Lecture # 2

Outline:
1) Review from last time
2) A word on Mathematical rigor
3) Surfaces
4) Polygonal Complexes
5) The Euler characteristic

Section 1: Review

Remark: 1) Topology = the study of shape

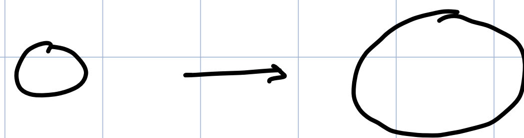
2) Two objects have the same shape if we can continuously deform / rearrange, in an invertible manner, the points of one object to obtain the other object.

↳ Continuous: takes points infintesimally close together to points infintesimally close together

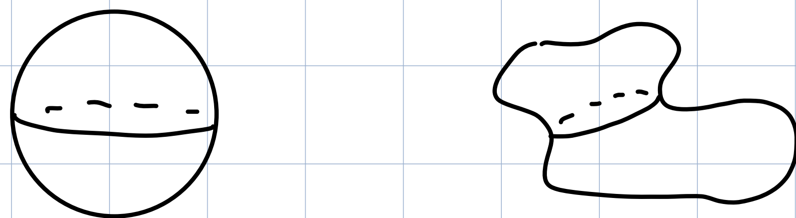
↳ Invertible: Continuous undoing of rearrangement such that every point goes back to where it started.

3) If two spaces/ objects have the same shape, then we say they are <u>homeomorphic</u>

↪ Small circle vs Big square



↪ Deflated Beach Ball vs Round Beach Bal

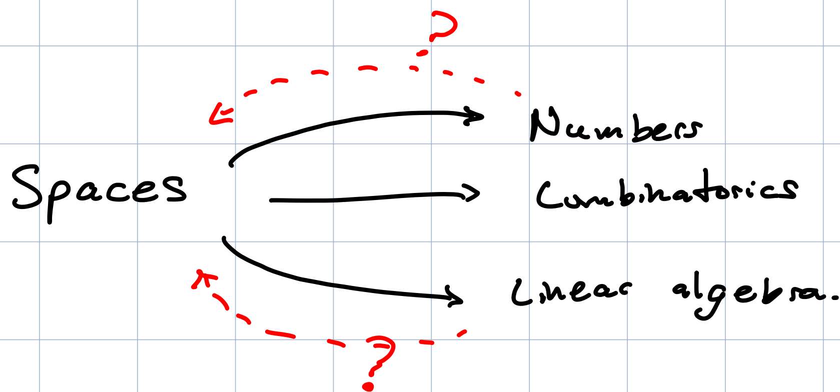


↪ Coffee Mug vs Donut

Goal :

1) create invarients of spaces that detect differences in shape

2) Use invariants to prove non-trivial results



Section: Mathematical Rigor

Remark: Need MATHEMATICAL means of studying shapes as oppose to a heuristical/visual means.

↳ There are spaces we can't visualize, but that we can nevertheless study.

Claim: The square root of 2 is irrational.

↳ $\sqrt{2} \neq p/q$ for some integers p and q

Remark: The proof goes via a technique called proof by contradiction

Remark:
- A statement is either true or false.
- To prove true, either prove that it is true or not false.
- To prove not false, we assume that the false statement holds and then under this assumption try to show that something we know (a priori) to be true to be false, under this assumption.
- We say that we arrive at a contradiction.
- Upshot is that our original assumption of falsehood was incorrect. So the statement must have been true.

Proof:

- Spse by way of contradiction that $\sqrt{2} = p/q$.
- We may assume $p = p_1 \cdots p_k$, $q = q_1 \cdots q_\ell$ w/ no common prime factors (ie p/q is in lowest-terms)
- $\Rightarrow 2q^2 = p^2$

  $\Rightarrow p^2$ is even

  $\Rightarrow p$ is even

  square of odd is odd

  $\Rightarrow p^2$ is divisible by 4, so $p^2 = 4 \cdot r$

  $\Rightarrow q^2 = 2r$

  $\Rightarrow q^2$ is even

  $\Rightarrow q$ is even
- So both p and q have a common factor of 2

  $\Rightarrow$ contradiction

$\square$

Remark: Why does higher math alway appear so alien?

To begin, the moduli spaces $\overline{\mathcal{M}}(\sigma, x_0, x_\ell)$, furnished with choices of implicit atlases $\mathcal{A}$ and coherent orientations $\mathfrak{o}$, can be assembled over the Kan complex $\mathcal{JH}(M, \Omega, \lambda)$ to define a flow category $\overline{\mathcal{M}}$ over $\mathcal{JH}(M, \Omega, \lambda)$. From this data, Pardon constructs a trivial Kan fibration

$$\widetilde{\mathcal{JH}}(M, \Omega, \lambda) \to \mathcal{JH}(M, \Omega, \lambda).$$

Roughly speaking, a section of this fibration is a coherent choice of virtual fundamental chains for the above moduli spaces. Given such data, Pardon constructs a diagram

$$\begin{tikzcd} \widetilde{\mathcal{JH}}(M, \Omega, \lambda)^{\mathrm{op}} \arrow[r, "\widetilde{\mathbb{H}}(\mathcal{M}, \mathcal{A}, \mathfrak{o})"] \arrow[d, "\pi"] & \mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\Lambda)) \arrow[d, "\text{forget}"] \\ \mathcal{JH}(M, \Omega, \lambda)^{\mathrm{op}} \arrow[r, dotted, "\mathbb{H}(\mathcal{M}, \mathcal{A}, \mathfrak{o})"] & H^0(\mathrm{Ch}(\Lambda)) \end{tikzcd} ,$$

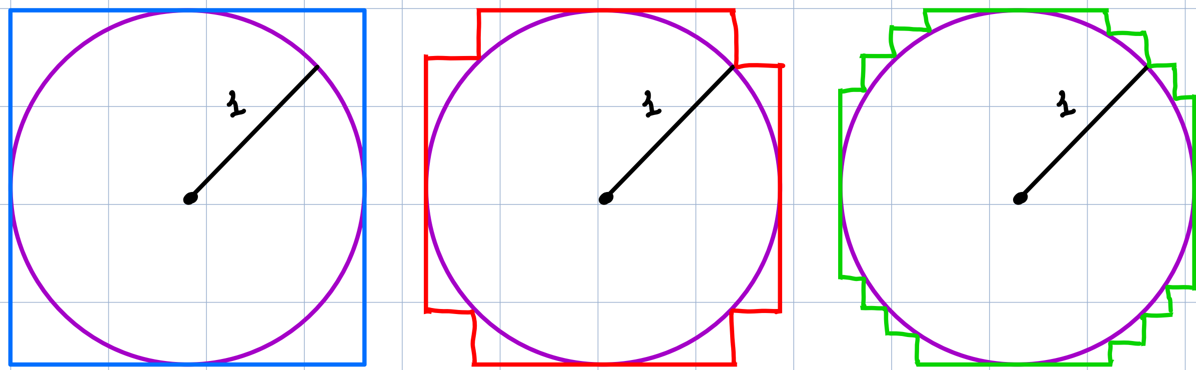
where $\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\Lambda))$ is the differential graded nerve of $\mathrm{Ch}(\Lambda)$ and $H^0(\mathrm{Ch}(\Lambda_{\geq 0}))$ is the associated homotopy category, see [Lur17, Construction 1.3.1.6]. To obtain the dashed arrow $\mathbb{H}(\mathcal{M}, \mathcal{A}, \mathfrak{o})$, one must choose a section of π. The functor $\mathbb{H}(\mathcal{M}, \mathcal{A}, \mathfrak{o})$ does not depend on the choice of section.

Jargon, notation, technical definitions, etc. allow mathematicians to concisely express and rigorously prove ideas.

For this class, Ideas/pictures > technical details.

Claim: $\pi = 4$

Proof:

- 

- $8 = \text{length}(\text{blue})$

  $= \text{length}(\text{red})$

  $= \text{length}(\text{green})$

  $\vdots$

  $= \text{length}(\text{purple})$

  $= 2\pi$

- $\Rightarrow \pi = 4$

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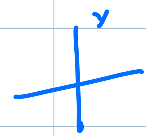
Section: Surfaces

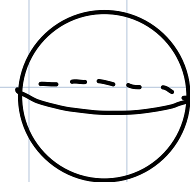


$= \{(x,y) \mid x, y \text{ real numbers}\}$

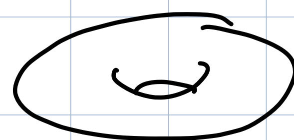
Definition: A <u>surface</u> is space that locally looks like $\mathbb{R}^2$

↳ ie, zoom in close it just looks like a piece of paper.

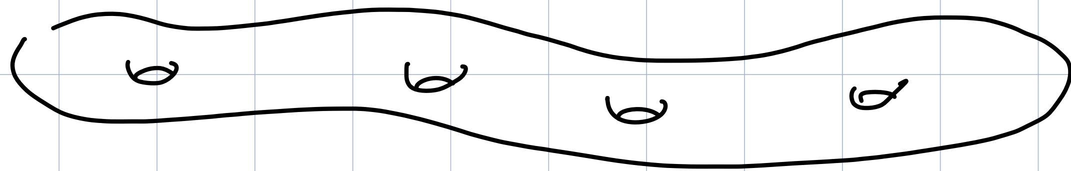
Examples: ① Beach ball = sphere = S^2.



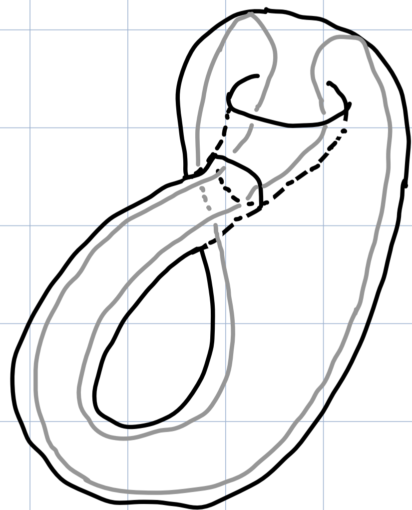
② Intertube = torus = T (T^2)



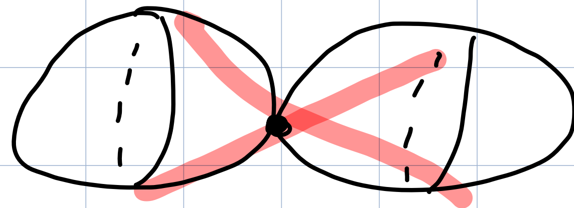
③ Complicated intertuble



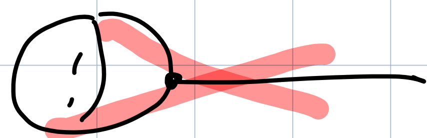
4 Klein Bottle



⑤ Kissing spheres



⑥ Ball on a stick



Remark: We want to view surfaces as combinatorial objects.

↳ We think about our globe as a map, which is combinatorial.

Our first attempt to study shape will be to turn topology into combinatorics.

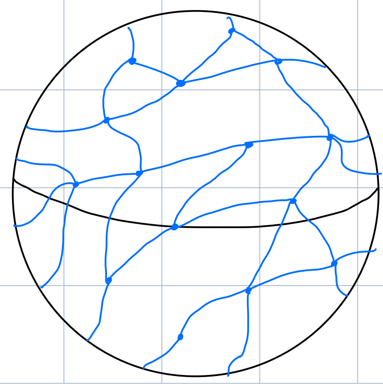
Section: Polygonal Complexes

Example: Divide the globe up according to regions.

That is, envision globe as polygons patched together

1. Faces of polygons $\longrightarrow$ Regions
2. Edges of polygons $\longrightarrow$ Borders
3. Vertices of polygons $\longrightarrow$ Borders of borders.

Picture:



Remark: This works equally as well w/ <u>any</u> surface

Remark: We will consider a broader class of spaces called polygonal complexes.

↪ surfaces broken up into polygons will give a specific class of examples.

vertex = •

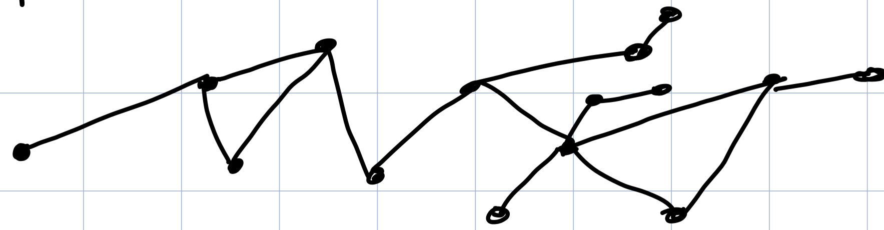
edge = •—•

polygon = 

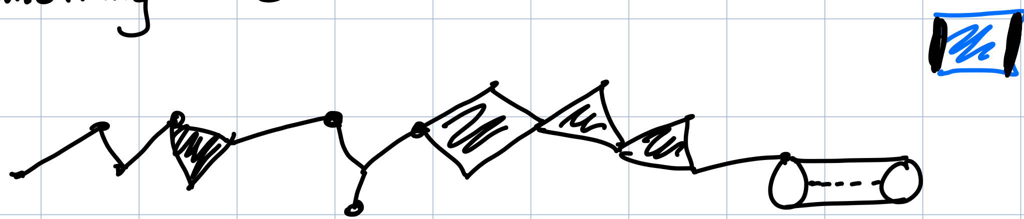
Definition: A polygonal complex is a space obtained by gluing together polygons, edges, and vertices, where by glue we mean that we identify edges w/ edges and vertices w/ vertices (could glue polygon to self)

Example:

① Graph

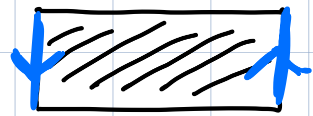


② Something Wild

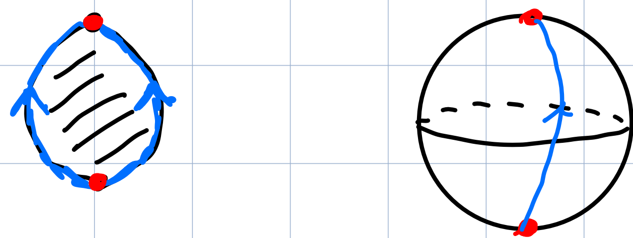


③ Non-example

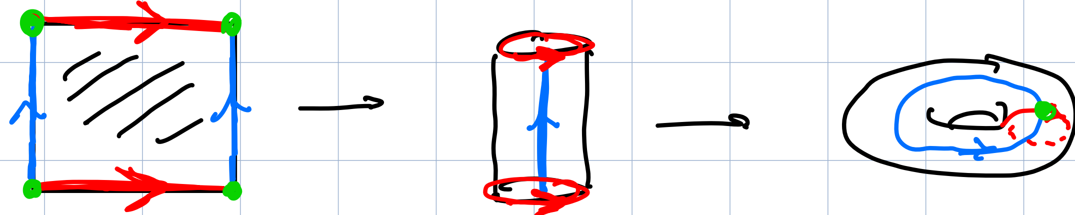
④ Möbius Band



⑤ Sphere



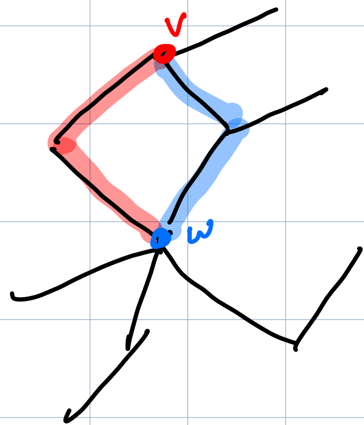
⑥ Torus



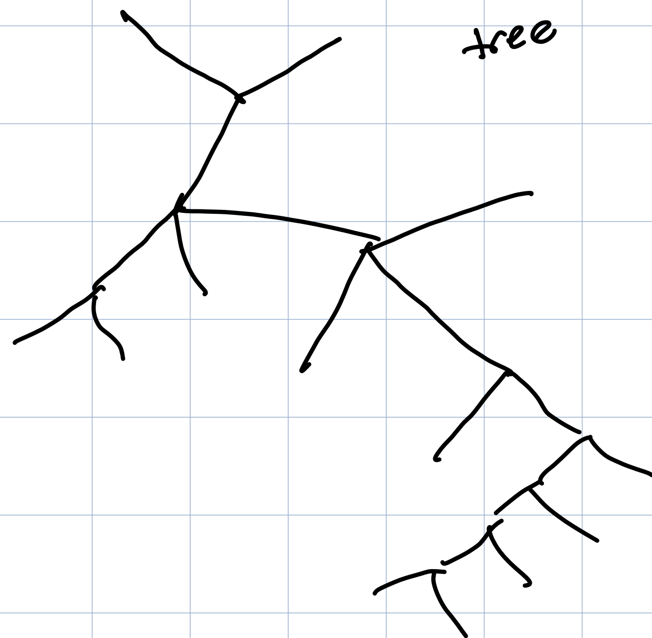
Definition:

- A graph is a polygonal complex composed of edges.
- A graph is a tree if every pair of vertices is connected via a unique sequence of edges.

Example:

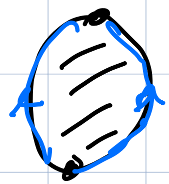


graph

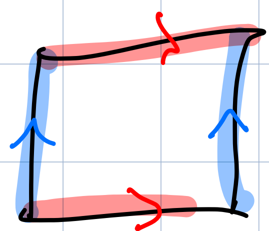


Definition: A planer diagram is a polygonal complex obtained by gluing together all pairs of edges of a single $2n$-polygon.

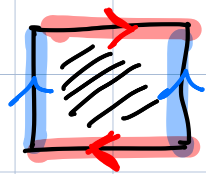
Example: 1) Sphere

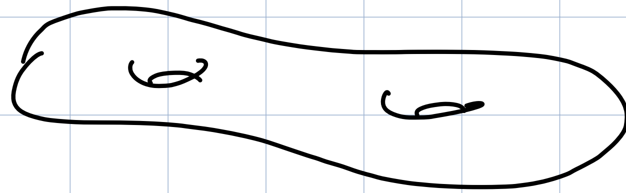


2) torus

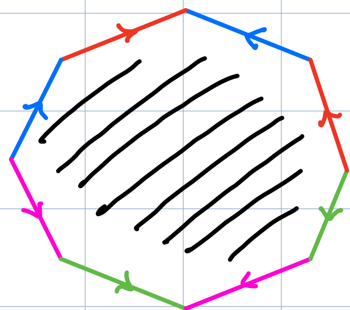


3) Klein bottle

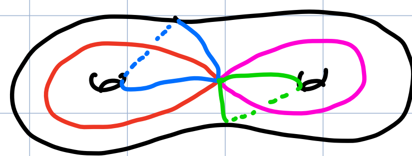




4) Genus 2 surface

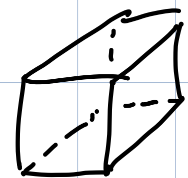


$\rightsquigarrow$



Remark: 1) We can always break surfaces up into polygonal cpxes

↪ A surface is homeomorphic to this associated polygonal complex



2) There are an infinite # of ways we could break it up

3) There are strictly infinitely many more polygonal cpxes than surfaces

Section: Euler Characteristic

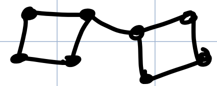
Definition: Let X = polygonal complex w/

- $V(X)$ = # of vertices
- $E(X)$ = # of edges
- $F(X)$ = # of faces

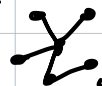
The Euler characteristic of X is

$$\chi(X) = V(X) - E(X) + F(X)$$

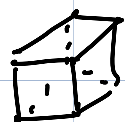
Examples: 1) Graph

$\chi\Big($  $\Big) = 8-9 = -1$, $\quad \chi\Big($  $\Big) = 0$

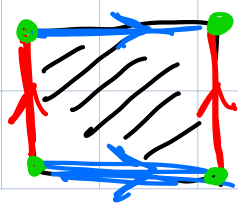
2) Tree

$\chi\Big($  $\Big) = 6-5 = 1$

3) Sphere

$\chi\Big($  $\Big) = 8 - 12 + 6 = 14 - 12 = 2$

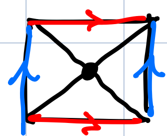
4) Torus

$\chi\Big($  $\Big) = \chi\Big($  $\Big) = 1 - 2 + 1 = 0$

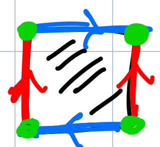
5) Sphere 2

$\chi($  $) = 2 - 1 + 1 = 2$

6) Torus 2

$\chi($  $) = 2 - 6 + 4 = 0$

7) Klein bottle

$\chi($  $) = 1 - 2 + 1 = 0$

Next time: 1) We will prove that the Euler characteristics of any two polygonal cpxes associated to a surface agree

2) Study Euler characteristics of graphs

3) Planarity of graphs

4) Colorings of Maps Theorem