Lecture # 5

Outline:

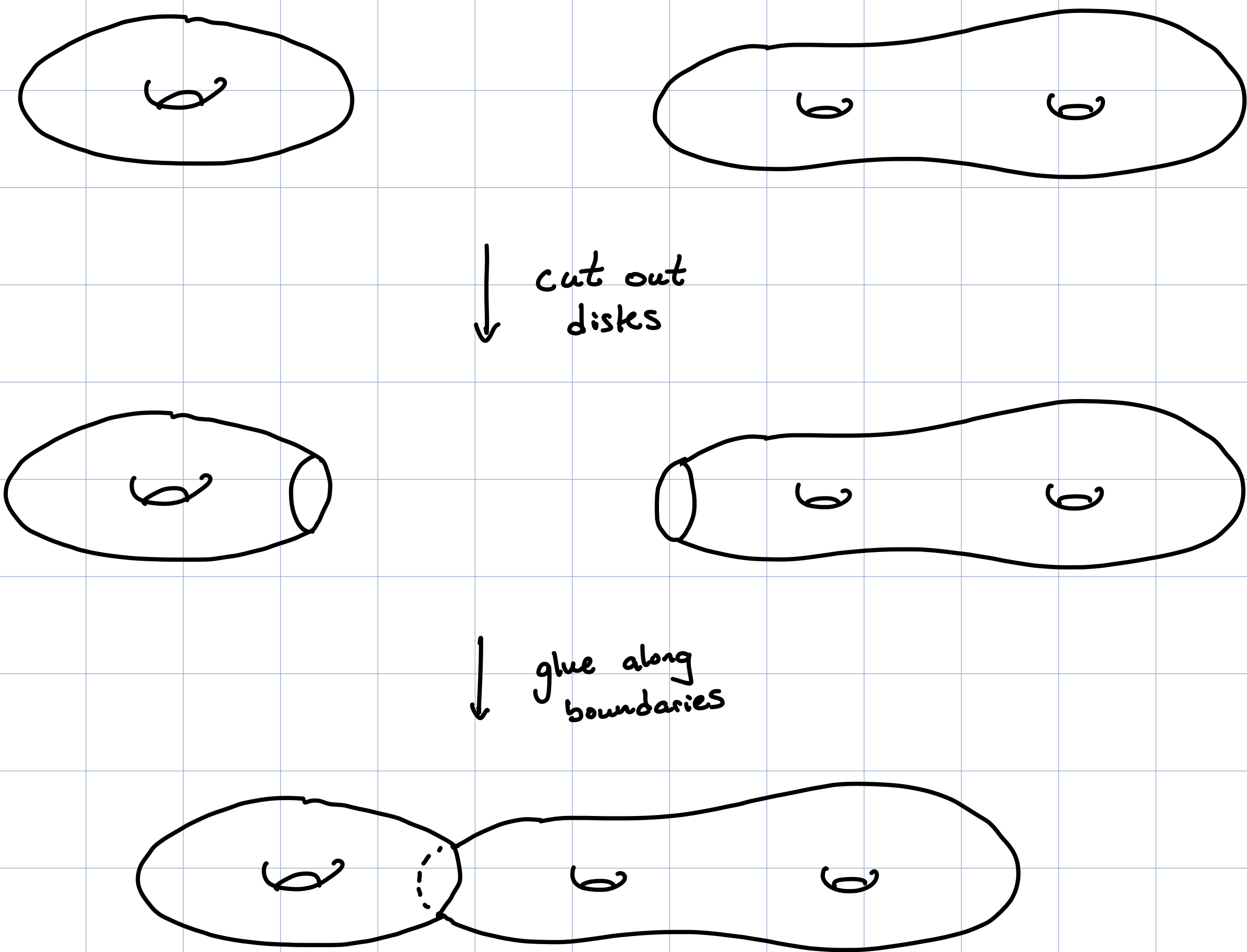
1) Connected Sums
2) Curves in Surfaces and Orientability
3) Preliminaries on Graphs
4) 2-dimension Poincare Conjecture
5) Classification of Surfaces

Section 1: Review

Definition: Given two surfaces X and Y, the connect sum of X and Y, denoted $X \# Y$, is obtained via

1) Remove an open disk from both X and Y to create two surfaces w/ "boundaries"
2) Glue the resulting boundaries together to create the new surface $X \# Y$.

Picture :



Example:

1) $T^2 \# T^2 =$ genus 2 surface

2) $S^2 \# S^2 = S^2$

3) $S^2 \# T^2 = T^2$

4) $\left.T^2 \# \dots \# T^2\right\}$ g-times $=$ genus g surface

Proposition: $\chi(X \# Y) = \chi(X) + \chi(Y) - 2$

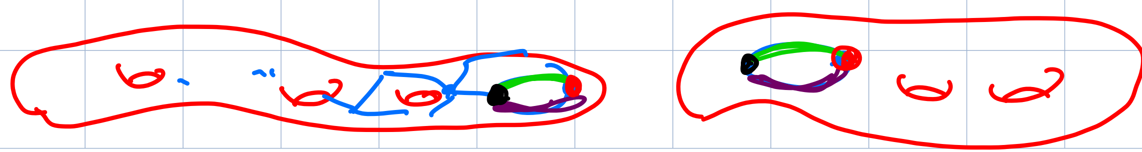
Proof:

1) Recall, we can compute the Euler characteristic of a surface by using <u>any</u> polygonal cpx associated to it.

2) Pick poly cpxes for X and Y that both have at least one face that is a 2-polygon w/ unique edges and vertices.

3) Removing said 2-polygons gives removal of disks from X and Y.

4) To glue, we glue together the boundaries of these removed 2-polygons.



5) This gluing gives poly cpx for $X \# Y$ w/

- Vertices $(X \# Y) = V(X) + V(Y) - 2$
- Edges $= E(X) + E(Y) - 2$
- Faces $= F(X) + F(Y) - 2$

6) $$\chi(X \# Y) = V(X) + V(Y) - 2 - (E(X) + E(Y) - 2) + F(X) + F(Y) - 2$$

$$= \chi(X) + \chi(Y) - 2 \qquad \square$$

Section 2: Curves in Surfaces and Orientability

Definition:

- A closed curve in a surface Σ is a ② continuous ① map $\gamma : S^1 = \text{circle} \longrightarrow \Sigma$.

  ① We send every pt in S^1 to a point in Σ.

  ② "Continuous" = we send points infintesimally close together in S^1 to points infintesimally close together in Σ.

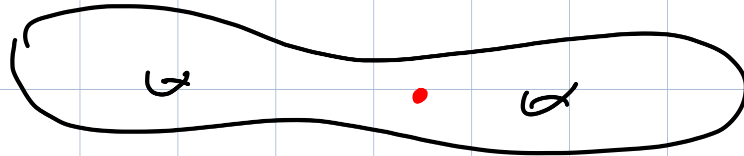
  ↳ We map S^1 into Σ w/out ripping or cutting it

- A curve is simple if the image of the curve in Σ does not cross/meet itself and the circle can be "pushed"/deformed to look like a seq. of edges

Examples:

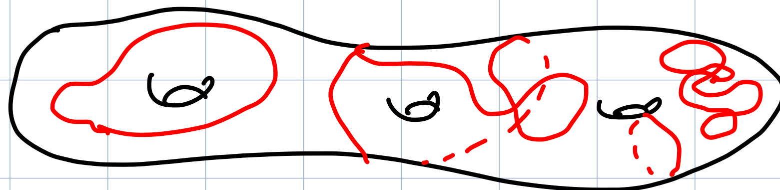
1) Constant curve

curve



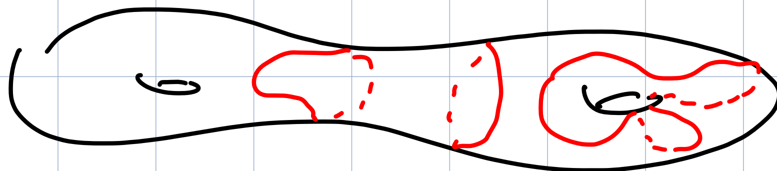
2) Crossing curve

curves.

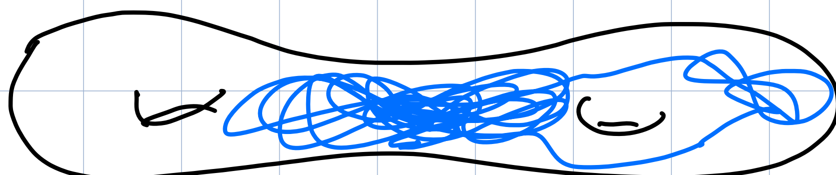


3) Simple closed curves

simple closed curves



4) Crazy curves

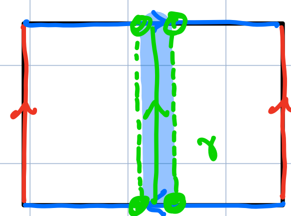


Definition:

- A simple closed curve is <u>1-sided</u> if a small thickening of the curve in Σ is a Möbius band.
- A simple closed curve is <u>2-sided</u> if a small thickening of the curve in Σ is a cylinder

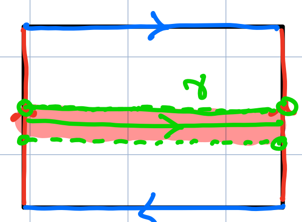
Examples:

1)



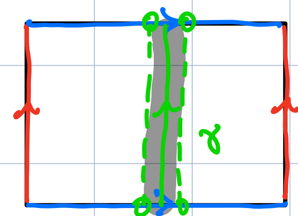
1-sided curve

2)



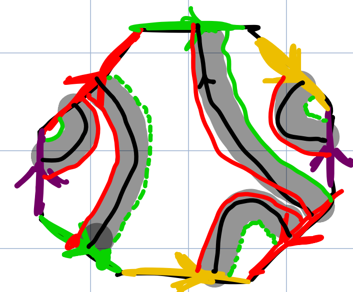
2-sided curve

3)



2-sided curve.

4)



2-sided curve.

Remark: If there are 1-sided curves on Σ, then we don't know what is up/down or in/out. We have no reference outward direction.

Definition: A surface is orientable if it has no 1-sided curves.

Example:

1) Connect sums of tori = orientable
2) Klein bottle is non-orientable

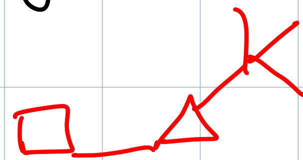
Definition: A surface is <u>compact</u> if it admits a polygonal complex structure w/ a finite # of vertices, edges, and faces.

Theorem: Every compact orientable surface is homeomorphic to a connect sum $T^2 \# \cdots \# T^2 \# S^2$ for some # of T^2's.

Section 3: Preliminaries on Graphs

Definition:

- A graph is a polygonal complex composed of edges.
- A graph is a tree if every pair of vertices is connected via a unique sequence of edges.
  - ↪ A tree is a graph w/ <u>no</u> loops

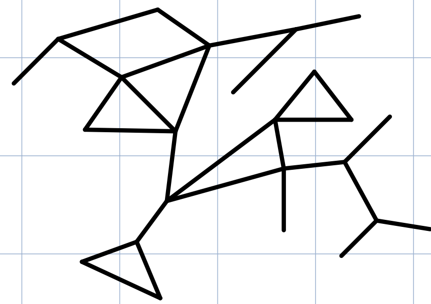


Proposition: Let Γ = connected graph. There exists a subcollection of edges of Γ that form a tree T that touches every vertex in Γ. T is called a <u>spanning tree</u> for Γ

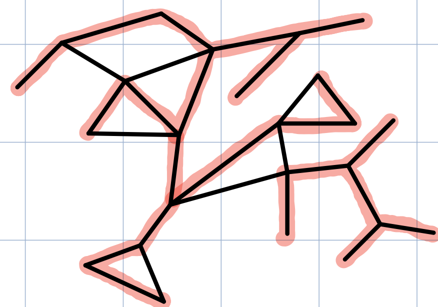
Remark: A graph can have multiple spanning trees.

Example:

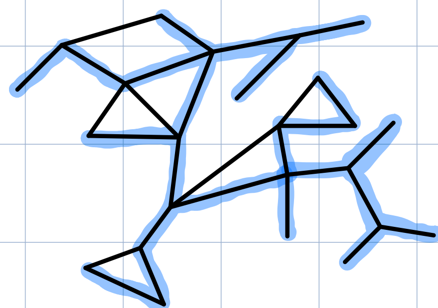
Graph



Spanning Tree



Another spanning tree



Proof:

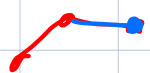
1) Buildup Γ one edge at a time.

$$\Gamma_0 \xrightarrow[\text{edge}]{\text{Add}} \Gamma_1 \xrightarrow[\text{edge}]{\text{Add}} \Gamma_2 \xrightarrow[\text{edge}]{\text{Add}} \cdots \xrightarrow[\text{edge}]{\text{Add}} \Gamma_n = \Gamma$$

2) We sequentially build spanning trees T_i for Γ_i.

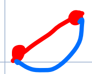
3) $\Gamma_0 = \text{edge}$, $T_0 = \Gamma_0$

4) $\Gamma_0 \longrightarrow \Gamma_1$: either



a) A new vertex is added to Γ_0 to create Γ_1

↪ create new "step"



b) No new vertex is " " " " " "

↪ create a loop

5) If a) => Set $T_1 = T_0 \cup$ new edge

If b) => Set $T_1 = T_0$

6) $\tilde{\Gamma}_i \longrightarrow \tilde{\Gamma}_{i+1}$: either

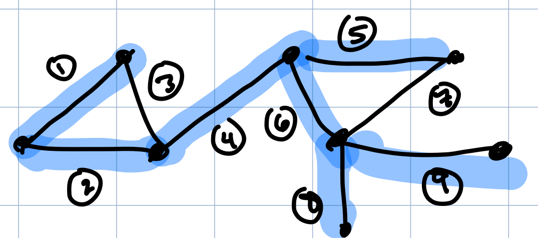
a) A new vertex is added to $\tilde{\Gamma}_i$ to create $\tilde{\Gamma}_{i+1}$

b) No new vertex is " " " " " "

7) If a) $\Rightarrow$ Set $T_{i+1} = T_i \cup$ new edge

If b) $\Rightarrow$ Set $T_{i+1} = T_i$

8) By construction, each T_i is a tree and touches every vertex of $\tilde{\Gamma}_i$. So repeated we obtain the result. $\square$



Lemma: Let Γ = connected graph. We have

$$\chi(\Gamma) \leq 1$$

w/ equality iff Γ is a tree.

Proof: 1) If Γ = tree, then we claim that $\chi(\Gamma) = 1$

i) Build up Γ sequentially: $\Gamma_1, \Gamma_2, \Gamma_3, \ldots, \Gamma_n = \Gamma$.

ii) Since Γ is a tree each time we add an edge, we also add another vertex

↳ if not, we would conn. two vertices via at least 2 different seqs of edges

iii) So Γ_1 = edge $\Rightarrow \chi(\Gamma_1) = 2 - 1 = 1$

$$\Gamma_2 = V(\Gamma_1) - E(\Gamma_1) + 1 - 1 = 1$$

iv) Repeatedly, $\chi(\Gamma_{i+1}) = V(\Gamma_i) - E(\Gamma_i) + 1 - 1 = 1$

v) $\Rightarrow \chi(\Gamma = \text{tree}) = 1$

2) Spse Γ is not necessarily a tree.

Let T = spanning tree for Γ.

$$\begin{aligned}
\chi(\Gamma) &= V(\Gamma) - E(\Gamma) \\
&= V(T) - E(T) - E(\text{not in } T) \\
&= \chi(T) - E(\text{not in } T) \\
&\leq 1
\end{aligned}$$

3) Note, if $E(\text{not in } T) = 0$, then $\Gamma = T$.

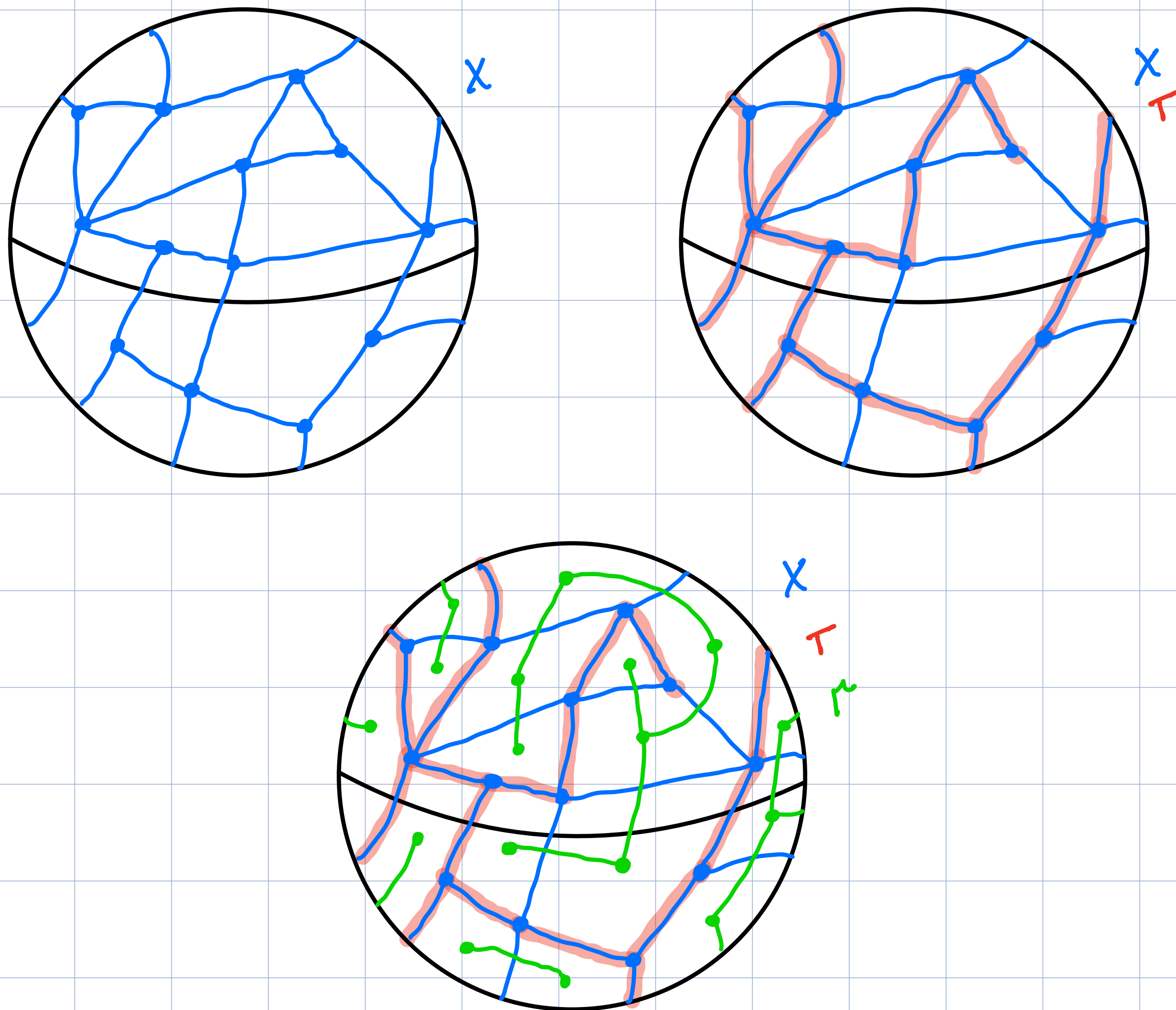
$\Rightarrow \chi(\Gamma) = 1$ if and only if Γ = tree.

Theorem: Let Σ = compact surface. Then $\chi(\Sigma) \leq 2$ and $\chi(\Sigma) = 2$ if and only if Σ is homeomorphic to S^2.

Proof:

1) Fix a polygonal cpx that gives Σ.
2) Let T = spanning tree for the graph that is made up of the edges of X.
3) Define a graph Γ (that can be drawn on X) via:
   a) place a vertex in the center of each face of X.
   b) Connect two vertices via an edge for each edge in X that is not in T that their faces share

Picture :



4) $\chi(\Sigma) = \chi(X)$

$\quad \overset{\shortparallel}{2}$

$$= V(X) - E(X) + F(X)$$

$$= V(T) - E(T) - E(\Gamma) + V(\Gamma)$$

$$= \chi(T) + \chi(\Gamma)$$

$$\leq 2$$

↪ This gives the first claim

5) Spse $\chi(\Sigma) = 2$, then $\chi(\Gamma) = 1$

6) $\Rightarrow$ Γ is a tree

7) Thicken T and Γ into weird looking disks, which are trees, until they fill out Σ.

8) $\Rightarrow$ Σ is gluing of two disks along their boundaries

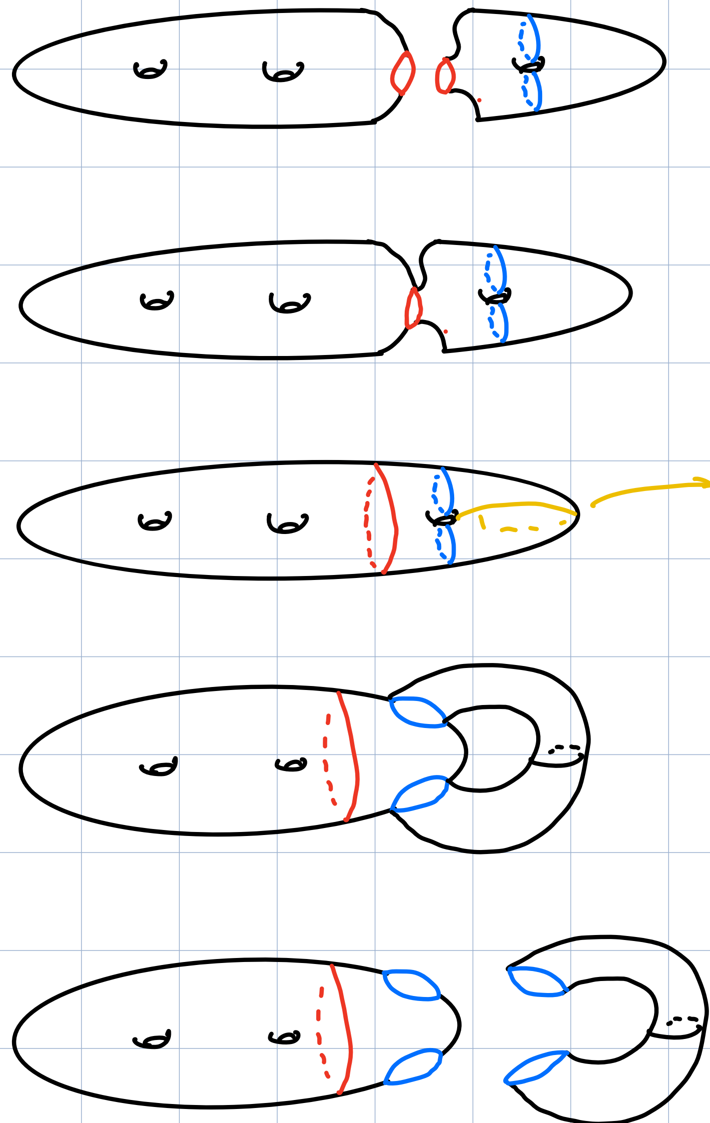
9) $\Rightarrow$ Σ is homeomorphic to S^2. $\square$

Lemma: If a surface Σ has a 2-sided curve that does not separate Σ into two pieces, then Σ is homeomorphic to $\Sigma' \# T^2$ for some surface Σ'.

Proof:

1) Let γ = 2-sided curve in Σ.
2) Thicken γ to cylinder in Σ.
3) Note, $\Sigma' \# T^2$ can also be obtained via:
   i) Remove two disjoint disks from Σ'.
   ii) Connect these boundaries via gluing in a cylinder.
4) So removing γ from Σ and capping off the boundaries w/ disks undoes a connect sum.
5) Upshot, γ let's us realize Σ as connect sum w/ T^2.

Picture :



2-sided curve that realizes/sees this connect sum.

Proof:

1) Let X = poly. cpx for Σ

2) Let T and Γ be defined as before.

3) If Γ = tree, then as argued before $\Sigma = S^2$.

4) So we assume Γ is not a tree.

$\Rightarrow$ Γ has a loop γ = 2-sided curve

5) We claim that γ does not separate Σ.

$\hookrightarrow$ If not, $\Sigma - \gamma = \Sigma_0 \cup \Sigma_1$ two separate pieces

$\hookrightarrow$ If we remove the faces and edges that γ touches in X, then this divides X into poly. cpxes X_0 and X_1 for Σ_0 and Σ_1

↳ γ doesn't meet T

$\Rightarrow$ T is completely contained in, say, X_0

↳ But T contains all the vertices of X.

$\Rightarrow$ X_1 has no vertices and thus no polygons

$\Rightarrow$ X_1 is empty, a contradiction.

6) By previous lemma, $\Sigma = \Sigma' \# T^2$ for some surface Σ'.

$\chi(X \# Y) = \chi(X) + \chi(Y) - 2$

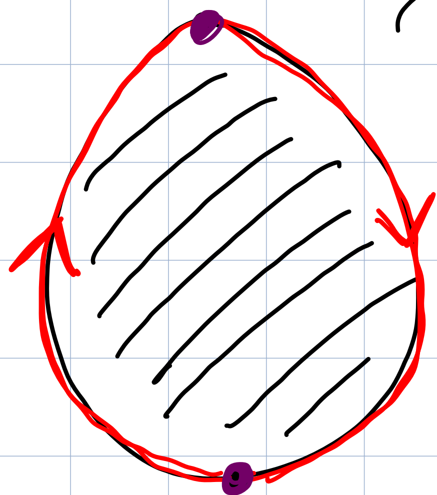
7) $\chi(\Sigma') = \chi(\Sigma) + 2$

8) $\Rightarrow$ Repeating this setup w/ Σ replaced by Σ' realizes $\Sigma' = \Sigma'' \# T^2$.

9) Eventually, this will terminate as $\chi(\Sigma'') = \chi(\Sigma') + 2$, ie, eventually $\chi(\Sigma'') = 2$ and thus $\Sigma'' = S^2$. $\square$

Nextime:

- ?????



$\chi(\;) = 1$