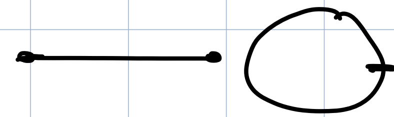
Lecture # 8

Outline:

1) Review
2) Some complex analysis
3) Fundamental Theorem of Algebra
4) Complex algebraic varieties

Section 1: Review



Definition: A closed curve in S^1 = circle is a ② continuous ① map $\gamma : S^1 \longrightarrow S^1$.

① We send every pt in S^1 to a point in S^1.

② "Continuous" = we map S^1 into S^1 w/ out ripping or cutting it

Remark: Equivalently, $\gamma : S^1 \longrightarrow S^1$ may be viewed as a continuous map $\gamma : [0, 2\pi] \longrightarrow S^1$ w/ $\gamma(0) = \gamma(2\pi)$

↪ i.e., a map of a circle is just a map of an interval that connects up at its end points.

Lemma: (Curve Lifting) Given a closed curve $\gamma : S^1 \longrightarrow S^1$, there exists a function $f : [0, 2\pi] \longrightarrow \mathbb{R}$ st

1) $f(0) = f(2\pi) + 2\pi \cdot n$ for some integer n
2) $\gamma(t) = (\cos(f(t)), \sin(f(t)))$

↪ f is called a <u>lift</u> of γ to $\mathbb{R}$.

Idea: $f(t)$ = Accumulated angle of rotation of $\gamma(t)$ measured w/ respect to $(1,0)$

↪ rotate clockwise angle decreases

↪ rotate counter clockwise angle increases

Definition: The <u>degree</u> of a closed curve $\gamma : S^1 \to S^1$ is

$$\deg(\gamma) = \left(f(2\pi) - f(0) \right) / 2\pi$$

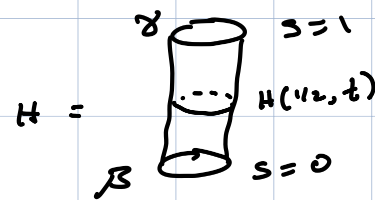
where f is <u>any</u> lift of γ to $\mathbb{R}$.

↪ This did not depend on the choice of lift

Remark: $\deg(\gamma)$ = signed # of times γ wraps around the circle

Definition: Two closed curves $\beta : S^1 \to S^1$ and $\gamma : S^1 \to S^1$ are <u>homotopic</u> if there is a continuous map $H : [0,1] \times S^1 \to S^1$ satisfying

1) $H(0,t) = \beta(t)$
2) $H(1,t) = \gamma(t)$



Remark: Intuitively, H parameterizes how we can push, compress, deform the image of β in S^1 to the image of γ in S^1.

Theorem: Two closed curves $\beta : S^1 \to S^1$ and $\gamma : S^1 \to S^1$ are homotopic if and only if $\deg(\beta) = \deg(\gamma)$

Section 2: Some complex analysis

Definition:

- A real polynomial is a fcn $f: \mathbb{R} \longrightarrow \mathbb{R}$ of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

where each a_i is a real number.

- When $a_n \neq 0$, we say the degree of f is

$$\deg(f) = n$$

- If $f(x_0) = 0$, then we say x_0 is a root of f.

Example: $f(x) = x^{77} - 17x^{66} + 42x - 26$

$\hookrightarrow$ $\deg(f) = 77$

$\hookrightarrow$ $f(1) = 0$

Remark:

- Not all real polynomials have real roots
- $f(x) = x^2 + 1$

  If $f(x) = 0$, then $0 = x^2 + 1 \Rightarrow x^2 = -1$.

  But the square of a real number is never negative

  $\Rightarrow$ f has <u>no</u> roots
- There just aren't enough real numbers.
- If $i = \sqrt{-1}$, then $f(i) = 0$ so f would have a root.
- Need to make sense of such numbers.

Definition: The complex numbers $\mathbb{C}$ is the set

$$\mathbb{C} = \{(x,y) \text{ in } \mathbb{R}^2\} = \{x+iy \mid (x,y) \text{ in } \mathbb{R}^2\}$$

↪ ie, a complex number is a formal sum $x+iy$ where x and y are real numbers.

↪ x is called the real part of $x+iy$

↪ y " " " imaginary " "

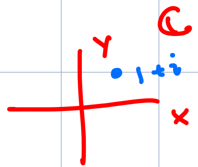
Notation: We will often write $z = x+iy$ to denote a complex number.

Remark: We can add complex numbers

$$(x_0 + iy_0) + (x_1 + iy_1) = (x_0 + x_1) + i(y_0 + y_1)$$

$$\Leftrightarrow \quad (18 + 7i) + (-25 - 2i) = -7 + 5i$$

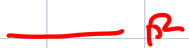
Remark: We can multiply complex numbers by requiring $i^2 = -1$



$$(x_0 + iy_0)\cdot(x_1 + iy_1)$$

$$= x_0x_1 + i(x_0y_1) + i(y_0x_1) + \overset{-1}{\cancel{i^2}}\, y_0y_1$$

$$= x_0x_1 - y_0y_1 + i(x_0y_1 + x_1y_0)$$



$$\Leftrightarrow \quad (2+i)\cdot(7-7i) = 14 - 14i + 7i - 7i^2 = 21 - 7i$$

Definition: The norm of a complex number $x + iy$ is

$$|x + iy| = \sqrt{x^2 + y^2}$$

Remark: If $|u+iv| \neq 0$, then we can divide $x+iy$ by $u+iv$

$$\frac{x+iy}{u+iv} \cdot \frac{u-iy}{u-iv} = \frac{x+iy}{u+iv} \cdot \frac{u-iy}{u-iv}$$

$$= \frac{(x+iy)\cdot(u-iv)}{u^2 \cancel{-iuv} + \cancel{iuv} - i^2 v^2}$$

$$= \frac{(x+iy)\cdot(u-iv)}{u^2+v^2}$$

$$= \frac{(x+iy)\cdot(u-iv)}{|u+iv|^2} \quad (*)$$

We can make sense of $(*)$ since we can just scale the real and imaginary parts of numerator by the denominator, which is a real number

Remark: Just as we can talk about fcns from $\mathbb{R}$ to $\mathbb{R}$, we can talk about fcns from $\mathbb{C}$ to $\mathbb{C}$.

Definition: A fcn $f : \mathbb{C} \to \mathbb{C}$ is an assignment of a complex number z to the complex number $f(z)$.

↳ e.g. $f(z) = z^2 - 17$

Definition:

- A <u>complex polynomial</u> is a fcn $f: \mathbb{C} \to \mathbb{C}$ of the form

$$f(x) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$$

where each a_i is a complex number.

- When $a_n \neq 0$, we say the <u>degree</u> of f is

$$\deg(f) = n$$

- If $f(z_0) = 0$, then we say z_0 is a root of f.

Remark:

- One way to define the fcn $e^x : \mathbb{R} \to \mathbb{R}$ is via Taylor's series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

- Terms of a Taylor's series gives sucessive approximations to the actual fcn. e^1

$\hookrightarrow$ $\sum_{n=0}^{0} (1)^n/n! = 1$

$\sum_{n=0}^{1} (1)^n/n! = 1 + 1 = 2$

$\sum_{n=0}^{2} (1)^n/n! = 1 + 1 + \frac{1}{2} = 2.5$

$\sum_{n=0}^{3} (1)^n/n! = 1 + 1 + \frac{1}{2} + \frac{1}{6} = 2.\overline{666}$

$\sum_{n=0}^{4} (1)^n/n! = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} = 2.708\overline{333}$

Remark: So a Taylor series is approximated by a seq. of polynomials. These Taylor series have to satisfy some "convergence" properties, ie, this infinite sum alway needs to converge to something finite.

↪ So some calculus is required to make this rigorous.

↪ The calculus also carries over to the complex case.

⇒ Use Taylor series w/ complex numbers.

Definition: The complex exponential fcn is the fcn $e^z : \mathbb{C} \to \mathbb{C}$ given by

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Lemma: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ for θ a real number.

Proof: We use the Taylor series for sin and cos and compute.

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}$$

$i \cdot i^{2\ell} = i(i^2)^{\ell} = i(-1)^{\ell}$

$$= \sum_{k=0}^{\infty} \frac{i^{2k}\theta^{2k}}{(2k)!} + \sum_{\ell=0}^{\infty} \frac{i^{2\ell+1}\theta^{2\ell+1}}{(2k+1)!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{k}\theta^{2k}}{(2k)!} + i\sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}\theta^{2\ell+1}}{(2k+1)!}$$

$$= \cos(\theta) + i\sin(\theta)$$

$\square$

Cor: $e^{i\pi} = -1$

Remark:

- When we identify $\mathbb{C}$ w/ $\mathbb{R}^2$ via $x+iy \longleftrightarrow (x,y)$, the norm of $x+iy$ agrees w/ the norm of (x,y)

  $\hookrightarrow$ $|(x,y)|$ = distance from (x,y) to the origin

- When we identify $\mathbb{C}$ w/ $\mathbb{R}^2$ the unit circle in $\mathbb{R}^2$ becomes

$$\{z \mid |z|=1\} = \{\cos(t) + i\cdot\sin(t) \mid 0 \le t \le 2\pi\}$$
$$= \{e^{it} \mid 0 \le t \le 2\pi\}$$

Section 3: Fundamental Theorem of Algebra

Theorem: Every complex polynomial w/ degree $\overset{= n}{>}$ 0 has $\overset{n}{\text{a}}$ root.

Proof:

1) Consider a polynomial

$$f(z) = \overset{1}{a_n} z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$$

Note $f(z) = 0$ if and only if $f(z)/a_n = 0$

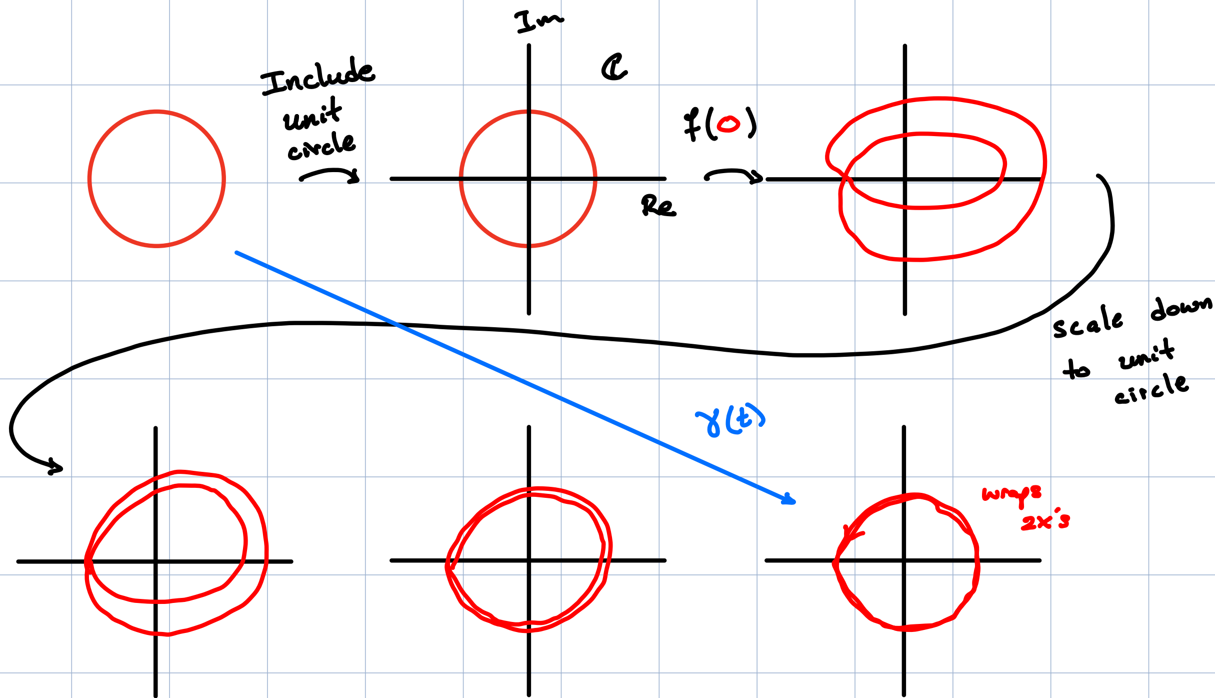
So it suffices to assume $a_n = 1$

2) Spse by way of contradiction that f has no roots.

3) Define $\gamma : \overset{\mathbb{C}}{\overset{\cup}{S^1}} \to \overset{\mathbb{C}}{\overset{\cup}{S^1}}$ via

$$\gamma(t) = \frac{f(\cos(t) + i\sin(t))}{|f(\cos(t) + i\sin(t))|} = \frac{f(e^{it})}{|f(e^{it})|}$$

Picture:



4) Define $H : [0,1] \times S^1 \to S^1$ via

$$H(s,t) = \frac{f(s \cdot e^{it})}{|f(s \cdot e^{it})|}$$

5) Notice that

$H(0,t) = f(0)/|f(0)| = \text{constant}$

$H(1,t) = \gamma(t)$

$\Rightarrow$ γ is homotopic to a constant curve

$\Rightarrow$ $\deg(\gamma) = 0$

6) Notice that $\quad z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$

$$s^n \cdot f(z/s) \quad \cancel{s^n}\left(\frac{z^n}{\cancel{s^n}} + a_{n-1} \frac{z^{n-1}\, s}{\cancel{s^{n}}} + \dots + a_1 \frac{z}{s} + a_0 \right)$$

$$= z^n + a_{n-1} \cdot z^{n-1} \cdot s + a_{n-2} z^{n-2} \cdot s^2 + \dots + a_1 z s^{n-1} + a_0 s^n$$

So when $s = 1$, $\quad s^n f(z/s) = f(z)$

So when $s = 0$, $\quad s^n f(z/s) = z^n$

7) Define $G : \overset{s}{[0,1]} \times \overset{t}{S^1} \longrightarrow S^1$ via

$$G(s,t) = \frac{s^n \cdot f(e^{it}/s)}{|s^n \cdot f(e^{it}/s)|}$$

8) $$G(0,t) = (e^{it})^n / |(e^{it})^n|$$

$$= (e^{int}) / |(e^{int})|$$

$$= \frac{\cos(nt) + i\sin(nt)}{\cancelto{1}{|\cos(nt) + i\sin(nt)|}}$$

$$= \frac{\cos(nt) + i\sin(nt)}{\cancelto{1}{\cos^2(nt) + \sin^2(nt)}}$$

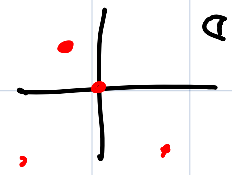
$$= (\cos(nt), \sin(nt)) = \gamma_n$$

9) $$G(1,t) = \gamma(t)$$

10) $$\Rightarrow \deg(\gamma) = n = \deg(f)$$

$$\Rightarrow 0 = \deg(\gamma) = n \neq 0$$, a contradiction. $\square$

Section 4: Complex algebraic varieties (sets)



Remark: Given a polynomial $f : \mathbb{C} \longrightarrow \mathbb{C}$, the set of zeros

$$\mathbb{V}(f) = \{ z \text{ in } \mathbb{C} \mid f(z) = 0 \}$$

is some finite set of points.

↪ Domain is 2-dim'l, but the constraint cuts down the dimension by 2.

Definition:

- Let $z_1, \dots, z_n$ be a set of variables.
- A monomial in z_i is a polynomial of the form

$$a \cdot z_i^m$$

where a is a complex number and m is a non-neg. integer.

- A polynomial in $z_1, \dots, z_n$ is a finite product and sum of monomials in the z_i.

↳ $f(z_1, z_2) = \boxed{z_1^2}\boxed{z_2^1} + \boxed{7 z_1^8} + \boxed{16 z_2^{32}}\boxed{z_1^1}$.

$f(x,y,z) = x^2 + y^2 + z^2$

$f(z_1, \dots, z_n) = z_2 + z_3$

- $\mathbb{C}[z_1, \dots, z_n]$ = set of polynomials in $z_1, \dots, z_n$

↳ Add, multiply polynomials ⟹ algebraic structure.

Remark: Notice that a polynomial in $z_1, \dots, z_n$ gives a fcn

$$f : \underset{\mathbb{C}\times\dots\times\mathbb{C}}{\mathbb{C}^n} \longrightarrow \mathbb{C}$$

via evaluating f at $(z_1, \dots, z_n)$ in $\mathbb{C}^n$.

Remark: The zero locus of f is the subset of $\mathbb{C}^n$ given by

$$\mathbb{V}(f) = \{(z_1, \dots, z_n) \mid f(z_1, \dots, z_n) = 0\}$$

↳ w/ probability 1, $\mathbb{V}(f)$ for a random f will be $(2n-2)$-dim'l and, in fact, a $(2n-2)$-manifold

- locally looks like $\mathbb{R}^{2n-2} \cong \mathbb{C}^{n-1}$

↳ surface is a 2-manifold

Example: Let $h(z)$ be a degree n polynomial.

Then w/ probability 1, $f(z_1, z_2) = z_2^2 - h(z_1)$, $\mathbb{V}(f)$ will be a surface (w/ some open ends) w/ g donut holes, where

- $g = \frac{n-1}{2}$, $n = \text{odd}$
- $g = \frac{n-2}{2}$, $n = \text{even}$

Definition: The ideal generated by f is the subset

$$\mathbb{I}(f) \subseteq \mathbb{C}[z_1, \ldots, z_n]$$

given by

$$\mathbb{I}(f) = \{ g \text{ in } \mathbb{C}[z_1, \ldots, z_n] \mid g = f \cdot h \text{ for some poly. } h \}$$

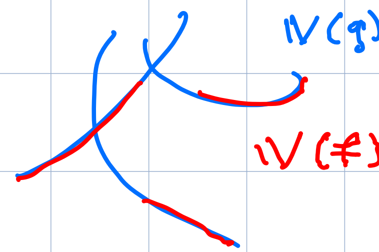
Remark: • If g is in $\mathbb{I}(f)$, then $\mathbb{I}(g) \subseteq \mathbb{I}(f)$

↳ p is in $\mathbb{I}(g) \Rightarrow p = g \cdot h_1$

g is in $\mathbb{I}(f) \Rightarrow g = f \cdot h_2$

$\Rightarrow p = f \cdot h_1 \cdot h_2$

$\Rightarrow p$ is in $\mathbb{I}(f)$



• If $\mathbb{I}(g) \subseteq \mathbb{I}(f)$ then $\mathbb{V}(g) \supseteq \mathbb{V}(f)$.

↳ $\mathbb{I}(g) \subseteq \mathbb{I}(f) \Rightarrow g \in \mathbb{I}(f)$

So if $f(z_1, \dots, z_n) = 0$, then

$$g(z_1, \dots, z_n) = f(z_1, \dots, z_n) \cdot h(z_1, \dots, z_n) = 0$$

(Google/wiki: Hilbert's Nullstellensatz) ⟶ Hard stuff...

Theorem: If $\mathbb{V}(g) \supseteq \mathbb{V}(f)$, then $\mathbb{I}(g^k) \subseteq \mathbb{I}(f)$ for some k.

Remark:

$$\begin{array}{ccc}\text{Topology} & \xleftrightarrow[\text{each others}]{\text{Determine}} & \text{Algebra}\\ \mathbb{V}(f) & & \mathbb{I}(f)\end{array}$$