to empty word.

Example: $\langle e_0, e_1, e_2 \rangle \xrightarrow{\partial} \langle v_0, v_1, v_2 \rangle$ as before

Any word in LHS is of the form $e_0^n e_1^m e_2^\ell$ for $n, m, \ell \in \mathbb{Z}$.

$$\partial(e_0^n e_1^m e_2^\ell) = v_2^n v_1^{-n} v_2^m v_0^{-m} v_1^\ell v_0^{-\ell}$$
$$= v_0^{-m-\ell} v_1^{\ell - n} v_2^{n+m}$$

so zero v_0, v_1, v_2 in word ie empty word

$$0 \iff -m - \ell = 0 = \ell - n = 0 = n + m$$
$$\iff \ell = n = -m$$

$$\mathrm{Ker}(\partial) = \{ e_0^n e_1^{-n} e_2^n \} = \langle e_0 e_1^{-1} e_2 \rangle$$

Claim: $Ker(\varphi)$ is a group. (17)

Proof: $Ker(\varphi) \subseteq G$, so we add as we would in G.

(Comm + Ass)

Its comm/assoc. in G, so it is when restricting to elements in Kernel.

(Unital)

By prev. notes, $\varphi(e_G) = e_H$

$\Rightarrow Ker(\varphi)$ has unit.

(Inverses)

By prev. notes, $\varphi(g^{-1}) = \varphi(g)^{-1}$

So $\varphi(g) = 0 \Rightarrow \varphi(g^{-1}) = \varphi(g)^{-1} = \underbrace{0^{-1}}_{=0}$

$\Rightarrow Ker(\varphi)$ has inverses. $\square$

Remark: $\varphi: \langle w_1, \dots, w_n \rangle \longrightarrow \langle v_1, \dots, v_\ell \rangle$ (18)

$\Rightarrow Ker(\varphi) = \langle x_1, \dots, x_k \rangle$ st

x_i = words in w_i's or w_i^{-1}'s st

$\varphi(x_i) = $ empty word $= 0$

Question: What are the Kernels of the following:

1) $\langle a, b, c \rangle \xrightarrow{\varphi} \langle e, f \rangle$ by

$a \mapsto e$, $b \mapsto e^{-1}$, $c \mapsto e^{-1}$.

Answer: Write elm of LHS as $a^n b^m c^\ell$.

$\varphi(a^n b^m c^\ell) = e^n e^{-m} e^{-\ell} = e^{n-m-\ell}$

Zero iff empty word

$\Leftrightarrow$ all powers are zero.

$\Rightarrow 0 = n - m - \ell$

$Ker(\varphi) = \{ a^n b^m c^{n-m} \}$.

Example: $\langle a, b, c \rangle \xrightarrow{\varphi} \langle e, f \rangle$

$\varphi(a) = e f^{-1}$

$\varphi(b) = f$

$\varphi(c) = e^{-1}$

$$\varphi(a^n b^m c^l) = e^n f^{-n} f^m e^{-l} = e^{n-l} f^{m-n}.$$

Zero iff $n - l = 0 = m - n$

$\Rightarrow n = l = m$

$\mathrm{Ker}(\varphi) = \{a^n b^n c^n\} = \langle abc \rangle$

Remark: If $B \subseteq \langle w_1, \dots, w_n \rangle = G$ is a collection of words obtained from w_i,

$$B = \langle b_0, \dots, b_k \rangle$$

Then we obtain new word group G/B

$$G/B = \langle w_1, \dots, w_n \mid b_0, \dots, b_k \rangle$$

Example: 1) $\langle a, b, c, d \rangle = G$

$B = \langle acccdddd \rangle$

$G/B = \langle a, b, c, d \mid \quad \rangle$

2) $\langle a, b \rangle = G$

$B = \langle ab^{-1} \rangle$

$G/B = \langle a, b \mid ab^{-1} \rangle$ $\quad a = b$

$= \langle a \rangle$

Section: Chain complexes and Polygonal Homology

Definition: A chain complex consist of

1) 3 word groups

$$C_0 = \langle v_0, \ldots, v_\ell \rangle$$

$$C_1 = \langle e_0, \ldots, e_n \rangle$$

$$C_2 = \langle f_0, \ldots, f_m \rangle$$

2) 2 group homomorphisms

Called boundary operators or "paritial"-"one" etc.

$$\partial_2 : C_2 \to C_1$$

$$\partial_1 : C_1 \to C_0$$

st $\partial_1 \circ \partial_2 (x) = 0$

Denote this by $(C_\bullet, \partial_\bullet)$

words hit by ∂_2

Lemma: $Im(\partial_2) \subseteq Ker(\partial_1)$

Proof: y = word in image of ∂_2.

$\Rightarrow \exists\ x \in C_2$ st $\partial_2(x) = y$

$\partial_1(y) = \partial_1 \circ \partial_2(x) = 0$

$\Rightarrow y \in Ker(\partial_1)$

$\Rightarrow Im(\partial_2) \subseteq Ker(\partial_1)$ as desired. $\square$

Definition: The homology groups of $(C_\bullet, \partial_\bullet)$ are

$$H_0(C_\bullet, \partial_\bullet) = C_0 / Im(\partial_1)$$

$$= \langle v_0, \ldots, v_\ell \mid \partial_1(e_0), \ldots, \partial_1(e_n) \rangle$$

$$H_1(C_\bullet, \partial_\bullet)$$

$$= Ker(\partial_1) / Im(\partial_2)$$

$$= \langle x_0, \ldots, x_k \mid \partial_2(f_0), \ldots, \partial_2(f_m) \rangle$$

where $\partial_1(x_i) = 0$

$$H_2(C_\bullet, \partial_\bullet) = Ker(\partial_2)$$

$$= \langle y_0, \ldots, y_a \rangle$$

where $\partial_2(y_i) = 0$

Example: $C_0 = \langle v_0 \rangle$

$C_1 = \langle e_0 \rangle$

$C_2 = \langle f_0 \rangle$

$\partial_1(e_0) = v_0^4$

$\partial_2(f_0) = 0$

We compute.

$H_0 = \langle v_0 \mid \partial_1(e_0) \rangle$

$= \langle v_0 \mid v_0^4 \rangle$

We claim $\mathrm{Ker}(\partial_1) = \langle 0 \rangle$.

$\partial_1(e_0^n) = v_0^{4n} = 0$ iff $n = 0$

$\Rightarrow e_0^n \in \mathrm{Ker}$ iff $n=0$ ie, $e_0^n = 0$

$\Rightarrow H_1 = \langle 0 \mid \partial_2(f_0) \rangle$

$= \langle 0 \mid 0 \rangle$

$= \langle \, \rangle$

By defn, $\partial_2(f_0) = 0$

$\Rightarrow H_2 = \mathrm{Ker}(\partial_2) = \langle f_0 \rangle$

Example: $C_0 = \langle v_0, v_1 \rangle$

$C_1 = \langle e_0, e_1 \rangle$

$C_2 = \langle f_0 \rangle$

$\partial_2(f_0) = 0$

$\partial_1(e_0) = v_1 v_0^{-1}$, $\partial_1(e_1) = v_0 v_1^{-1}$

We should check that this is actually a chain complex, ie, $\partial_1 \circ \partial_2 = 0$.

Indeed, $\partial_1 \circ \partial_2(f_0^n) = \partial_1(0) = 0 \ \forall n$.

We compute

$H_0 = \langle v_0, v_1 \mid \partial_1(e_0), \partial_1(e_1) \rangle$

$= \langle v_0, v_1 \mid v_1 v_0^{-1}, v_0 v_1^{-1} \rangle$

$= \langle v_0 \rangle$

$0 = \partial_1(e_0^n e_1^m) = v_1^n v_0^{-n} v_0^m v_1^{-m} = (v_0 v_1)^{n-m}$

iff $n = m \Rightarrow \mathrm{Ker}(\partial_1) = \langle e_0 e_1 \rangle$

$H_1 = \langle e_0 e_1 \mid \partial_2(f_0) \rangle$

$= \langle e_0 e_1 \mid 0 \rangle$

$= \langle e_0 e_1 \rangle$

$H_2 = \mathrm{Ker}(\partial_2) = \langle f_0 \rangle$

Construction: Let X = polygonal complex.

Pick direction for each edge

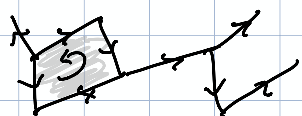
Pick orientation of each polygon

↳ way to sweep out edges.

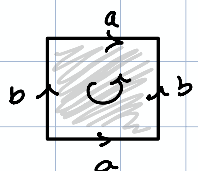
These need to be compatible w/ any gluings that we did. Recall diff between Torus and Klein bottle.

Remark: By "sweep", I mean clockwise or counter-clockwise direction to read of seq. of edges on boundary

Example: 1)



2)



Construction: (continued)

Label vertices : $v_0, \ldots, v_e$

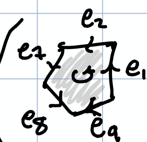
edges : $e_0, \ldots, e_n$

faces : $f_0, \ldots, f_m$

Define

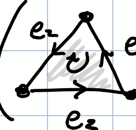
$$C_0(X) = \langle v_0, \ldots, v_e \rangle$$
$$C_1(X) = \langle e_0, \ldots, e_n \rangle$$
$$C_2(X) = \langle f_0, \ldots, f_m \rangle$$

Define

$\partial_2(\text{face}) = \partial_2\Big($  $\Big)$

$$= e_9^{-1} e_8 e_7 e_2 e_1$$

$$\partial_1(\text{edge}) = \partial_1(v_0 \xrightarrow{e_1} v_1)$$
$$= v_1 v_0^{-1}$$

Example: $\partial_2\Big($  $\Big)$ $= e_2^{-1} e_3^{-1} e_1^{-1}$

$= e_1^{-1} e_2^{-1} e_3^{-1}$ $\;\; \shortparallel \;\;$ starting loc. to read didn't matter.

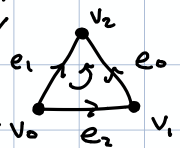
Claim: $(C_\bullet(X), \partial_\bullet)$ is a chain complex

Proof: We need

$$\partial_1 \circ \partial_2 \left(\text{polygon}\right) = 0$$

By example,

$$\partial_1 \circ \partial_2 \left(\quad \right)$$



$$= \partial_1 \left(e_0 e_2 e_1^{-1} \right)$$

$$= v_2 v_0^{-1} v_1 v_0^{-1} v_2^{-1} v_0^{+1}$$

$$= v_2 v_2^{-1} v_1 v_1^{-1} v_0 v_0^{-1}$$

$$= 0$$

$\square$

Definition: The homology of X is the homology of the chain complex $(C_\bullet(X), \partial_\bullet)$.

Denote it by $H_\bullet(X)$.

Example: $X = \text{point} = \bullet v_0$

$C_0 = \langle v_0 \rangle$

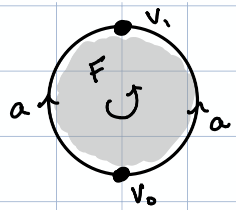
$C_1 = \langle \rangle$

$C_2 = \langle \rangle$

$H_0(pt) = \langle v_0 \rangle / \mathrm{Im}(\partial_1) = \langle v_0 \rangle$

No edges/faces, so $H_1 = 0 = H_2$.

Example: S^2 = sphere =



$C_0 = \langle v_0, v_1 \rangle$

$C_1 = \langle a \rangle$

$C_2 = \langle F \rangle$

$H_0(S^2) = \langle v_0, v_1 \mid \partial_1(a) \rangle$

$= \langle v_0, v_1 \mid v_1 v_0^{-1} \rangle$

$= \langle v_0 \rangle$

$H_1(S^2) = Ker(\partial_1) / Im(\partial_2)$

We compute $Ker(\partial_1)$.

$\partial_1(a^n) = v_1^n v_0^{-n} = 0$ iff $n = 0$

$\Rightarrow Ker(\partial_1) = 0$

$\Rightarrow H_1(S^2) = \langle \ \rangle$

We compute $Ker(\partial_2)$.

$\partial_2(F^n) = a^n a^{-n} = 0$

$\Rightarrow H_2(S^2) = Ker(\partial_2) = \langle F \rangle$

Remark: Topologically, what does each $H_i(X)$ mean?

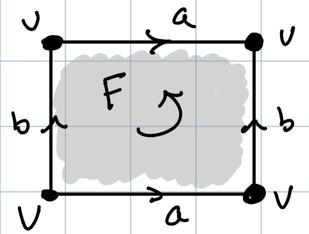
1) $H_0(X)$ = # of conn components

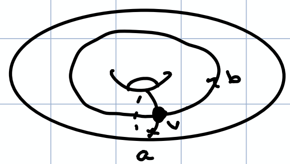
1) $H_1(X)$ = loops that are equivalent up to pushing across faces

2) $H_2(X)$ = 2-dim'l voids

↪ Will explain more next week!

Example: T^2 = Torus

= 

= 

$C_0 = \langle v \rangle$

$C_1 = \langle a, b \rangle$

$C_2 = \langle F \rangle$.

We compute

$$H_0 = \langle v \mid \partial_1(a), \partial_1(b) \rangle$$
$$= \langle v \mid vv^{-1}, vv^{-1} \rangle$$
$$= \langle v \mid 0, 0 \rangle$$
$$= \langle v \rangle$$

Need to determine $\operatorname{Ker}(\partial_1)$.

Note $\partial_1(a) = vv^{-1} = 0$

$\partial_1(b) = vv^{-1} = 0$

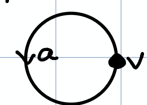
$\Rightarrow \operatorname{Ker}(\partial_1) = \langle a, b \rangle$

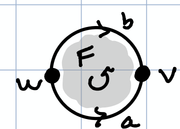
$$\Rightarrow H_1 = \langle a, b \mid \partial_2(F) \rangle$$
$$= \langle a, b \mid aba^{-1}b^{-1} \rangle$$
$$= \langle a, b \mid 0 \rangle$$
$$= \langle a, b \rangle$$

Note $\partial_2(F) = aba^{-1}b^{-1} = 0$

$\Rightarrow H_2 = \operatorname{Ker}(\partial_2) = \langle F \rangle$.

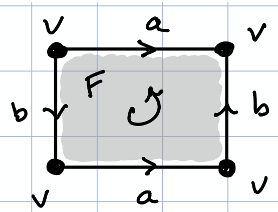
Example: You should compute H_* of

i) Circle 

ii) Disk 

just to get a feel for the method of computation.

Example: Klein bottle



$C_0 = \langle v \rangle$,

$C_1 = \langle a, b \rangle$

$C_2 = \langle F \rangle$

Sim. to torus case, $H_0 = \langle v \rangle$

Note,

$$\partial_1(a) = vv^{-1} = 0$$

$$\partial_1(b) = vv^{-1} = 0$$

$\Rightarrow \operatorname{Ker}(\partial_1) = \langle a, b \rangle$

$$\Rightarrow H_1 = \langle a, b \mid \partial_2(F) \rangle$$

$$= \langle a, b \mid aba^{-1}b \rangle$$

$$= \langle a, b \mid b^2 \rangle$$

Note,

$\partial_1(F^n) = a^n b^n a^{-n} b^n = b^{2n} = 0$

iff $n = 0$ $\Rightarrow \operatorname{Ker}(\partial_2) = 0$

$\Rightarrow H_2 = \langle \, \rangle$

Claim: X = surface, then $H(X)$ is independent of the choice of polygonal structure.

Proof: I may say more next week, but the idea is the following.
Given 2 different polygonal structures, say X_0 and X_1, we find some common refinement X_2.
Then we show $H_\bullet(X_0) = H_\bullet(X_2)$ and $H_\bullet(X_1) = H_\bullet(X_2)$, which implies the claim.
We get from X_0 to X_2 via performing 3 different types of modifications.
So one just needs to show that these modifications don't change $H_\bullet$.
The arguement is in a similar spirit to the proof that the Euler characteristic is ind. of poly. str. $\square$

Corollary: The torus is not homeomorphic to the Klein bottle.

Proof: If so, then a poly. str for torus determines a poly. str. for Klein bottle Since they have a common poly str. their homologies must be the same. However, by our above computations and the above claim, we see that this can't happen.

$\Rightarrow$ Cannot be homeomorphic. $\square$

Section: Exercises

① **Exercise:** Compute the homology of a collection of 42 disjoint points.

② **Exercise:** Compute homology of Möbius band.

③ **Exercise:** Show that $H_0(\text{tree}) \simeq \langle v_0 \rangle$ for v_0 a vertex in the tree.
Show that $H_i(\text{tree}) = \langle 0 \rangle$ for $i = 1, 2$.

④ **Exercise:** Show that $H_1(\text{Graph}) \neq \langle 0 \rangle$ if and only if the graph is not a tree.

⑤ **Exercise:** Explain why $H_2(\text{surface})$ is equiv to a free word group w/ either 0 or 1 word(s).

⑥ **Exercise:** Show that if X = orientable surface, then $H_1(X)$ is equivalent to a free word group w/ $2g$ words where g = # donut holes in X.
What do these words correspond to?
↪ Start w/ $g = 1$ or $g = 2$.

Remark: In the solutions below, we try to highlight the key points via . The is the takeaway that you should focus on. Everything else is just additional details to make it precise and rigorous.

① Solution: $C_0 = \langle v_1, \dots, v_{42} \rangle$

$C_1 = \langle\ \rangle$

$C_2 = \langle\ \rangle$ } No edges/faces (*)

(*) $\Rightarrow$ $H_1 = 0 = H_2$

Finally, $H_0 = \langle v_1, \dots, v_{42} \rangle / \mathrm{Im}(\partial_1)$

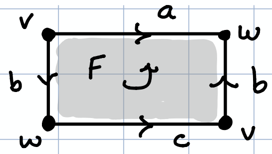
But no edges so $\mathrm{Im}(\partial_1) = 0$

$\Rightarrow$ $H_0 = \langle v_1, \dots, v_{42} \rangle$

↪ one letter for each connected component!

Key points

② Solution: Möbius band =



$C_0 = \langle v, w \rangle$

$C_1 = \langle a, b, c \rangle$

$C_2 = \langle F \rangle$

We compute

$$H_0 = \langle v, w \mid \partial_1(a), \partial_1(b), \partial_1(c) \rangle$$
$$= \langle v, w \mid wv^{-1}, wv^{-1}, vw^{-1} \rangle$$
$$\cong \langle v \rangle \quad (\text{or } \langle w \rangle \text{, same thing})$$

Compute $\mathrm{Ker}(\partial_1)$.

$$\partial_1(a^n b^m c^\ell) = w^n v^{-n} w^m v^{-m} v^\ell w^{-\ell} = 0$$

iff $n + m - \ell = 0 = \ell - n - m$

$$\Rightarrow \mathrm{Ker}(\partial_1) = \{ a^n b^m c^{n+m} \}$$
$$= \{ (ac)^n (bc)^m \}$$
$$= \langle ac, bc \rangle$$

$$\Rightarrow H_1 = \langle ac, bc \mid \partial_2(F) \rangle$$
$$= \langle ac, bc \mid cba^{-1}b \rangle \Rightarrow a = cb^2$$
$$= \langle c^2b^2, bc \rangle$$
$$= \langle bc \rangle$$

We compute $\partial_2(F) = bcba^{-1} \neq 0$

$\Rightarrow H_2 = \mathrm{Ker}(\partial_2) = 0$

So one connected component, one loop, and zero 2-dim'l voids.

Key Point

↳ Möbius band is connected, has no 2-dim'l voids and really only has the one loop that wraps around the core of the band. So these computations agree w/ our intuition.

3

Solution: The second part of the question follows from the solution to question 3. The first part we prove by induction.

Spse we build our tree up in steps: $T_0, \dots, T_n$, where we add the vertex v_i and the edge e_i to T_{i-1} to obtain T_i and $T_0 = v_0$.

Inductively, we claim that

$$H_0(T_i) \simeq \langle v_0 \rangle \simeq \dots \simeq \langle v_i \rangle$$

For $i = 0$, this vacuously follows.

So spse it holds for T_{i-1}, we want to show it holds for T_i.

By defn,

$$H_0(T_{i-1}) = \langle v_0, \dots, v_{i-1} \mid \partial_1(e_1), \dots, \partial_1(e_{i-1}) \rangle$$
$$H_0(T_i) = \langle v_0, \dots, v_i \mid \partial_1(e_1), \dots, \partial_1(e_i) \rangle$$

So — simplifies to say $v_0 = v_1 = \dots = v_{i-1}$

Now $\partial_1(e_i) = v_i v_j^{-1}$ for some $0 \leq j \leq i-1$

$\Rightarrow v_i = v_j = v_0 = \text{etc.}$ in $H_0(T_i)$

$\Rightarrow H_0(T_i) = \langle v_0 \rangle \simeq \cdots \simeq \langle v_i \rangle$. 43

So $H_0(T_n = \text{Tree}) = \langle v_0 \rangle$, as desired.

Key Point

Essentially, b/c the tree was connected and H_0 detects connected components, $H_0(\text{Tree})$ should be equiv to a free word group w/ one letter/word. $\square$

4 Solution: The punchline is that a graph is not a tree iff the graph has more than one seq of edges connecting two distinct vertices, using these two seq. we can form a non-trivial loop and thus $H_1 \neq 0$. Conversely, $H_1 \neq 0$ $\Rightarrow \exists$ a loop $\Rightarrow$ we can construct two different seq. of edges. 44

Key Point

Let Γ denote the graph.

First, we show that

$$\Gamma \text{ not a tree} \Rightarrow H_1(\Gamma) \neq 0$$

Since Γ is not a tree, $\exists$ two vertices w and v and two seq. of edges $e_0, \ldots, e_n$ and $d_0, \ldots, d_m$ that connect w to v

Since these are seq of edges the end of e_i agrees w/ the start of e_{i+1}. Sim. for d_i and d_{i+1}

So if we have

then $v_i = w_{i+1}$, $w_0 = w$, $v_n = v$ (*)

$x_i = y_{i+1}$, $y_0 = w$, $x_m = v$

Note,

$\partial_1(e_0 \cdots e_n d_0^{-1} \cdots d_m^{-1})$

$= \partial_1(e_0) \cdots \partial_1(e_n)\, \partial_1(d_0^{-1}) \cdots \partial_1(d_m^{-1})$

$= v_0 w_0^{-1} \cdots v_n w_n^{-1} x_0^{-1} y_0 \cdots x_m^{-1} y_m$

(*) $= w_0^{-1} v_n y_0 x_m^{-1}$

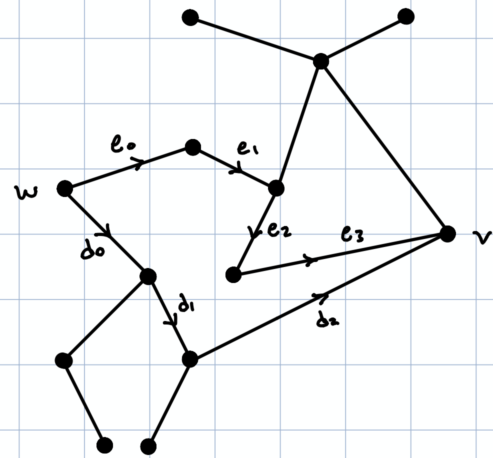
(*) $= w^{-1} v w v^{-1}$

$= 0$

$\Rightarrow \mathrm{Ker}(\partial_1) \neq 0$.

Since $\mathrm{Im}(\partial_2) = 0 \Rightarrow H_1(\Gamma) \neq 0$.

Picture:



Now we show

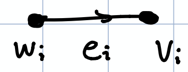
$H_1(\Gamma) \neq 0 \Rightarrow \Gamma$ not a tree

Since $H_1(\Gamma) = \mathrm{Ker}(\partial_1)$ $(\mathrm{Im}(\partial_2) = 0)$

$\Rightarrow \exists$ word $e_0 \cdots e_k$ st

$\partial_1(e_0 \cdots e_k) = 0$

Using notation from above, spse we have



then

$$0 = \partial_1(e_0 \cdots e_k)$$
$$= v_0 w_0^{-1} \cdots v_k w_k^{-1}$$

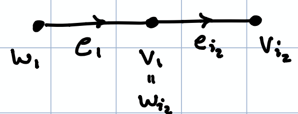
So each v_i has to cancel off w/ some w_j. Now we construct two seq. of edges that connect w_0 to v_0

The first seq is simple e_0.

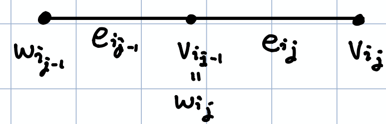
We inductively construct the other.

Set $d_1 = e_1$.

Set $d_2 = e_{i_2}$ where $0 \le i_2 \ne 1 \le n$



Ect, $d_j = e_{i_j}$ where $0 \le i_j \ne 1 \le n$



Since all vertices cancel off, eventually $v_{i_j} = w_0$. So the d_i give another sequence.

By defn, Γ is not a tree. $\square$

5

Solution: Using the connect sum operation w/ planar diagrams, we could create a planar dgm for any surface by gluing edges of a single polygon together in pairs

Key Point

$\Rightarrow$ every surface admits a polygonal structure w/ a <u>single</u> polygon.

$\Rightarrow C_2(X) = \langle F \rangle$ where F is the single polygon.

If $\partial_2(F) = 0$, then

$$H_2(X) = \ker(\partial_2) = \langle F \rangle$$

If $\partial_2(F) \neq 0$, then

$$H_2(X) = \ker(\partial_2) = \langle\ \rangle$$

This is what we needed to show. $\square$

6

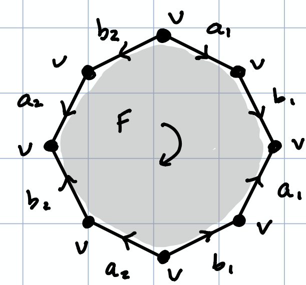
Solution: Recall that if X = surface, then $H(X)$ is independent of the choice of polygonal structure.

So to compute we should pick a nice one.

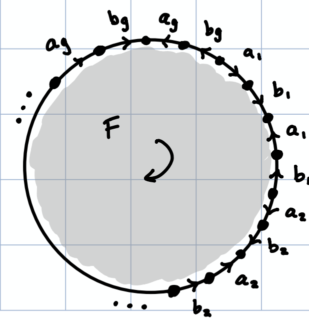
Key Point

Using the connect sum operation w/ planar diagrams, we could create a planar dgm for a genus g surface by gluing edges of $4g$-gon to themselves in pairs as illustrated below.

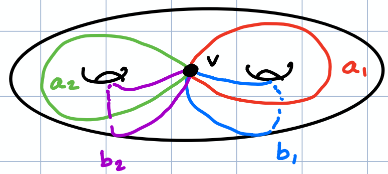
$g = 2$:



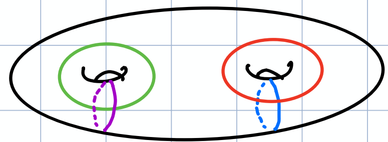
g = arbitrary:



For $g=2$, this look like



$\cong$ pushing around curves



$C_0 = \langle v \rangle$

$C_1 = \langle a_1, b_1, \dots, a_g, b_g \rangle$

$C_2 = \langle F \rangle$

$\partial_1(a_i) = vv^{-1} = 0$

$\partial_1(b_i) = vv^{-1} = 0$

$\Rightarrow \operatorname{Ker}(\partial_1) = \langle a_1, b_1, \dots, a_g, b_g \rangle$

$\partial_1(F) = a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$

$= 0$

$\Rightarrow \operatorname{Ker}(\partial_2) = \langle F \rangle$

Combining all of this yields

$H_0 = \langle v \mid \dots, \partial_1(a_i), \partial_1(b_i), \dots \rangle$

$= \langle v \mid \dots, vv^{-1}, vv^{-1}, \dots \rangle$

$= \langle v \mid \dots, 0, 0, \dots \rangle$

$= \langle v \rangle$

$H_1 = \operatorname{Ker}(\partial_1) / \operatorname{Im}(\partial_2)$

$= \langle a_1, b_1, \dots, a_g, b_g \mid \partial_2(F) \rangle$

$= \langle a_1, b_1, \dots, a_g, b_g \mid 0 \rangle$

$= \langle a_1, b_1, \dots, a_g, b_g \rangle$

$H_2 = \operatorname{Ker}(\partial_2) = \langle F \rangle$