Lecture #9

Outline:

1) Metrics and Isometries
2) Geodesics
3) Gaussian Curvature
4) Gauss-Bonnet Theorem

Section 1: Metrics and Isometries

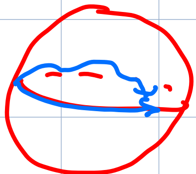
Definition: A surface is space that locally looks like $\mathbb{R}^2$

↪ ie, zoom in close it just looks like a "piece of paper."

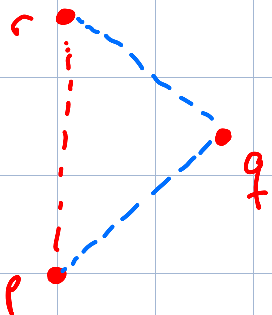
Definition: A metric on a surface Σ is a fcn d that assigns to every pair of points $p, q \in \Sigma$ a real #, $d(p,q)$. This function satisfies

1) $d(p,q) \geq 0$ w/ equality only when $p = q$
2) $d(p,q) = d(q,p)$
3) $d(p,r) \leq d(p,q) + d(q,r)$

Remark: Intuitively, $d(p,q)$ is the distance between p and q on Σ. So the above conditions translate to:



1) distance is always positive and is zero only when $p=q$

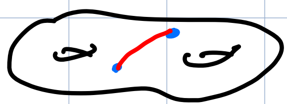


2) the distance from p to q is the distance from q to p

3) the distance from p to r is less than the distance from p to any intermediary point q plus the distance from r to the intermediary point q.

Notation: (Σ, d) = surface Σ w/ a choice of metric d.

Remark: We can obtain a metric d on any surface Σ as follows:



1) Embed Σ in $\mathbb{R}^3$

2) $d(p,q)$ = length of shortest path on Σ that connects p to q, where the length is measured wrt the usual distance in $\mathbb{R}^N$.

Definition: (Σ_0, d_0) and (Σ_1, d_1) are isometric if they are homeomorphic in such a way that preserves distance wrt the metrics.

↪ ie, take points that are distance C apart to points that are distance C apart.

Examples:

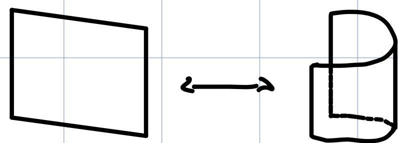
1) Inflating / deflating the beach ball

↳ <u>Not</u> isometry

2) Rotating beach ball

↳ isometry

3) Slightly rolled piece of paper



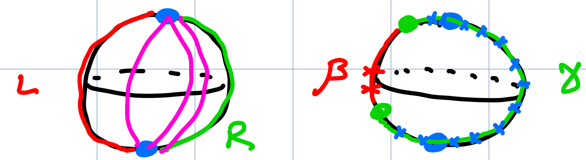
↳ isometry

Remark:

1) We have moved beyond topology and into geometry.

2) Now our deformations need not only preserve shape, but also distances/angles.

Section 2: Geodesics



Definition: A geodesic on (Σ, d) is a curve that is locally distance minimizing.

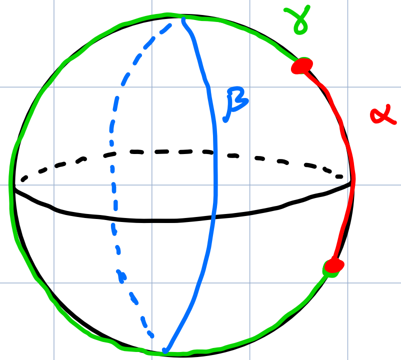
Remark: 1) In geometry or even real life, it is very hard to find and work w/ curves that are everywhere the shortest path.

↳ Best we can try is shortest path "locally", ie find shortest distance to the points we can actually see.

↳ geodesics generalize "straight-lines" to surfaces.

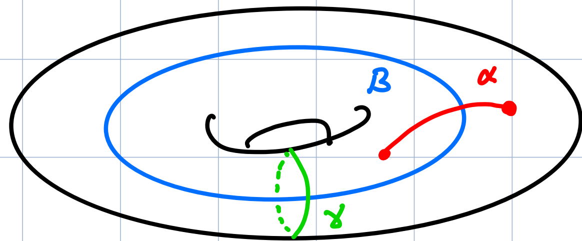
Examples:

1)



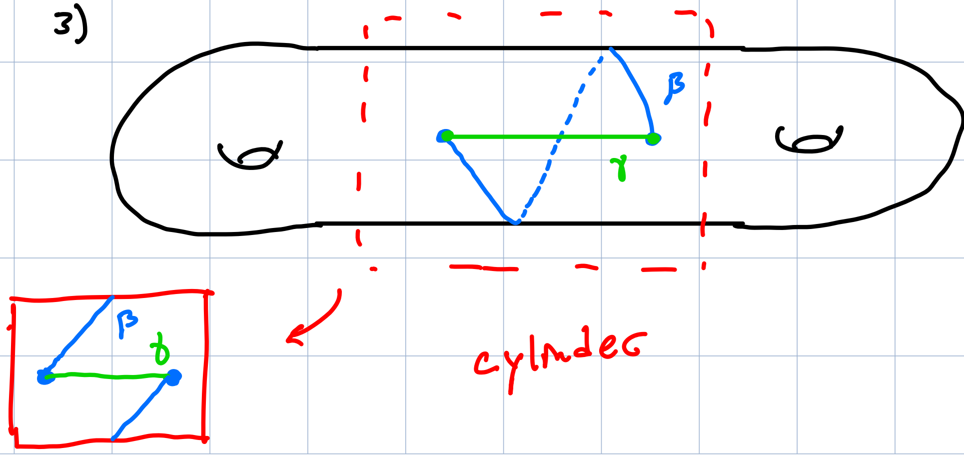
α, β, γ are all geodesics.

2)



α, β, γ are all geodesics.

3)



β, γ are all geodesics.

Section 3: Gaussian Curvature

Remark: 1) The geometry of a space is concerned w/ how curved the space (when are geodesics not straight lines).

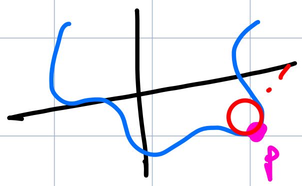
2) The topology/shape of a space doesn't care to some extent.

3) We will define Gaussian curvature, which will quantify this failure of surfaces to be flat.

Notation: We will assume (Σ, d) is a surface Σ that lives in $\mathbb{R}^3$ and $d(p,q)$ is the length of the shortest path in Σ connecting p to q, where "length" is measured wrt usual distance in $\mathbb{R}^3$.

Remark: All of the below defn/results generalize to orientable surfaces w/ more arbitrary metrics; however, we will just focus on the case above for ease/concreteness.

Definition:

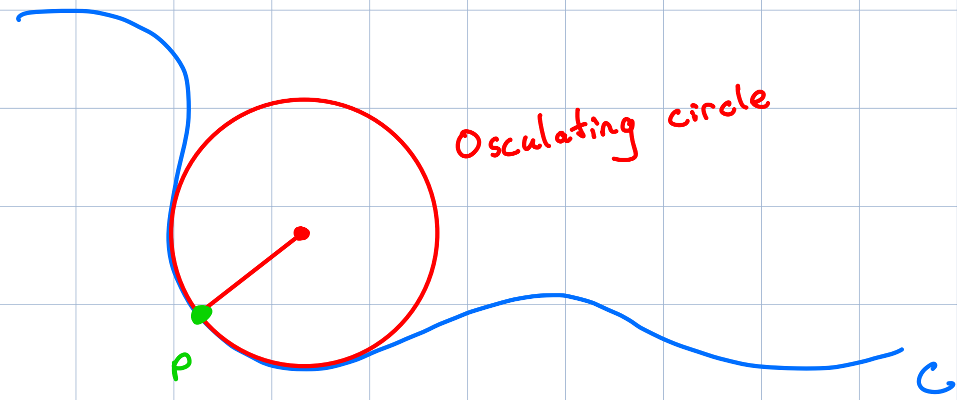


1) Let C be a curve in $\mathbb{R}^2$ and let p be a point on C. The osculating circle of C at p is the circle in $\mathbb{R}^2$ that is tangent to C at p and hugs the curve most tightly.

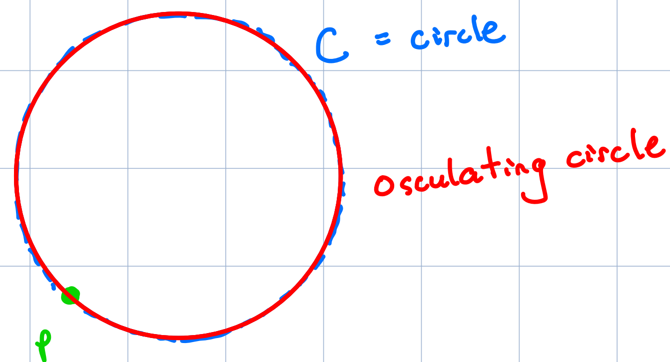
2) The curvature of C at p is $1/r$ where r = radius of the osculating circle.

Example:

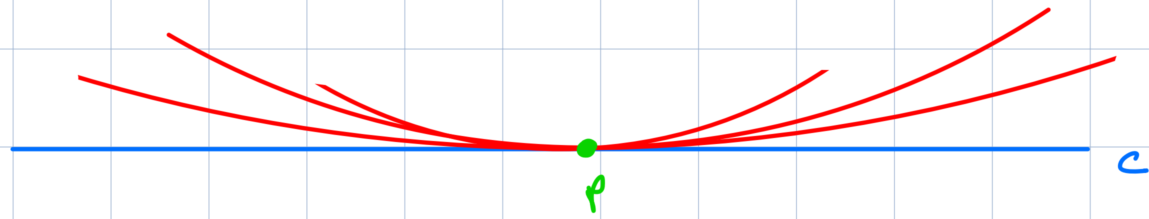
1)



2)



3)

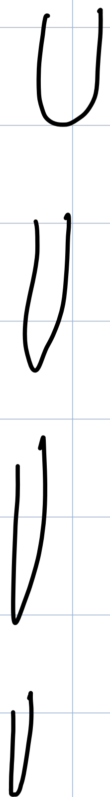


↪ radius of osculating circle is ∞ when C is a line.

$\Rightarrow$ curvature at p is $1/\infty = 0$.

Example:

$c \cdot x^2 \overset{\cdot\cdot}{\rightsquigarrow} 2c$



1) Given a fcn $f: \mathbb{R} \longrightarrow \mathbb{R}$, we obtain a curve in $\mathbb{R}^2$ by looking at the graph of f.

2) The radius of the osculating circle at $(x, f(x))$ is

$$r = \frac{(1 + f'(x)^2)^{3/2}}{|f''(x)|}$$

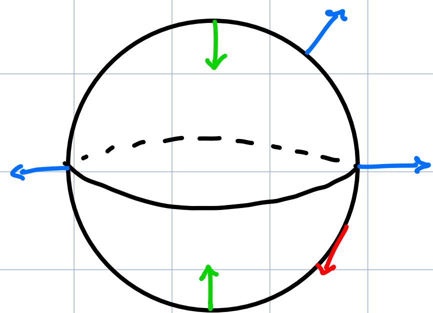
3) So the curvature is something seen by 2^{nd}-order derivatives.

4) Roughly, as $|f''|$ increases so does curvature.

Definition: An <u>outward normal vector</u> at p is a direction in $\mathbb{R}^3$ that is perpendicular to Σ at p and points outward from Σ

Picture:

1)

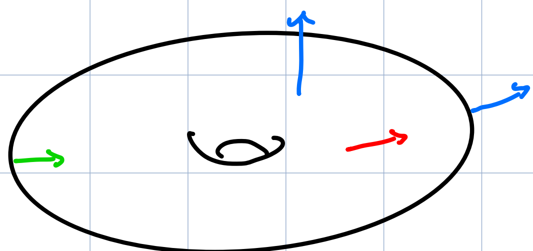


" $\rightarrow$ " = outward normal vectors

" $\rightarrow$ " = inward normal vectors

" $\rightarrow$ " = not normal vectors

2)

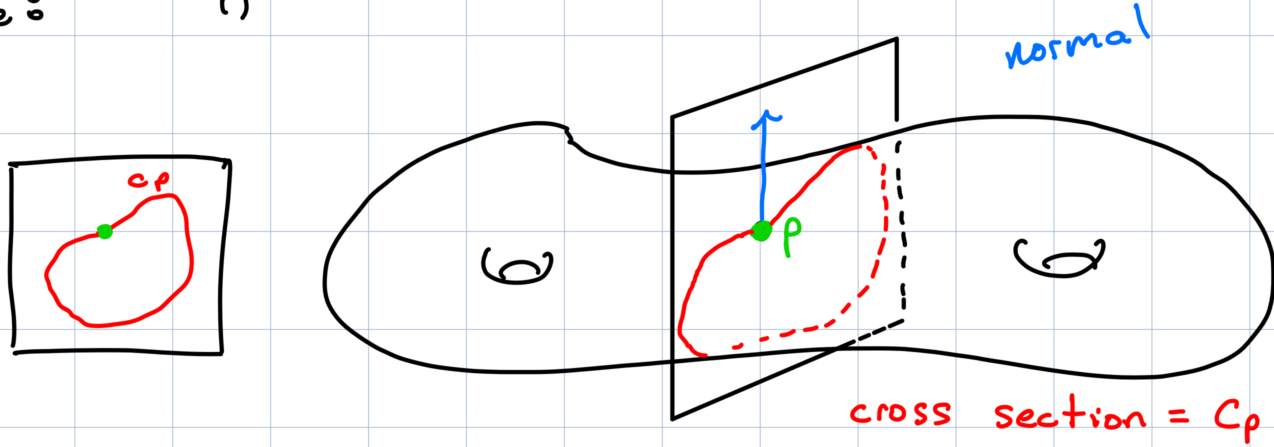


Definition: We define the Gaussian curvature of Σ at p as follows:

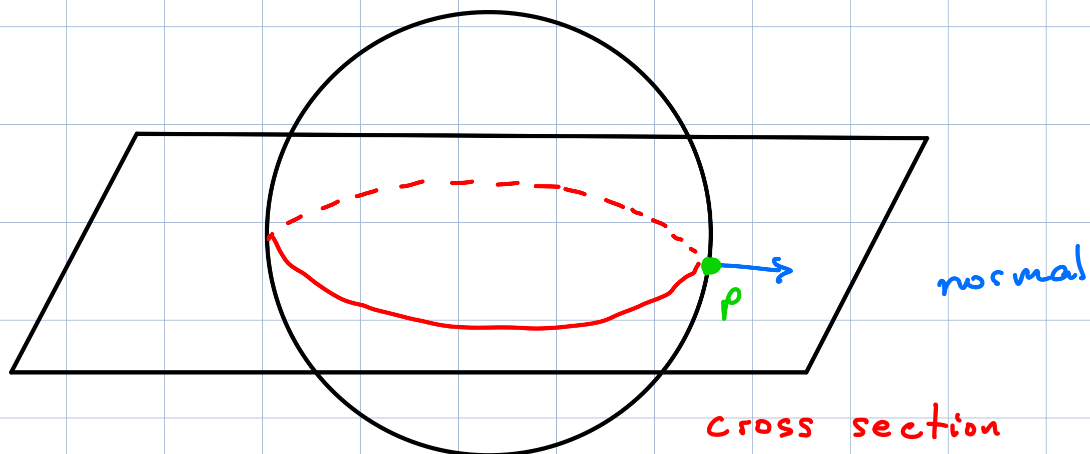
1) Fix an outward normal vector at p.
2) Consider a cross section Σ that contains p and the outward normal vector.
   ↳ ie, part of Σ that lies in a plane that contains p and outward normal vector.
   ↳ this cross-section of Σ defines a curve C_p in plane

Picture:

1)



2)



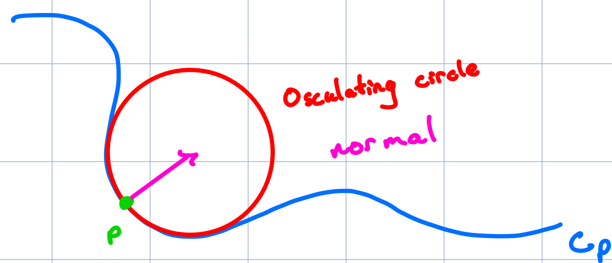
3) $\mathcal{K}(C_p) = \pm$ (curvature of C_p)

↪ $+$ when center of circle lies <u>above</u> p wrt outward normal direction

↪ $-$ when center of circle lies <u>below</u> p wrt outward normal direction

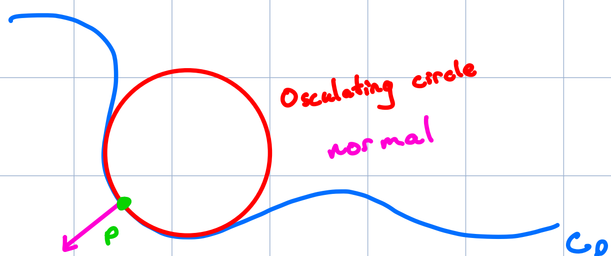
Picture:

1)



$\longleftrightarrow$ **+**

2)



$\longleftrightarrow$ **−**

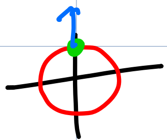
4) $\kappa_{max}(p)$ = maximum curvature among all possible cross-section curves

$\kappa_{min}(p)$ = minimum curvature among all possible cross-section curves

5) The curvature of Σ at p is

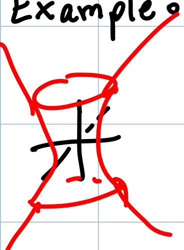
$$K(p) = \kappa_{max}(p) \cdot \kappa_{min}(p).$$

Example:

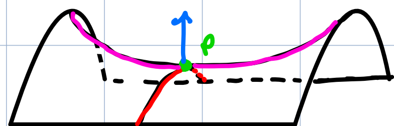


- Σ = sphere of radius r.
- Every cross-section is a great circle of radius r
- $\Rightarrow$ curvature of every cross-section is $-1/r$
- $\Rightarrow$ $K = 1/r^2$ for every point p in $\Sigma = S^2$

Example:



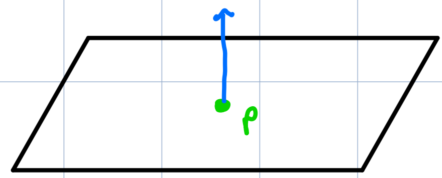
- Compute Gaussian curvature at center of hyperboloid



- κ_{min} will be negative and correspond to
- κ_{max} will be positive and correspond to
- $\Rightarrow$ $K(p) < 0$

Example:

- Compute Gaussian curvature at center of plane



- Every cross section is a straight line
- $\Rightarrow \kappa_{max} = \kappa_{min} = 0$
- $\Rightarrow K(p) = 0$

Theorem: If two surfaces are isometric, then they have the same Gaussian curvature.

Corollary: Any map of the earth must distort distances.

Proof:

1) Plane is flat $\Rightarrow$ Gaussian curvature $= 0$
2) Sphere is curved, Gaussian curvature $= 1$
3) Thm $\Rightarrow$ not isometric

$\Rightarrow$ no identification of points that preserves distance $\longrightarrow$ Even locally! $\square$

Section 4: Gauss-Bonnet

Definition: A curvi-linear triangle on (Σ, d) is a triangle whose edges are geodesics.

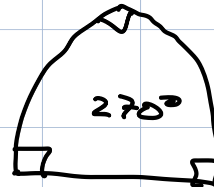
Example: 1) A curvi-linear triangle in the plane

↪ geodesics are straight lines

↪ so just normal triangle

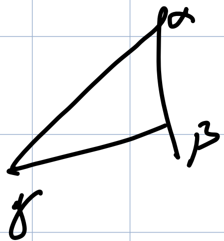
↪ sum of interior angles is π.

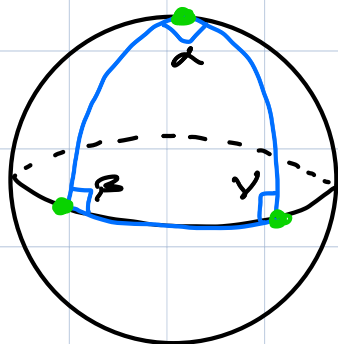
2) Sphere



↪ geodesics are great circles

↪

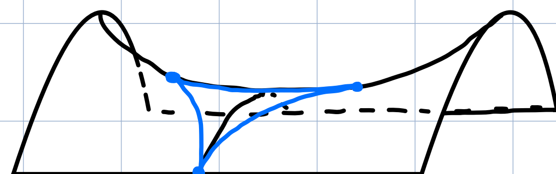




↪ sum of interior angles is $> \pi$

3) Center of hyperboloid

↪



↪ sum of interior angles is $< \pi$

Theorem: Let α, β, γ be the interior angles of a curvi-linear triangle Δ in (Σ, d). We have

$$\alpha + \beta + \gamma - \pi = \int_\Delta K$$

Remark: One can interpret $\int_\Delta K$ in two ways

1) K is a function on Σ.

   So we can integrate it over the region Δ.

   $\int_\Delta K$ is the surface integral of K over Δ

2) $\int_\Delta K = \text{area}(\Delta) \cdot (\text{average curvature over all } p \in \Delta)$

Example: curvi-linear triangle in the plane

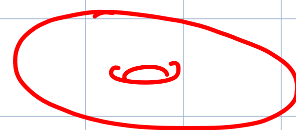
$\hookrightarrow K \equiv 0$

So theorem says: $\alpha + \beta + \gamma = \pi$

Theorem: (Gauss-Bonnet)

$$\int_\Sigma K = 2\pi \cdot \chi(\Sigma)$$

$$\chi = 2 - 2g$$



Remark: Again $\int_\Sigma K$ can be interpreted either as a surface integral <u>or</u>

$$\int_\Sigma K = \text{area}(\Sigma) \cdot (\text{average curvature over all } p \in \Sigma)$$

Example: Let's prove the theorem when Σ is sphere of radius r.

↳ Above, K is always $1/r^2$.

↳ Area of sphere of radius r is $4\pi r^2$

↳ $\chi(\Sigma) = 2$

So

$$
\begin{aligned}
&2\pi \cdot \chi(\Sigma) \\
&= 4\pi \\
&= 4\pi \cdot r^2 / r^2 \\
&= \text{area}(\Sigma) \cdot (\text{average curvature over all } p \in \Sigma) \\
&= \int_\Sigma K
\end{aligned}
$$

as desired. $\square$

Proof:

1) Pick a triangulation of Σ composed of curvilinear triangles that have no edges/vertices glued together.

2) Let $\Delta_1, \dots, \Delta_n$ be all the triangles w/ respective interior angles $\alpha_i, \beta_i, \gamma_i$

3) Note,

$$\int_{\Sigma} K = \sum_{i=1}^{n} \int_{\Delta_i} K$$

4) $2(\#\text{ Edges}) = 3(\#\text{ Faces})$

↳ "unglue" the triangulation as we've done before.

use that no each triangle does not have any of its edges/vertices glued together.

5) $2\pi \cdot (\#\text{ Vertices}) = \sum_{i=1}^{n} (\alpha_i + \beta_i + \gamma_i)$

6)
$$\begin{aligned}
\int_\Sigma K &= \sum_{i=1}^{n} \int_{\Delta_i} K \\
&= \sum_{i=1}^{n} (\alpha_i + \beta_i + \gamma_i - \pi) \\
&= \sum_{i=1}^{n} (\alpha_i + \beta_i + \gamma_i) - \pi \cdot F \\
&= 2\pi \cdot V - \pi \cdot F \\
&= 2\pi \cdot V - 2\pi E + 3\pi F - \pi \cdot F \\
&= 2\pi \cdot (V - E + F) \\
&= 2\pi \cdot \chi(\Sigma)
\end{aligned}$$

$\square$

The curvature of a triangle Δ w/ interior angles α, β, γ in a surface Σ is

$$K(\Delta) = \alpha + \beta + \gamma - \pi$$

The curvature of a surface Σ wrt a triangulation is the sum of the curvatures of each of the triangles in the triangulation.
We denote it by $K(X)$, where X is the triangulation of Σ.

Theorem :

$$\sum_{i=0}^{n} K(\Delta_i)$$

$$K(\Sigma) = K(X)$$