Lecture # 6

Outline:
1) Review from last time
2) Maps of S^1 to S^1
3) Lifting closed curves to $\mathbb{R}$
4) The degree of a map of S^1 to S^1
5) Homotopy classes of curves
6) Homotopy invarience of degree

Section 1: Review

Definition:

- A closed curve in a surface Σ is a ② continuous ① map $\gamma : S^1 = \text{circle} \longrightarrow \Sigma$.

  ① We send every pt in S^1 to a point in Σ.

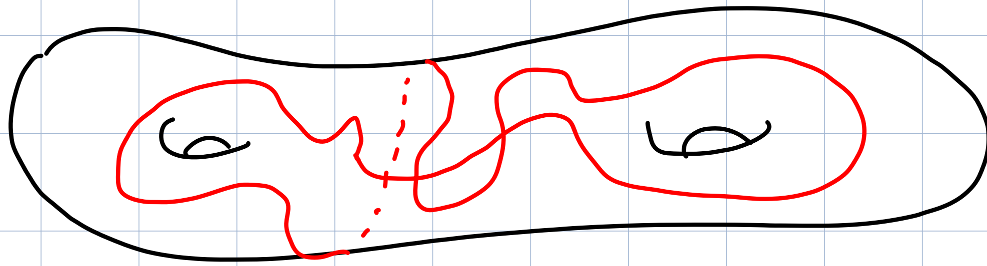
  ② "Continuous" = we send points infintesimally close together in S^1 to points infintesimally close together in Σ.

  ↪ We map S^1 into Σ w/ out ripping or cutting it

Example:

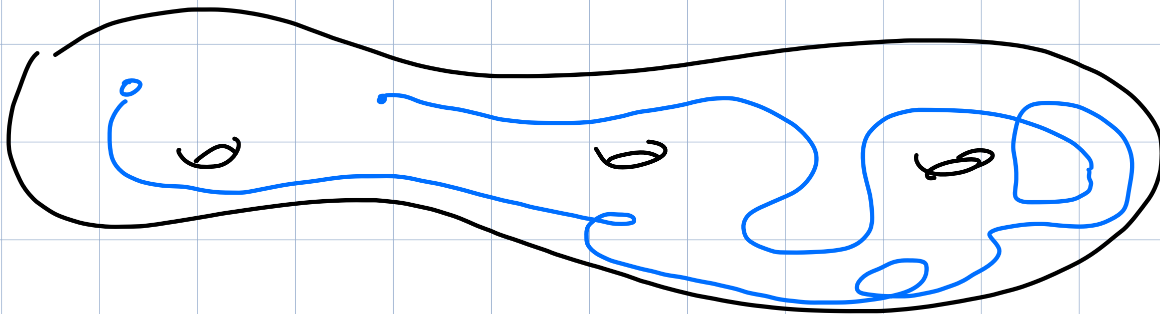
1)

closed curve



2)

not a continuous curve.



3)

Theorem: Every compact orientable surface is homeomorphic to a connect sum $T^2 \# \cdots \# T^2 \# S^2$ for some # of T^2's.

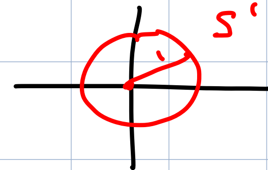
Up Next:

1) Brouwer's Fixed Point Theorem
2) Fundamental Theorem of Algebra

Section 2: Maps of S^1 to S^1

Definition: • $S^1 = \{(x,y) \text{ in } \mathbb{R}^2 \mid x^2+y^2 = 1\}$

= unit circle in the plane



Definition: A closed curve in S^1 = circle is a continuous ① map $\gamma : S^1 \longrightarrow S^1$.

① We send every pt in S^1 to a point in Σ.

② "Continuous" = we send points infintesimally close together in S^1 to points infintesimally close together in S^1.

↪ We map S^1 into S^1 w/ out ripping or cutting it

Remark: Equivalently, a map $\gamma : S^1 \longrightarrow S^1$ may be viewed as a continuous map

$$\gamma : [0, 2\pi] \longrightarrow S^1$$

w/

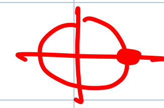
$$\gamma(0) = \gamma(2\pi)$$

↳ i.e., a map of a circle is just a map of an interval that connects up at its end points.

↳ Intuitively, $\gamma : S^1 \to S^1$ gives a way of wrapping/laying a string onto a circle such that you can tie together its ends.

Example: 1) $\gamma_n : [0, 2\pi] \longrightarrow S^1 \subseteq \mathbb{R}^2$ given by $(n = \text{integer})$

$$\gamma_n(t) = (\cos(nt), \sin(nt))$$

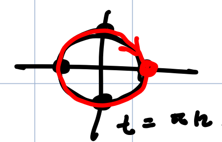


↪ What is γ_0?

$=$ constant $t \longmapsto (\cos(0 \cdot t), \sin(0 \cdot t)) = (1, 0)$

↪ What is γ_{-1}?

clockwise wrapping curve (neg)



↪ What is γ_n?

curve that wraps n-times around S^1 in the sign(n)-direction.

$n = 3$

$t = 0 \quad (1,0)$

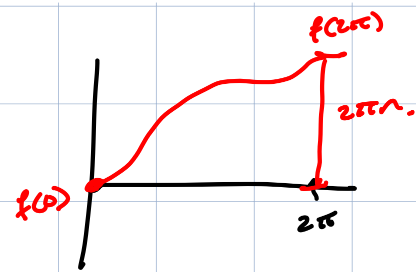
$t = \frac{2\pi}{3} \quad (1,0)$

$t = \frac{2 \cdot 2\pi}{3} \quad (1,0)$

$t = 2\pi \quad (1,0).$

2) Let $f : \mathbb{R} \to \mathbb{R}$ be any function st

$$f(0) = f(2\pi) + 2\pi \cdot n$$



for some n an integer.

$\gamma_f : [0, 2\pi] \longrightarrow S^1$ given by

$$\gamma_f(t) = (\cos(f(t)), \sin(f(t)))$$

↪ Note $\gamma_f(0) = (\cos(f(0)), \sin(f(0)))$

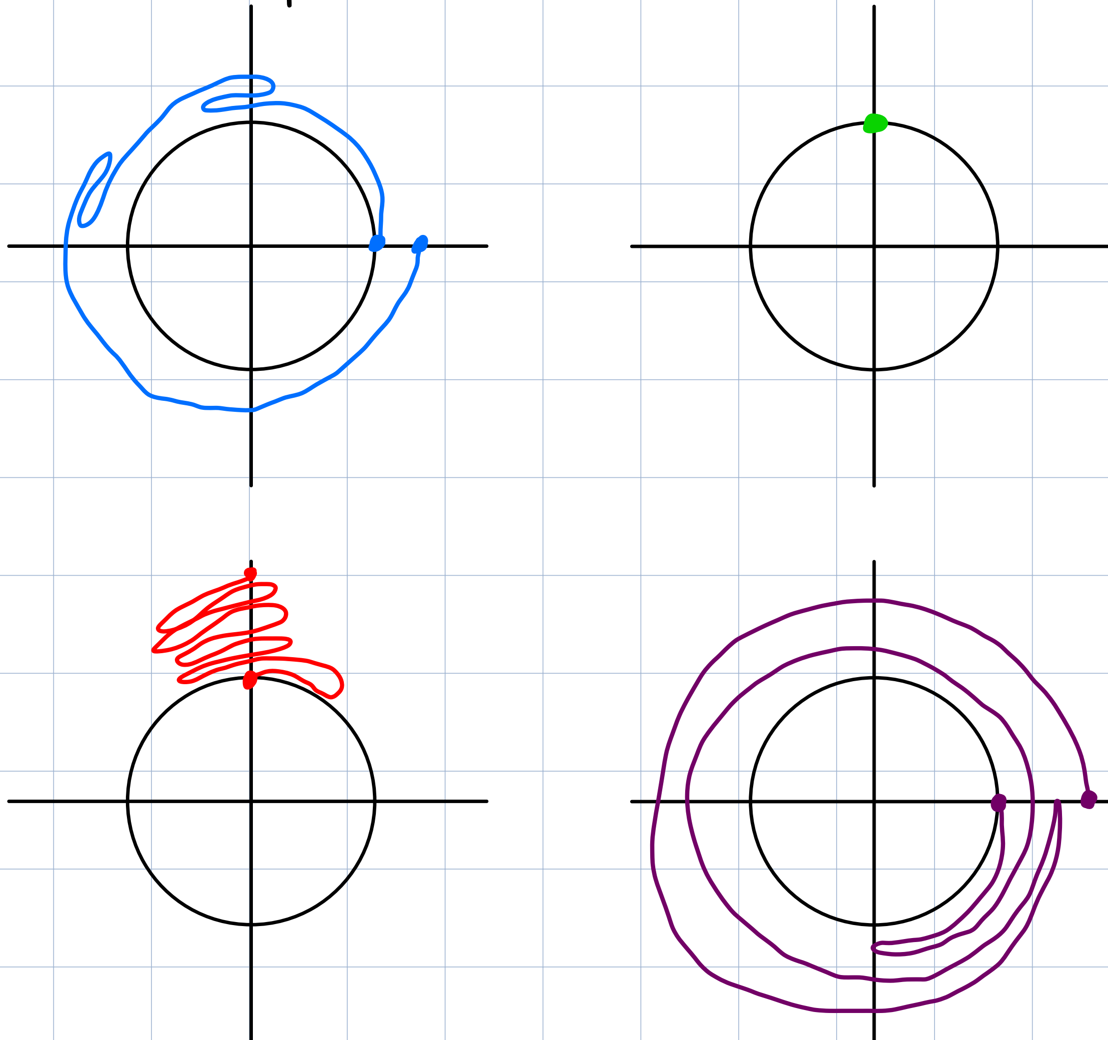
$$= (\cos(f(2\pi) + 2\pi n), \sin(f(2\pi) + 2\pi n))$$

$$= (\cos(f(2\pi)), \sin(f(2\pi)))$$

$$= \gamma_f(2\pi)$$

$\Rightarrow$ γ_f is a <u>closed</u> curve.

3) Crazier examples



Section 3: Lifting closed curves to $\mathbb{R}$

Lemma: (Curve Lifting) Given a closed curve $\gamma : S^1 \to S^1$, there exists a function $f : [0, 2\pi] \longrightarrow \mathbb{R}$ st

1) $f(0) = f(2\pi) + 2\pi \cdot n$ for some integer n
2) $\gamma(t) = (\cos(f(t)), \sin(f(t)))$

$\hookrightarrow$ f is called a <u>lift</u> of γ to $\mathbb{R}$.

Remark:

1) f need not be unique.
2) If f is a lift of γ, then $f + 2\pi \cdot k$ is a lift of γ for every integer k.
3) These are the <u>only</u> other lifts of γ.

Proof :

1) Idea : "Unwind" the curve by noting the angle.

2) Notice that $\gamma(t) = (\cos(f(t)), \sin(f(t)))$

if and only if $f(t) =$ angle between $\gamma(t)$ and $(1,0)$

3) $f(t) =$ Accumulated angle of rotation of $\gamma(t)$

measured w/ respect to $(1,0)$

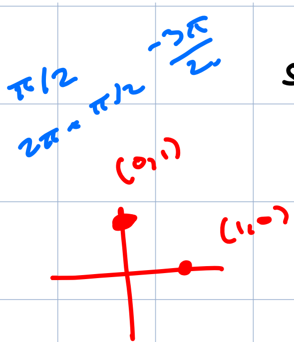
↪ rotate clockwise angle decreases

↪ rotate counter clockwise angle increases

↪ go around 5 times angle increases by

$10\pi = 5 \cdot (2\pi)$.

4) By construction and (2), $\gamma(t) = (\cos(f(t)), \sin(f(t)))$



5) The only part in defining f where we have any choice is picking $f(0)$, that is, the starting angle from $(1,0)$; any two choices differ by multiples of 2π.

So adding multiples of 2π to f gives all lifts

6) Notice that $\gamma(0) = \gamma(2\pi)$.

So total accumulated angle must be a multiple of 2π plus the starting angle.

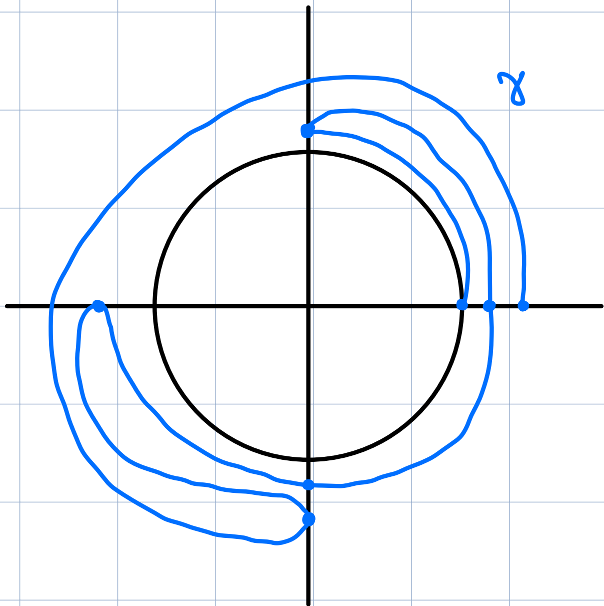
So $f(0) = f(2\pi) + 2\pi \cdot n$ for some n.

Example:

1) A lift of γ_n to $\mathbb{R}$ is $f(t) = n \cdot t$.

2) A lift of γ_f to $\mathbb{R}$ is f.

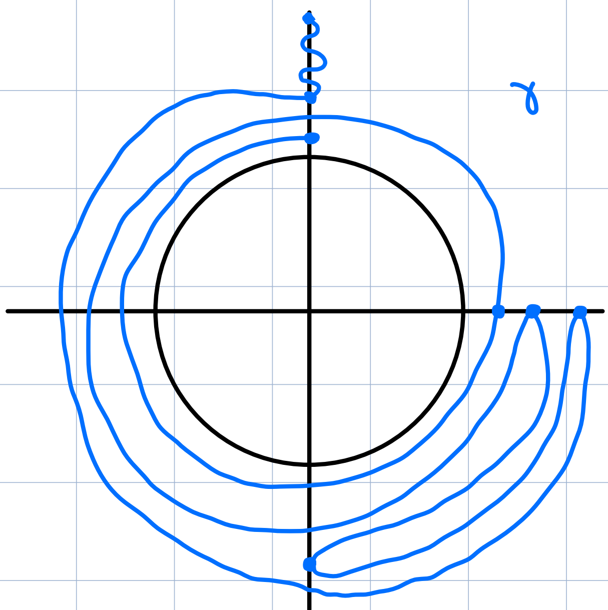
3)

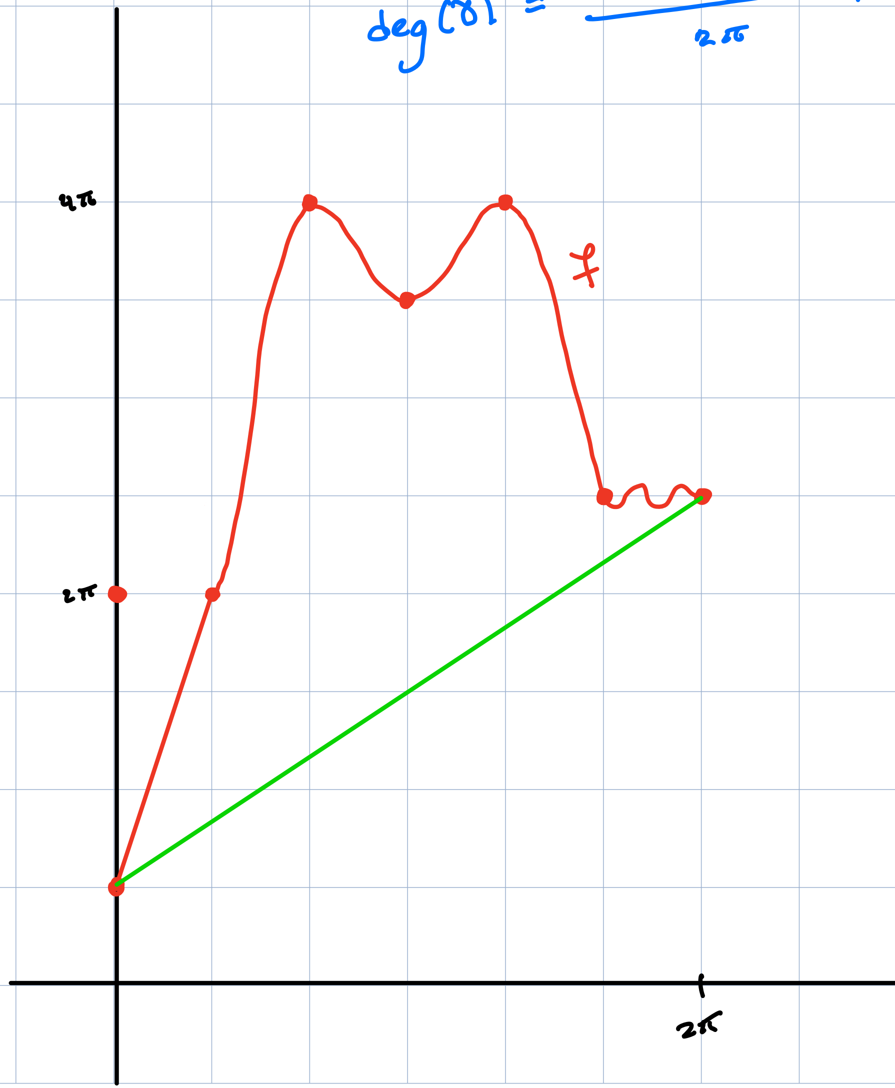


$$\deg = \frac{-2\pi - 0}{2\pi} = -1$$

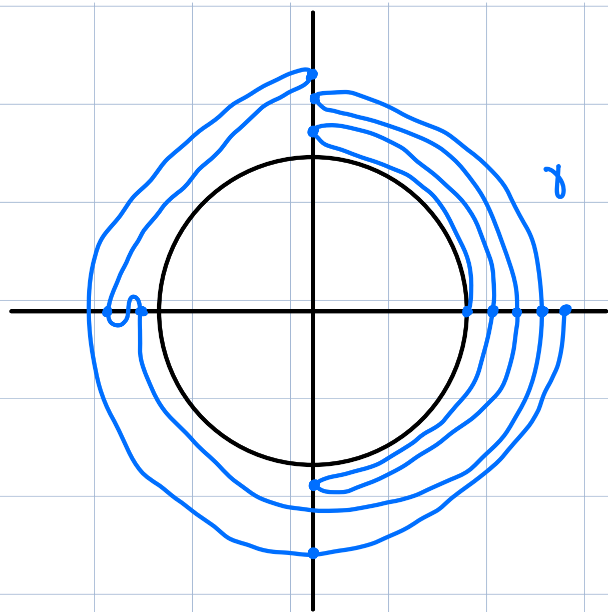
4)

$$\deg(\gamma) = \frac{f(2\pi) - f(0)}{2\pi} = ($$

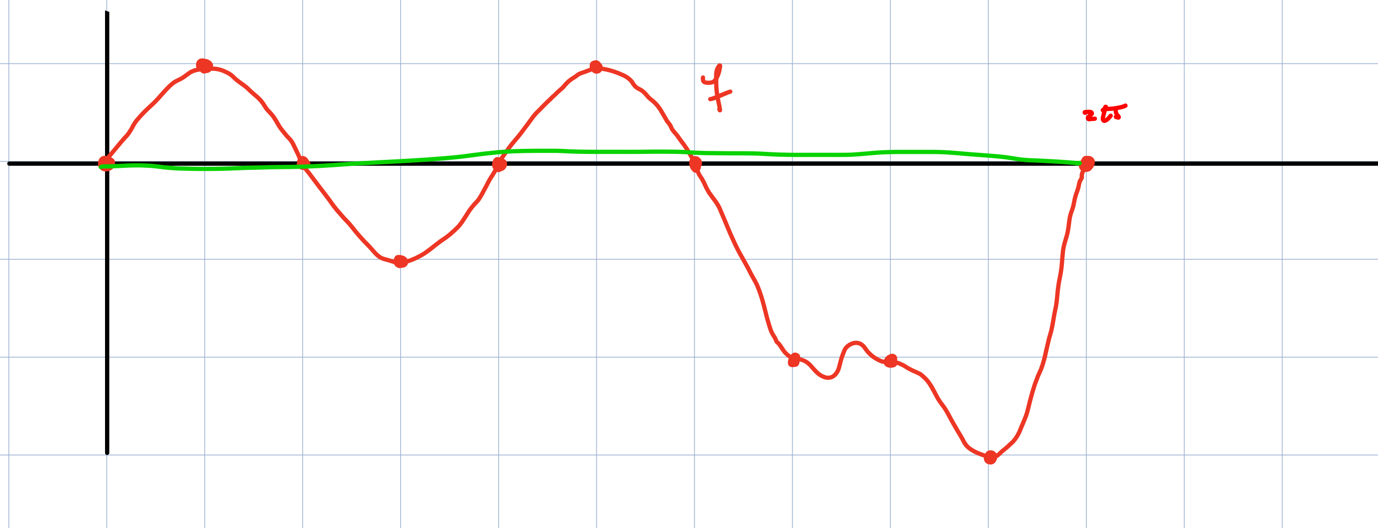




5)



$\deg(\gamma) = 0$



Section 4: The degree of a map of S^1 to S^1

Definition: The degree of a closed curve $\gamma : S^1 \to S^1$ is

$$\deg(\gamma) = \left(f(2\pi) - f(0)\right)/2\pi \quad \hookrightarrow \text{integer}$$

where f is any lift of γ to $\mathbb{R}$.

Remark: 1) $\deg(\gamma)$ is independent of the choice of lift f.

$\hookrightarrow$ If g is another lift of γ, then

$$f = g + 2\pi \cdot k$$

So $$\frac{f(2\pi) - f(0)}{2\pi} = \frac{g(2\pi) - g(0)}{2\pi}.$$

Remark:

1) Intuitively, $\deg(\gamma)$ = signed # of times γ completely wraps around the circle

↪ signed: wraps clockwise = negative wrap

wraps counter clockwise = positive wrap

Example:

1) $\deg(\gamma_n) = n$

2) See above examples.

$f(t) = n \cdot t$

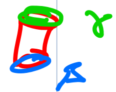
$$\deg(\gamma_n) = \frac{f(2\pi) - f(0)}{2\pi}$$

$$= \frac{2\pi n - 0}{2\pi}$$

$$= n.$$

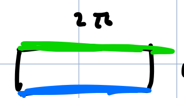
Section 5: Homotopy classes of curves

Definition: Two closed curves $\beta : S^1 \to S^1$ and $\gamma : S^1 \to S^1$ are homotopic if there is a continuous map $H : [0,1] \times S^1 \to S^1$ satisfying



1) $H(0,t) = \beta(t)$

2) $H(1,t) = \gamma(t)$



Remark: Equivalently, $H : [0,1] \times [0,2\pi] \to S^1$ w/

1) $H(0,t) = \beta(t)$

2) $H(1,t) = \gamma(t)$

3) $H(s,0) = H(s,2\pi)$

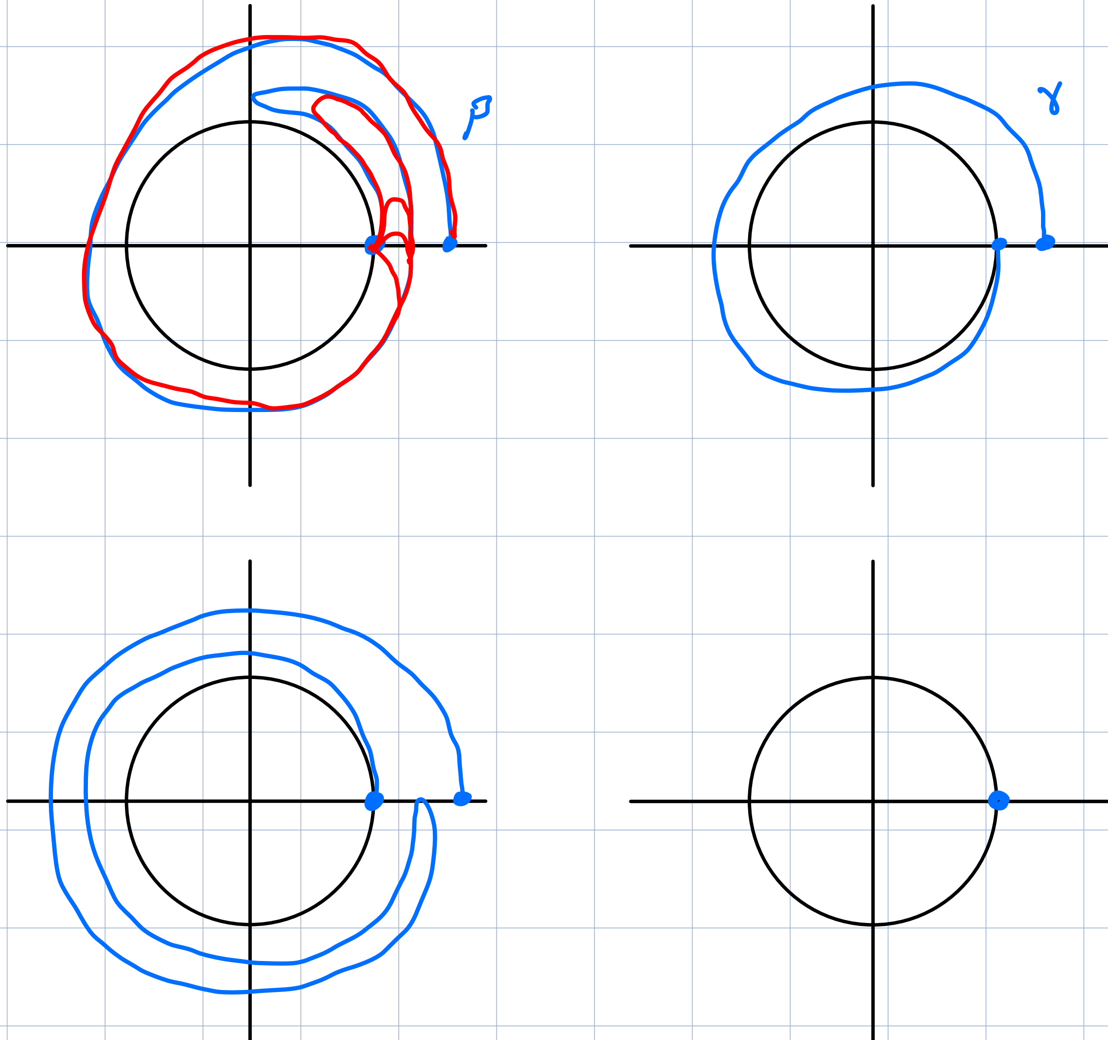
Remark:

1) For each s_0 in $[0,1]$, $H(s_0, t)$ defines a closed curve in S^1.

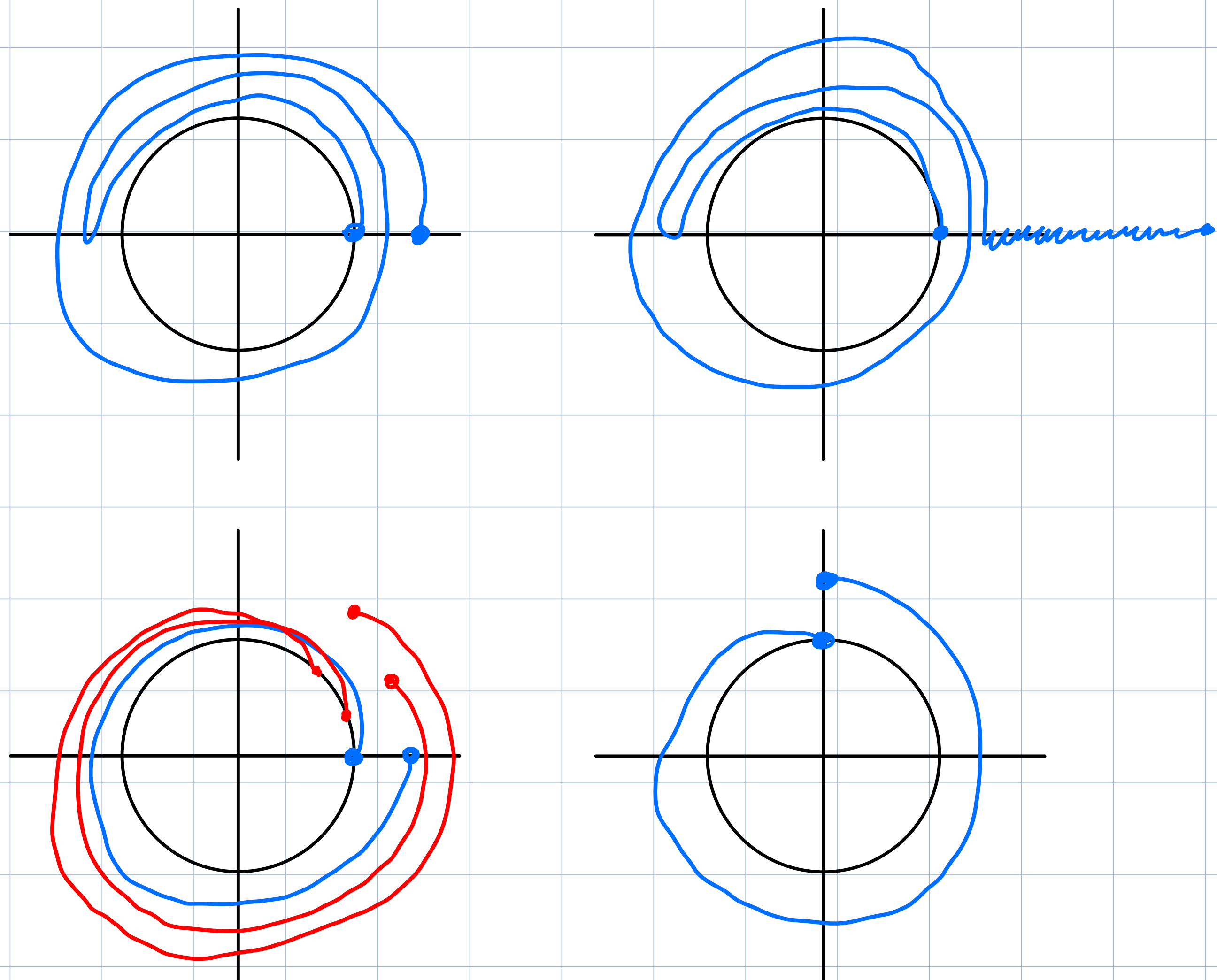
2) H parameterizes a family of curves that interpolates between β and γ.

3) Intuitively, H parameterizes how we can push, compress, deform the image of β in S^1 to the image of γ in S^1.

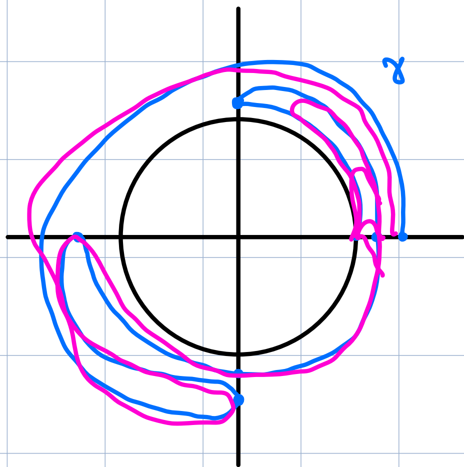
Example:

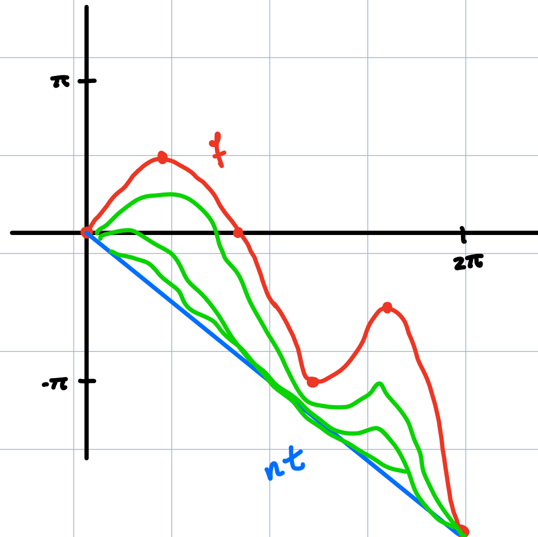


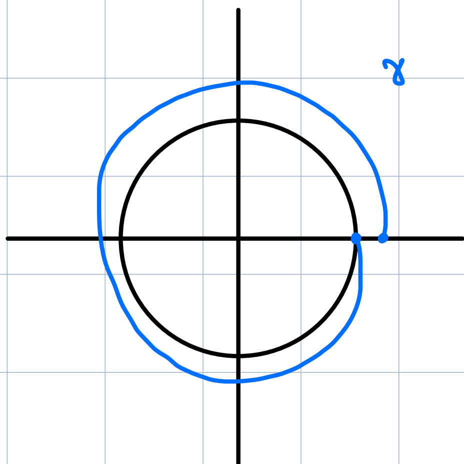
Example :

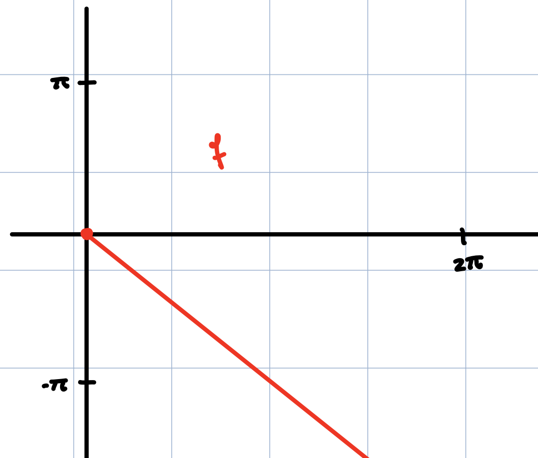


Example:









Section 6: Homotopy invarience of degree

Theorem: Two closed curves $\beta : S^1 \to S^1$ and $\gamma : S^1 \to S^1$ are homotopic if and only if $\deg(\beta) = \deg(\gamma)$

Remark: 1) Notice that rotating the image of a curve in S^1 defines a continuous family of curves and thus a homotopy

$\Rightarrow$ Any curve is homotopic to a curve w/ $\gamma(0) = (1,0)$

2) If we rotate γ by θ, then the lift changes by $f + \theta$

$(f(2\pi) + \theta - f(0) - \theta)/2\pi$

$= \frac{f(2\pi) - f(0)}{2\pi}$

$\Rightarrow \deg(\gamma) =$ deg of rotated γ.

Claim 1: If $\deg(\beta) = \deg(\gamma)$, then β is homotopic to γ.

Lemma: Spse $\gamma(0) = (1,0)$. If $\deg(\gamma) = n$, then γ is homotopic to γ_n.

Proof:

1) Let f be a lift of γ w/ $f(0) = 0$.

↪ we can do this since $\gamma(0) = (1,0)$.

2) Define $\tilde{H} : [0,1] \times [0, 2\pi] \longrightarrow \mathbb{R}$ by

$$\tilde{H}(s,t) = (1-s) \cdot f(t) + s \cdot (n \cdot t)$$

3) Define $H : [0,1] \times [0, 2\pi] \longrightarrow S^1$ by

$$H(s,t) = \left(\cos(H(s,t)), \sin(H(s,t)) \right)$$

4) We claim that H is a homotopy from γ to γ_n.

5) Check H glues up at ends

i) $\tilde{H}(s,0) = (1-s)\cdot f(0) = 0$

ii) $\tilde{H}(s,2\pi) = (1-s)\cdot f(2\pi n) + s\cdot 2\pi n$

$$= (1\cancel{-s})\cdot 2\pi \deg(\overset{=n}{\gamma}) + \cancel{s\cdot 2\pi n}$$

$$= 2\pi n$$

iii) $\Rightarrow H(s,0) = (\cos(0), \sin(0))$

$$= (\cos(2\pi), \sin(2\pi))$$

$$= H(s,2\pi)$$

6) Check H is a homotopy from γ to γ_n

7) $\tilde{H}(0,t) = f(t)$

$\Rightarrow H(0,t) = (\cos(f(t)), \sin(f(t))) = \gamma(t)$

8) $\tilde{H}(1,t) = n \cdot t$

$\Rightarrow H(1,t) = (\cos(nt), \sin(nt)) = \gamma_n(t)$

of Claim 1

Proof:

1) Spse $\deg(\gamma) = n = \deg(\beta)$

2) $n = \deg(\gamma) = \deg(\text{rotated } \gamma \text{ w/ starting pt } (1,0))$

By lemma and fact that rotation is a homotopy,

$\gamma \simeq \text{rotated } \gamma \simeq \gamma_n$ homotopic

3) Similarly, $\beta \simeq \text{rotated } \beta \simeq \gamma_n$

4) Chain together these homotopies to get $\beta \simeq \gamma$.

Claim 2: If β is homotopic to γ, then $\deg(\gamma) = \deg(\beta)$

Theorem: (Homotopy Lifting) Let β and γ be closed curves in S^1. Given a homotopy $H : [0,1] \times [0, 2\pi] \longrightarrow S^1$ w/ $H(0,t) = \beta(t)$ and $H(1,t) = \gamma(t)$.

There exists a continuous map

$$\tilde{H} : [0,1] \times [0, 2\pi] \longrightarrow \mathbb{R}$$

that satisfies

1) $H(s,t) = (\cos(\tilde{H}(s,t)), \sin(\tilde{H}(s,t)))$

2) For all s, $\tilde{H}(s, 2\pi) - \tilde{H}(s, 0) = 2\pi n$

3) $\tilde{H}$ is unique up to adding a multiple of 2π.

Remark:

1) Prove the theorem by "unwrapping" in families.

$\tilde{H}(s,t)$ = lift of the curve $H(s,t)$ w/ s fixed.

↳ This essentially will imply (1), (3) in theorem

2) Since H parameterizes some family of curves, as we vary s, the accumulated amount of rotation can't jump (it is continuous).

3) For any lift, $\tilde{f}(2\pi) - \tilde{f}(0)$ is a multiple of 2π.

So $\tilde{H}(s, 2\pi) - \tilde{H}(s, 0)$ is a multiple of 2π for each s.

But it can't jump as s varies => must be constant.

Proof:

1) Let H be homotopy from β to γ.

2) Let $\tilde{H}$ be a lift of H

3) $\tilde{H}(0,t)$ is a lift of β $\qquad 0 \times [0,2\pi] \to \mathbb{R}$

$$\Leftrightarrow \left(\cos(\tilde{H}(0,t)), \sin(\tilde{H}(0,t))\right) = H(0,t) = \beta(t)$$

4) $\tilde{H}(1,t)$ is a lift of γ

5) So
$$\begin{aligned} \deg(\beta) &= (\tilde{H}(0,2\pi) - \tilde{H}(0,0))/2\pi \\ &= (\tilde{H}(1,2\pi) - \tilde{H}(1,0))/2\pi \\ &= \deg(\gamma). \end{aligned}$$