Lecture # 3

Outline:

1) Review from last time

2) More on Planar diagrams

3) The Euler characteristic

4) Planarity of graphs

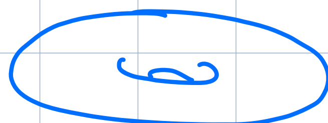
Section 1: Review

Definition: A <u>surface</u> is space that locally looks like $\mathbb{R}^2$

↳ ie, zoom in close it just looks like a "piece of paper."

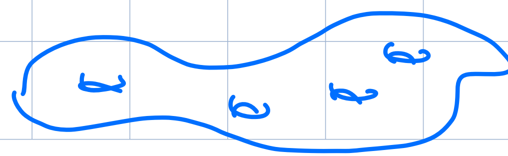
Examples: ① Sphere $= S^2$

② Torus $= T^2$



③ Klein bottle

④ Intertubes w/ multiple holes.



Remark:

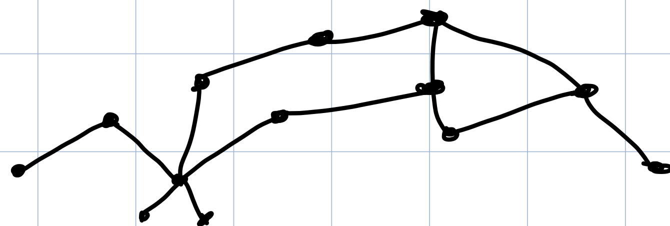
- We want a way of viewing surfaces as combinatorial objects.
-

$$\begin{tikzcd}
\text{Spaces} \arrow[r] & \text{Combinatorics} \arrow[l, dashed, bend right, "?"'] \\
\text{Surfaces} \arrow[r] & \text{Polygonal complexes}
\end{tikzcd}$$

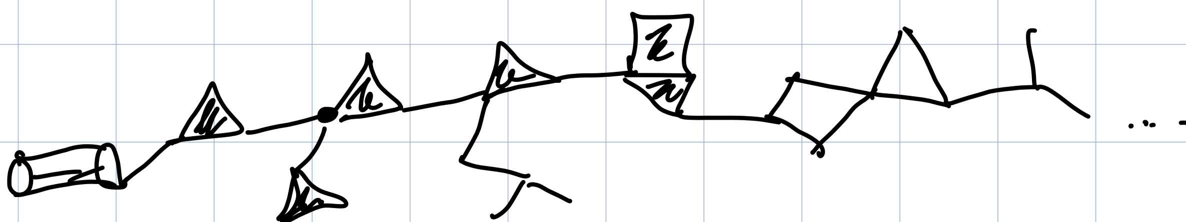
Definition: A polygonal complex is a space obtained by gluing together polygons, edges, and vertices, where by glue we mean that we identify edges w/ edges and vertices w/ vertices (could glue polygon to self)

Example:

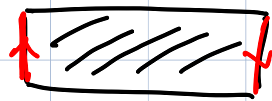
① Graph:

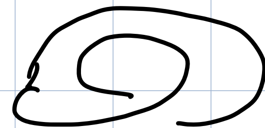


② Something Wild:

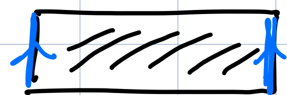


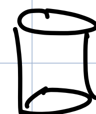
④ Möbius Band





⑤ Cylinder





Remark: 1) We can always break surfaces up into polygonal cpxes

↪ A surface is homeomorphic to this associated polygonal complex

2) There are an infinite # of ways we could break it up

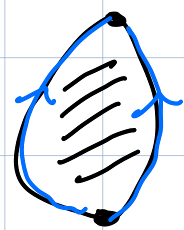
3) There are strictly infinitely many more polygonal cpxes than

Section: More on Planar Diagrams

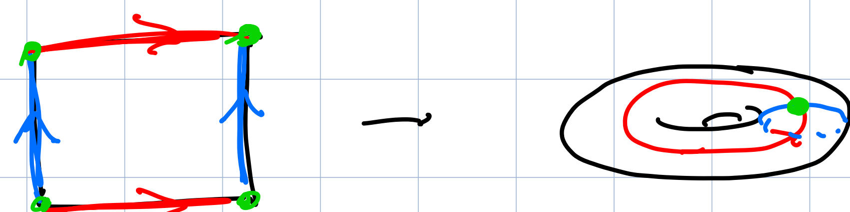
Definition: A planer diagram is a polygonal complex obtained by gluing together all pairs of edges of a single $2n$-polygon.

Examples:

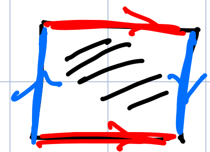
1) Sphere:



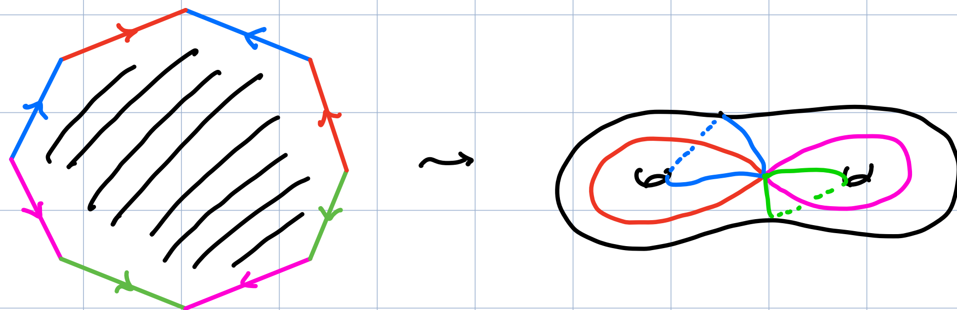
2) Torus



3) Klein bottle



4) Genus 2 surface



5) Genus g surface = "intertube w/ g-holes"

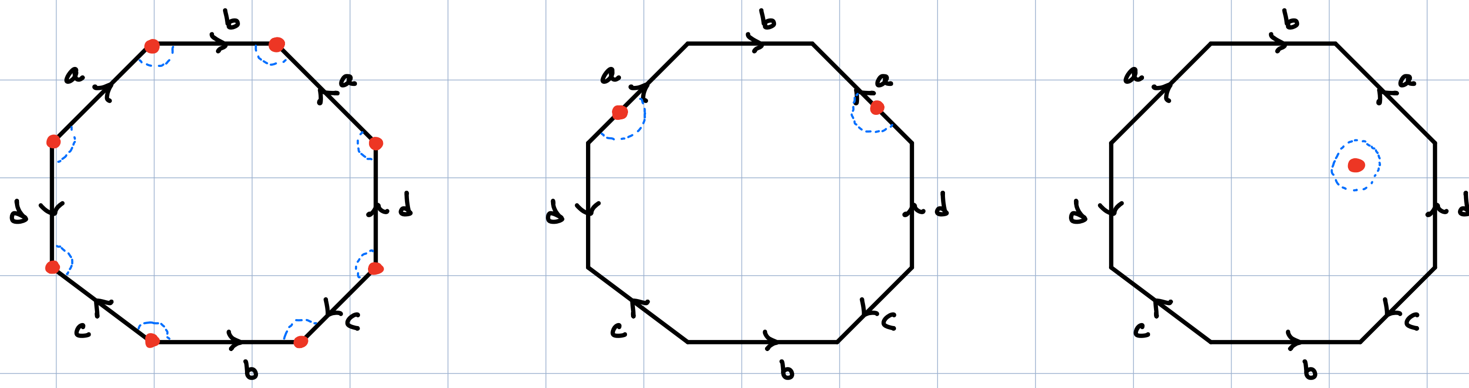
Need $4g$-polygon for planar dgm

⑤

Proposition: Every planar diagram is homeomorphic to a surface.

Proof:

- Need to show that locally about every point in the planar diagram the space looks like $\mathbb{R}^2$.
- We have 3 possibilities
    1) the point is a vertex
    2) the point is in an edge
    3) the point is in the polygon.
- We think about each case



- Punchline: the gluing required for a planar diagram forces us to patch together regions that don't look like $\mathbb{R}^2$ in the "preglued" polygon into new regions that do look like $\mathbb{R}^2$ in the "glued" polygon.

Section: Euler Characteristic

Definition: Let X = polygonal complex w/

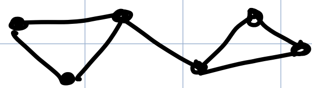
- $V(X)$ = # of vertices
- $E(X)$ = # of edges
- $F(X)$ = # of faces = # of polygons

The Euler characteristic of X is

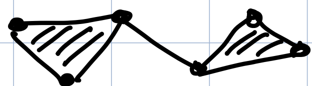
$$\chi(X) = V(X) - E(X) + F(X)$$

Examples:

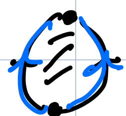
1) Graph

$\chi\Big($  $\Big) = 6 - 7 = -1$

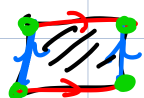
2) Random

$\chi\Big($  $\Big) = 6 - 7 + 2 = 1$

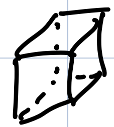
3) Sphere

$\chi\Big($  $\Big) = 2 - 1 + 1 = 2$

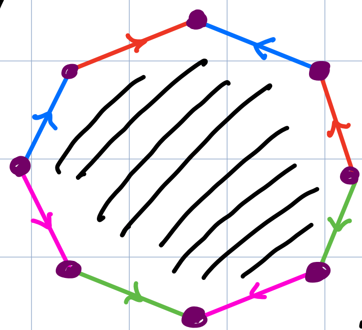
4) Torus

$\chi\Big($  $\Big) = 1 - 2 + 1 = 0$

5) Sphere 2

$\chi\Big($  $\Big) = 8 - 12 + 6 = 2$

6)

$\chi\Big($  $\Big) = 1 - 4 + 1 = -2$

7) $\chi(\text{genus } g \text{ surface}) = 2 - 2g$

Remark: It appears that the Euler characteristics of polygonal complexes that are homeomorphic to the same surface always agree

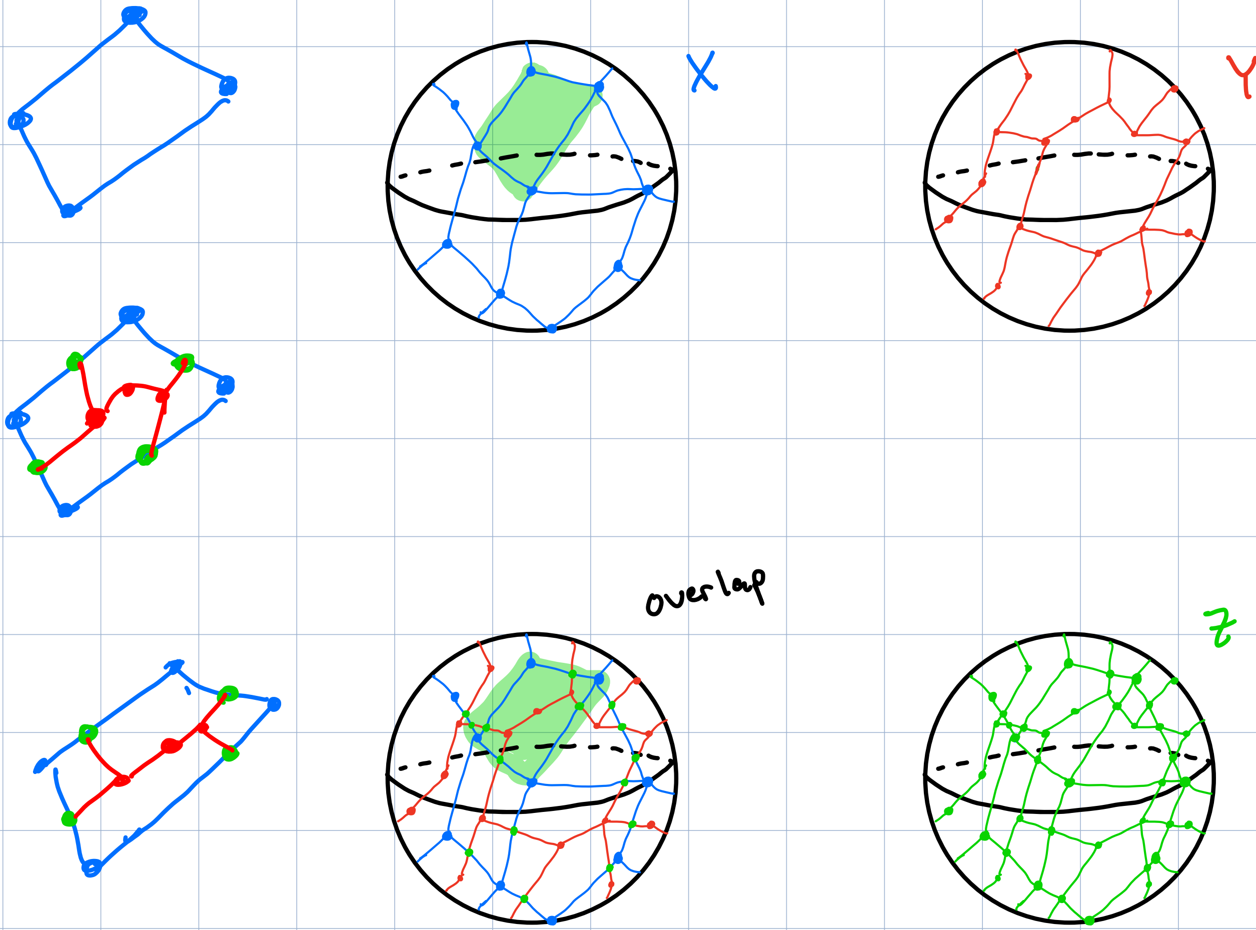
↳ Think: Given two different maps/ways of breaking up a surface into regions, they will have the same Euler characteristic.

Proposition: Let X and Y be polygonal complexes that are homeomorphic to the same surface. Then their Euler characteristics agree.

$$\chi(X) = \chi(Y)$$

Proof:

- X and Y give two different ways of breaking our surface up into polygon-like regions
- We can "overlap" X and Y on our surface, adding vertices where the edges of X intersect the edges of Y, to produce a new polygonal cpx for the surface. Call it Z.


X
Y
overlap
Z

- Note that one can obtain Z from X (similarly from Y) by
    1) Adding edge between two vertices in a polygon
    2) Adding vertex to interior of an edge
    3) Adding vertex to the interior of a polygon and connecting it to an existing vertex via an edge.
- If these don't change the Euler characteristic, then repeatedly applying them to X to get Z will give

$$\chi(X) = \chi(Z).$$

Similarly for Y, $\chi(Y) = \chi(Z)$.

- Type 1 => 1 new edge, 1 face divided into 2

$$\chi(\text{new}) = V(\text{old}) - (E(\text{old}) + 1) + (F(\text{old}) + 1) = \chi(\text{old})$$

- Type 2 => 1 new vertex, 1 edge divided into 2

- Type 3 => 1 new vertex, 1 new edge.

Definition: The Euler characteristic of a surface Σ is the Euler characteristic of any polyogonal cpx that is homeomorphic to Σ.

Remark:
- To compute $\chi(\Sigma)$, break Σ up into regions and count the # of vertices, edges, and faces.
- This allows us to prove that we are "logical beings".

Examples:

1) $\chi(S^2) = 2$
2) $\chi(T^2) = 0$
3) $\chi(\text{klein bottle}) = 0$
4) $\chi(\text{genus 2 surface}) = 2 - 2g$

Definition:

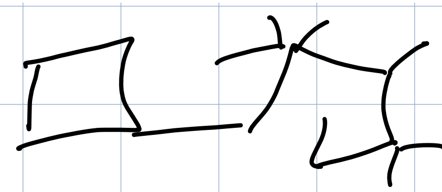
- A graph is a polygonal complex composed of edges.
- A graph is a tree if every pair of vertices is connected via a unique sequence of edges.

Definition: A graph is planar if it is given by the edges of a polygonal complex for S^2.

Fact: A graph is planar if it may be drawn in $\mathbb{R}^2$ w/out having edges intersecting / laying over each other

Proof: Remove a face from sphere and lay the remainder flat on the plane

Question: Is every graph planar?

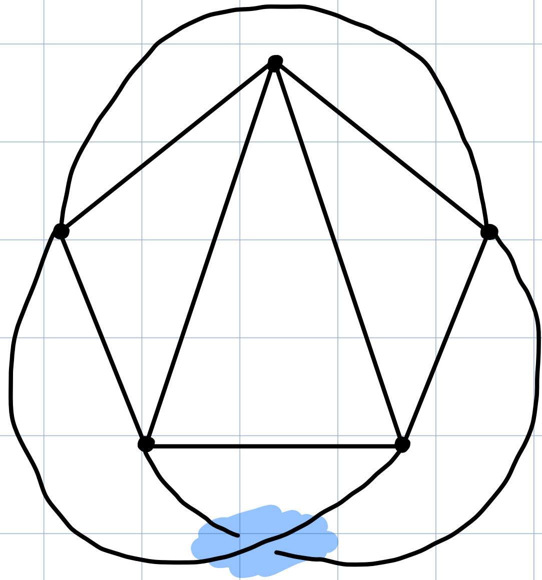


Answer: No

Reason: The Euler characteristic of the sphere puts restrictions on how edges can come together.

Notn: Let K_5 = graph w/ 5 vertices and 10 edges st every pair of vertices is connected by a unique edge.

Claim: K_5 is not a planar graph.

$K_5 =$ 

Proof:

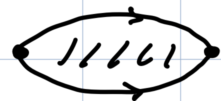
- We use proof by contradiction. So we assume K_5 is planar and derive a contradiction. Thus our assumption will be wrong and K_5 must be non-planar.
- If K_5 is planar $\Rightarrow$ determines poly. cpx for S^2, say X.
- By the Euler characteristic proposition from today,

$$
\begin{aligned}
2 &= \chi(S^2) \\
&= V(X) - E(X) + F(X) \\
&= V(K_5) - E(K_5) + F(X) \\
&= 5 - 10 + F(X)
\end{aligned}
$$

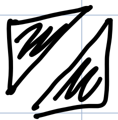
$\Rightarrow F(X) = 7$

- * Note every face of X has at least 3 unique edges.

If not, then the two vertices on the face are connected via 2 different edges



But this can't happen for K_5



$\tilde{X}$

$= X$

- We claim that $\boxed{3F \leq 2E}$

  Let $\tilde{X}$ = denote the "preglued" collection of polygons that we glue together to form X.

- Note $\tilde{X}$ is itself a polygonal complex.

- Note

$$\boxed{2E(X)} = E(\tilde{X}) \geq 3F(\tilde{X}) = \boxed{3F(X)}$$

↪ Used #.

- $21 = 7 \cdot 3 = 3 \cdot F(X) \leq 2E(X) = 2\overset{10}{E(K_5)} = 20$

$\Rightarrow$ contradiction

Nexttime: 1) Colorings of Maps Theorem (4 Colors Theorem)

2) Preliminaries for the Classification of surfaces.

Exer: If Γ = connected graph, then $\chi(\Gamma) \leq 1$ w/ equality if and only if Γ = tree.

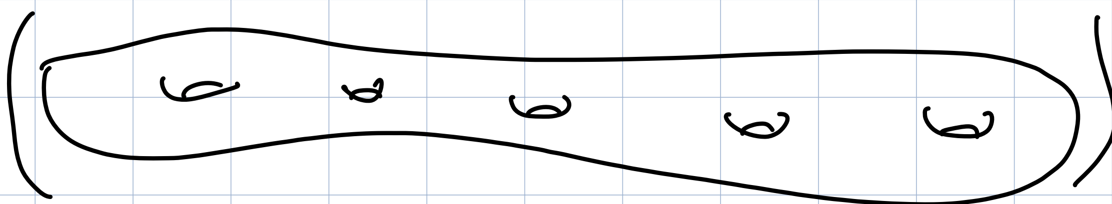
χ = "Chi" $\quad 1 \geq \chi = V - E$

$E + 1 \geq V$

Exer: Is K_5 the set of edges of a polygonal cpx on some genus g surface?

Defn: A surface of genus g is "an intertube" w/ g holes

Ex: genus



$= 5$