Lecture # 4

Outline:

1) Review from last time
2) Colorings of Maps Theorem
3) The connect sums of surfaces
4) Oriented surfaces and curves

Section 1: Review

Definition: A <u>surface</u> is space that locally looks like $\mathbb{R}^2$

↳ ie, zoom in close it just looks like a "piece of paper."

Definition: A polygonal complex is a space obtained by gluing together polygons, edges, and vertices, where by glue we mean that we identify edges w/ edges and vertices w/ vertices (could glue polygon to self)

Definition: Let X = polygonal complex w/

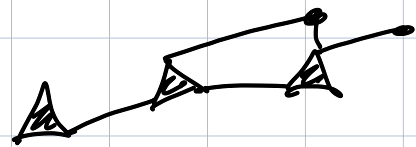
- $V(X)$ = # of vertices
- $E(X)$ = # of edges
- $F(X)$ = # of faces

The Euler characteristic of X is

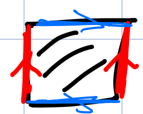
$$\chi(X) = V(X) - E(X) + F(X)$$

Examples:

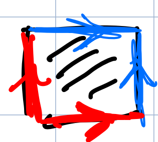
1) Random Polygonal Complex

$\chi($  $) = 11 - 14 + 3 = 0$

2) Torus

$\chi($  $) = 1 - 2 + 1 = 0$

3) Sphere

$\chi($  $) = 3 - 2 + 1 = 2$

Proposition: Let X and Y be polygonal complexes that are homeomorphic to the same surface. Then their Euler characteristics agree.

$$\chi(X) = \chi(Y)$$

Definition: The Euler characteristic of a surface Σ is the Euler characteristic of any polygonal cpx that is homeomorphic to Σ.

Remark: To compute $\chi(\Sigma)$, break Σ up into regions and count the # of vertices, edges, and faces.

Examples:

1) $\chi(S^2) = 2$

2) $\chi(T^2) = 0$

3) $\chi(\text{klein bottle}) = 0$

4) $\chi(\text{genus } g \text{ surface}) = 2-2g.$

↳ g donut holes

Section 2: Colorings of Maps (4 Colors Theorem)

Question:

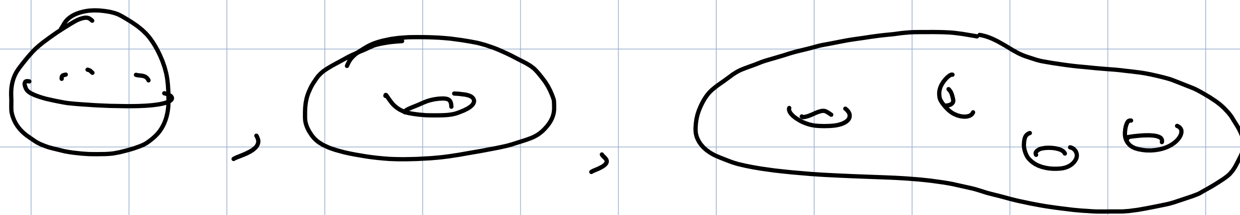
- What is the minimum number of colors needed to color any map of the <u>globe</u> so that no two adjacent regions are colored the same color?
- What is the minimum number of colors needed to color any map of a <u>surface</u> so that no two adjacent regions are colored the same color?

Definition: A surface is <u>compact</u> if it admits a polygonal complex structure w/ a finite # of vertices, edges, and faces.

↳ Secretly, we needed to assume that our surfaces were compact when we defined their Euler characteristics.

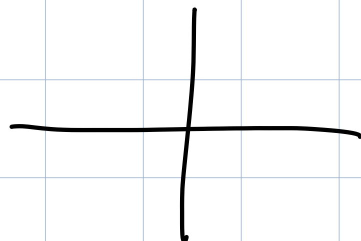
Example: 1) Sphere, torus, multiple donut holes

Compact

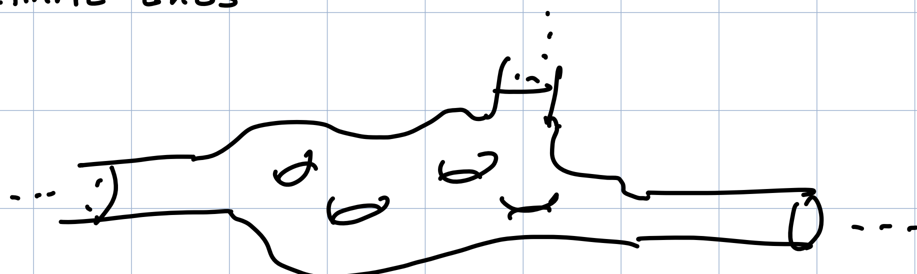


non compact.

2) $\mathbb{R}^2$



3) Infinite Ends



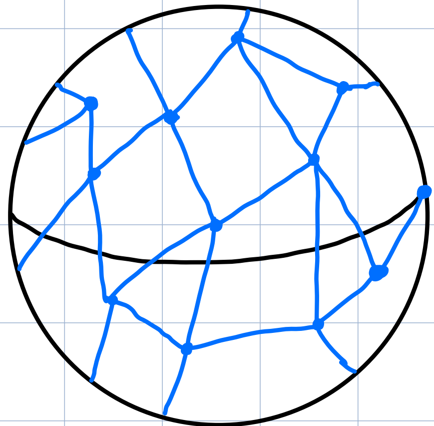
Definition: A geographic complex associated to a compact surface Σ is a polygonal complex that is homeomorphic to Σ and satisfies:

1) Every face does not meet itself
2) Any two faces that meet/share a unique edge
3) At least three faces meet at each vertex.

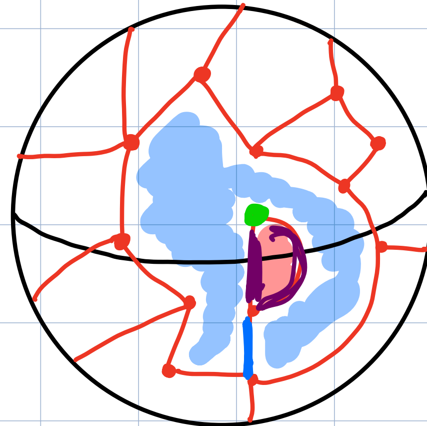
Remark: Intuitively, a geographic cpx is a map of the surface that satisfies

1) A region cannot boarder itself
2) Two regions can only share one unique boarder
3) Vertices are where 3 or more regions meet

Example:



Geo cpx.



Not geo cpx

1) A region cannot boarder itself

2) Two regions can only share one unique boarder

3) Vertices are where 3 or more regions meet

Definition:

- A legal coloring of a geo. cpx. is an assignment of a color to each face st no two adjacent faces have the same color.

- The coloring number of a geo cpx X

$$N(X) = \begin{array}{l}\text{minimum \# of colors needed to} \\ \text{produce a legal coloring of } X.\end{array}$$

- The coloring number of a compact surface Σ is

$$N(\Sigma) = \begin{array}{l}\text{minimum \# of colors needed to} \\ \text{produce a legal coloring of all geo.} \\ \text{cpx associated to } \Sigma.\end{array}$$

Theorem:

$$N(\Sigma) \leq \frac{7 + \sqrt{49 - 24 \cdot \chi(\Sigma)}}{2}$$

Corollary: Any geo. cpx on Σ can be legally colored using

$$\frac{7 + \sqrt{49 - 24 \cdot \chi(\Sigma)}}{2}$$

colors.

Example:

1) $\Sigma = S^2 \Rightarrow$ need at most 4 colors

2) $\Sigma = T^2 \Rightarrow$ " " " 7 "

3) $\Sigma =$ genus 4 surface $\Rightarrow$ need at most 10 colors

Remark: To prove the theorem for $\Sigma = S^2$ is extremely difficult. We will prove it for $\chi(\Sigma) \leq 1$.

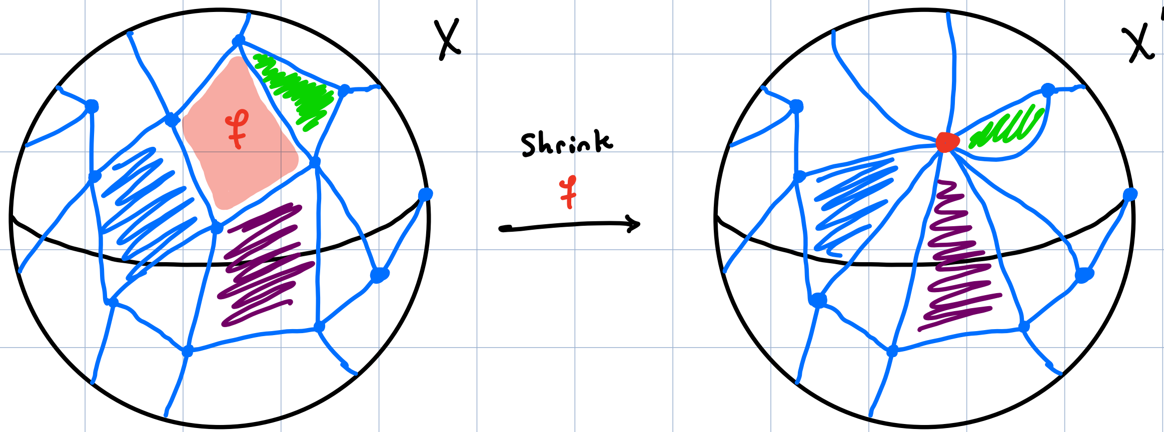
Notation: Let X be the geo. cpx associated to Σ that satisfies:

1) $N(X) = N(\Sigma)$
2) If Y is another geo. cpx associated to Σ w/ $N(Y) = N(X)$, then $F(X) \leq F(Y)$.

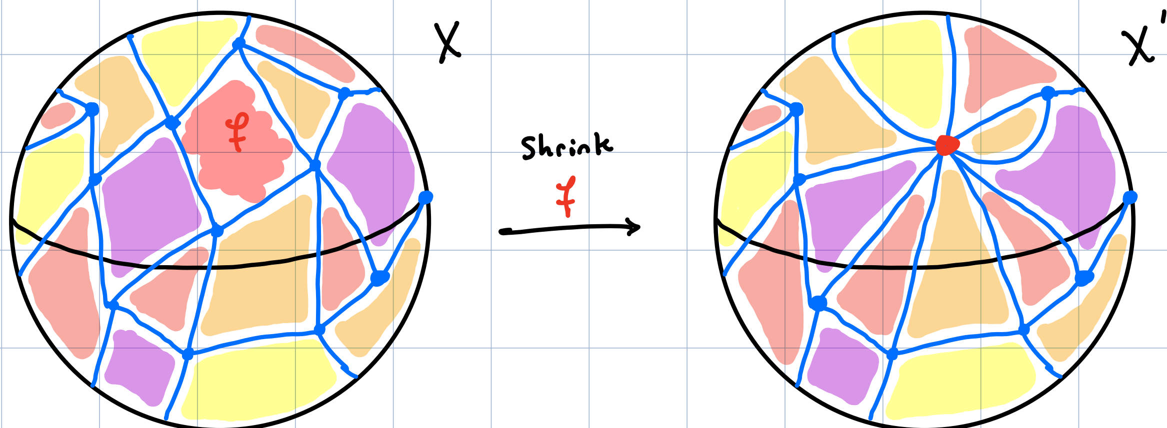
Lemma 0: Every face of X has at least $N(X)-1$ edges.

Proof:

1) We suppose by way of contradiction that there exists a face in X w/ strictly less than $N(X)-1$ edges.
2) Denote this face by f.
3) Shrink f and all of its edges down to a single vertex. This produces a new geo. cpx X'.



4) Since this proceedure does not produce any new edges any coloring of X gives rise to a coloring of X'



5) So $N(X') \le N(X)$

6) If $N(X') = N(X)$, then by assumption on X,

$F(X) \le F(X') = F(X) - 1$

$\Rightarrow$ we actually must have $N(X') < N(X)$

7) So we may color X' w/ $N(X)-1$ colors.

But this allows us to color X w/ $N(X)-1$ colors.

Namely, we color X', then since f has less than $N(X)-1$ edges, it has at most $N(X)-2$ adjacent faces. So we can always pick on of the $N(X)-1$ colors to color f differently than all its adjacent faces. $\Rightarrow$ $N(X) \le N(X)-1$, a contradiction.

8) $\Rightarrow$ Every face of X has at least $N(X)-1$ edges

Lemma 1: $(N(X)-1)\cdot F(X) \leq 2\cdot E(X)$

$N(X) \leq \frac{2E}{F}+1$

Proof:

1) Every edge touches two unique faces.

=> Average # of edges per face is $2E(X)/F(X)$

2) By Lemma 0, each face has at least $N(X)-1$ edges

=> Average # of edges per face $\geq N(X)-1$

3) Combining these inequalities,

$$N(X)-1 \leq 2E(X)/F(X)$$

Lemma 2: $3V(X) \leq 2E(X)$

Proof:

1) Let $\tilde{X}$ = preglued collection of polygons that we glue together to produce X.

2) Note, $2E(X) = E(\tilde{X})$

3) Since at least 3 faces meet at each vertex,

$$3V(X) \leq V(\tilde{X})$$

4) Since $\tilde{X}$ is disjoint collection of polygons,

$$E(\tilde{X}) = V(\tilde{X})$$

5) Combining,

$$2E(X) = E(\tilde{X}) = V(\tilde{X}) \geq 3V(X) \qquad \square$$

Lemma 3: $N(X) = N(\Sigma) \leq 7 - 6 \cdot \chi(\Sigma)/F(X)$

Proof:

$$
\begin{aligned}
N(\Sigma) &= N(X) \\
&\leq 1 + 2E(X)/F(X) && \text{Lemma 1} \\
&\leq 1 + (6E(X) - 6V(X))/F(X) && \text{Lemma 2} \\
&= 1 + (-6\chi(X) + 6F(X))/F(X) \\
&= 7 - 6 \cdot \chi(X)/F(X) \\
&= 7 - 6 \cdot \chi(\Sigma)/F(X)
\end{aligned}
$$

$\square$

$6\chi = 6V - 6E + 6F$

$6E - 6V = 6F - \chi \cdot 6$

Proof: $(\chi(\Sigma) = 1)$:

$$N(\Sigma) \leq 7 - 6/F(X)$$
$$\leq 6$$
$$= \frac{7 + \sqrt{49 - 24 \cdot 1}}{2}$$
$$= \frac{7 + \sqrt{49 - 24 \cdot \chi(\Sigma)}}{2}$$

$\square$

Proof: $(\chi(\Sigma) \leq 0)$

1) $N(\Sigma) = N(X) \leq F(X)$

2) $N(\Sigma) \leq 7 - 6 \cdot \chi(\Sigma)/F(X)$

$$\leq 7 - 6 \cdot \chi(\Sigma)/N(\Sigma)$$

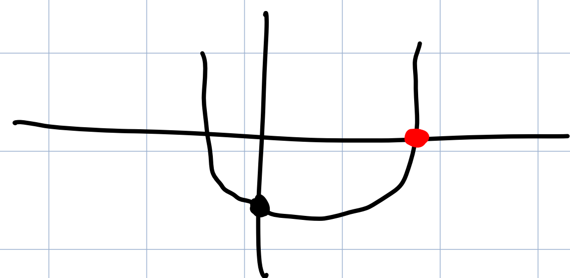
3) $\Rightarrow$ $N(\Sigma)^2 - 7N(\Sigma) + 6 \cdot \chi(\Sigma) \leq 0$

4) This polynomial in $N(\Sigma)$ is upwards opening w/ at least one point on $N(\Sigma)$-axis.

5) $\Rightarrow$ Largest $N(\Sigma)$ for which this holds is largest zero of poly.

6) $\Rightarrow$ $$N(\Sigma) \leq \frac{7 + \sqrt{49 - 24 \cdot \chi(\Sigma)}}{2}$$

$\square$

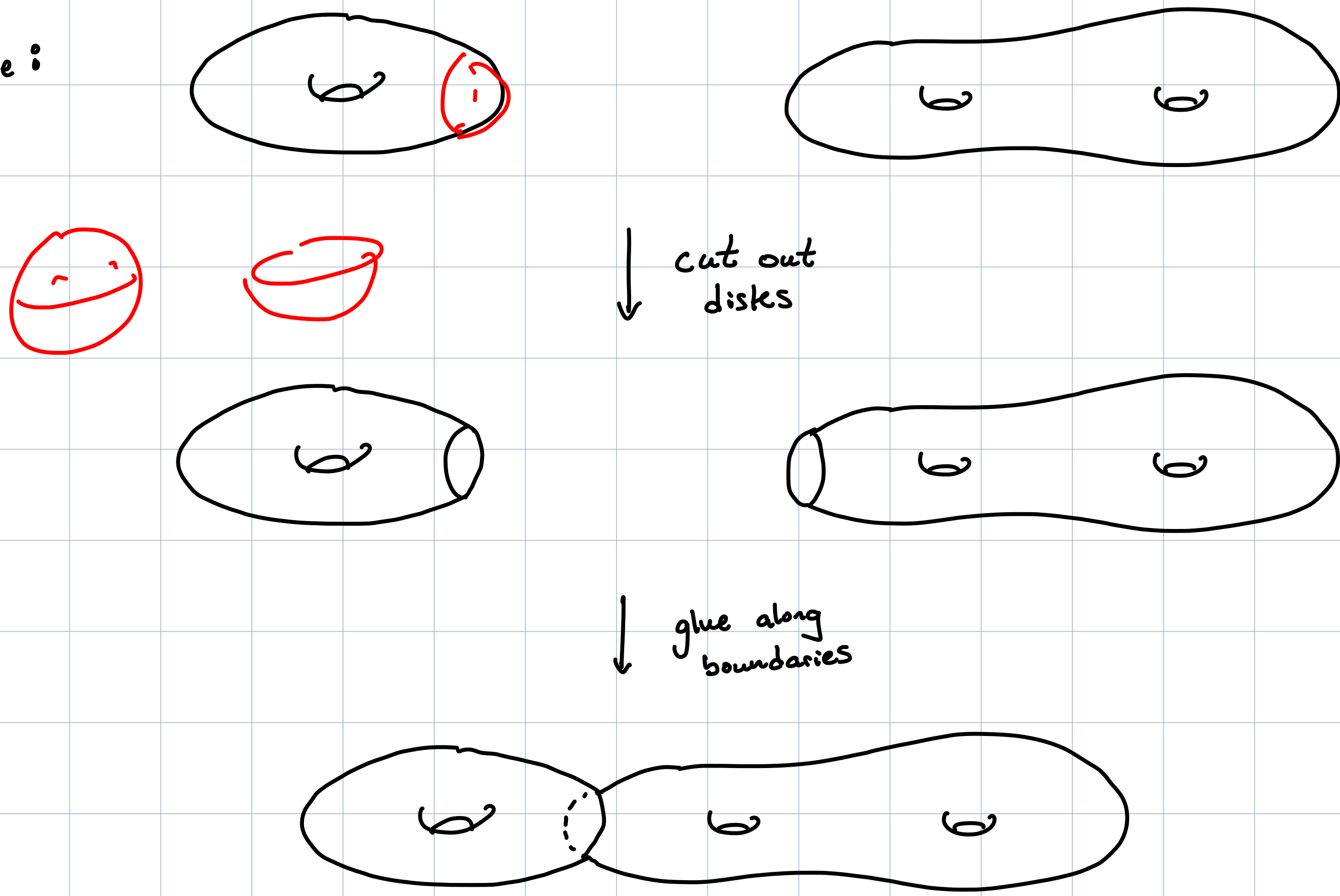


Section: The connect sum of surfaces

Definition: Given two surfaces X and Y, the connect sum of X and Y, denoted $X \# Y$, is obtained via

1) Remove an open disk from both X and Y to create two surfaces w/ "boundaries"
2) Glue the resulting boundaries together to create the new surface $X \# Y$.

Picture :



Example:

1) $T^2 \# T^2 = 2$ holed surface $=$ genus 2 surface.

2) $S^2 \# S^2 = S^2$

3) $S^2 \# T^2 = T^2$

4) $T^2 \# \dots \# T^2 \}$ g-times $=$ genus g surface.

Proposition: $\chi(X \# Y) = \chi(X) + \chi(Y) - 2$

Ex:

1) $\chi(T^2 \# T^2) = \chi(T^2) + \chi(T^2) - 2 = 0 + 0 - 2 = -2$

2) $2 = \chi(S^2) = \chi(S^2) + \chi(S^2) - 2 = 2 + 2 - 2 = 2.$ ✓

3) ✓

4) $\chi(T^2 \# \dots \# T^2) = 2 - 2g.$